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AUTOMORPHISMS GENERATED BY UMBRAL CALCULUS ON A NUCLEAR SPACE OF ENTIRE TEST FUNCTIONS

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Abstract. In this paper we show that Sheffer operators, mapping monomials to certain Sheffer polynomial sequences, such as falling and rising factorials, Charlier, and Hermite polynomials extend to continuous automorphisms on the space of entire functions of first order growth and minimal type.

1. Introduction

In the 1970s, the theory of generalized functionals of infinitely many variables with a dual pairing between spaces of test and generalized functions generated by Gaussian measures was introduced independently by Yu. M. Berezansky [2], Yu. G. Kondratiev [12], and T. Hida [10]. The underlying principle for this analysis is the construction of suitable Gelfand triples of test and generalized functions. A very effective approach is to embed polynomials into a countably Hilbert space. Depending on the specific choice of these Hilbert spaces one thus obtains e.g. the spaces of Hida or the Kondratiev test functions. The latter in particular extend the polynomials to a topological space of entire functions [13]. Moreover, a characterization of the spaces considered in terms of analytic and growth properties of the corresponding S-transforms, was established [11]. Kondratiev test and generalized functions of one complex variable were studied in [6].

Umbral calculus is a kind of spatial combinatorics, a mathematical tool developed systematically by G. C. Rota (see e.g., [14, 16, 19]) and S. Roman [18] in the 1970s (see also a survey paper [5] for a list of references and applications). It deals with Sheffer sequences, polynomials systems with generating functions of exponential type from which we will elaborate in Section 2. Inspired by the construction of Kondratiev test functions in infinite dimensional analysis we construct a countable family of Hilbert spaces $\{\mathcal{H}_p\}_{p=0}^{\infty}$ of entire functions such that their intersection $\mathcal{E} = \bigcap_p \mathcal{H}_p$ is a space of entire functions of exponential type, endowed with the projective limit topology. Sheffer operators will transform the monomials into polynomials, we consider in particular the Charlier polynomials, the "rising", and the "falling factorials", and the Hermite polynomials. While an extension of these results to infinite dimensional analysis is now on the way [7] we shall advance further in the one dimensional case, also to pave the way for further developments in infinite dimension.

Our principal result is that these maps induce continuous automorphisms on the space $\mathcal{E}_0^1$ of entire functions of order one and minimal type. To this end, in Section 2, we shall collect necessary concepts from umbral calculus and present pertinent examples, as well as regarding countably Hilbert spaces, and properties of entire functions . Section 3 is dedicated to the automorphism theorems and their proof.

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2. Preliminaries

2.1. **Elements of umbral calculus** [18]**.** Denote by $\mathcal{P}$ the vector space of all polynomials in the single variable z over the field $\mathbb{C}$ of complex numbers. A polynomial is said to be a monic polynomial if the leading coefficient of its highest degree term is equal to one. A polynomial sequence $\{p_n(z)\}_{n\in\mathbb{N}_0}$ is a sequence of monic polynomials such that $p_n(z)$ has degree exactly n.

Definition 2.1. A sequence $\{p_n(z)\}_{n\in\mathbb{N}_0}$ is called a Sheffer sequence if its generating function has the form

$$\sum_{n=0}^{\infty} \frac{p_n(z)}{n!} t^n = A(t) e^{zB(t)},$$

where $A(t) = \sum_{n=0}^{\infty} A_n t^n$ and $B(t) = \sum_{n=1}^{\infty} B_n t^n$ are formal power series with $A_0 \neq 0$ and $B_1 \neq 0$.

Furthermore, the linear operator on $\mathcal{P}$ given by $\lambda : z^n \to p_n(z)$ is called the Sheffer operator for the sequence $\{p_n(z)\}_{n\in\mathbb{N}_0}$.

2.1.1. *Examples.* We consider the following Sheffer sequences.

1. For $z \in \mathbb{C}$, *monomials* $p_n(z) = z^n$ have a generating function

$$e^{tz} = \sum_{n=0}^{\infty} \frac{z^n}{n!} t^n.$$

2. The *falling factorials*

$$(z)_n = \frac{\Gamma(z+1)}{\Gamma(z-n+1)} = \begin{cases} 1, & \text{if } n = 0, \\ z(z-1)\dots(z-n+1), & \text{if } n \in \mathbb{N} \end{cases}$$

are given by the generating function

$$e_+(t,z) := e^{z\ln(1+t)} = \sum_{n=0}^{\infty} \frac{(z)_n}{n!} t^n \quad \text{for} \quad |t| < 1.$$

One extends the notation $\binom{m}{n}$ to $z \in \mathbb{C}$ by

$$\binom{z}{n} := \frac{(z)_n}{n!}.$$

3. The *rising factorials*

$$(z)^n = \begin{cases} 1, & \text{if } n = 0, \\ z(z+1)\dots(z+n-1), & \text{if } n \in \mathbb{N} \end{cases}$$

can be defined via the generating function

$$e_-(t,z) := e^{-z\log(1-t)} = \sum_{n=0}^{\infty} \frac{(z)^n}{n!} t^n, \quad t \in \mathbb{C}, \quad |t| < 1.$$

4. The *Charlier polynomials*

$$C_n^a(z) = \sum_{k=0}^{n} \binom{n}{k} \binom{z}{k} k!(-a)^{n-k}$$

have the generating function

$$C(z,t) := e^{-at}(1+t)^z = \sum_{n=0}^{\infty} \frac{C_n^a(z)}{n!} t^n, \quad a \neq 0.$$

For simplicity we use the notation

$$C_n(z) := C_n^1(z).$$

5. The *Hermite polynomials*[1]

$$He_n(z) = (-1)^n e^{z^2/2} \frac{d^n}{dz^n} e^{-z^2/2} = n! \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^m}{2^m m!(n-2m)!} z^{n-2m}$$

have the generating function

$$e^{zt-\frac{1}{2}t^2} = \sum_{n=0}^{\infty} \frac{He_n(z)}{n!} t^n.$$

In what follows, we enumerate some of the well-known [18] relations among the polynomial sequences under consideration.

For monomials and falling factorials there is a relation

$$(z)_n = \sum_{k=0}^{n} s(n,k) z^k, \tag{1}$$

where $s(n,k)$ are the (signed) Stirling numbers of the first kind, with bound [3]

$$|s(n,k)| \leq \frac{n!}{(1-e^{-1})^n k!} \quad \text{for} \quad 1 \leq k \leq n. \tag{2}$$

Conversely,

$$z^n = \sum_{k=0}^{n} S(n,k)(z)_k, \tag{3}$$

where $S(n,k)$ is the Stirling number of the second kind, with bound [17]

$$0 < S(n,k) \leq \frac{1}{2} \binom{n}{k} k^{n-k}. \tag{4}$$

For rising factorials the corresponding formulas follow from the identity

$$(z)^n = (-1)^n (-z)_n. \tag{5}$$

For the representations of falling factorials in terms of Charlier polynomials and vice versa there are the following relations:

$$C_n^a(z) = \sum_{k=0}^{n} \binom{n}{k} (-a)^{n-k} (z)_k \tag{6}$$

and

$$(z)_n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} C_k^a(z). \tag{7}$$

Finally, we note

$$z^n = n! \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{2^m m!(n-2m)!} He_{n-2m}(z). \tag{8}$$

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[1] we choose the Hermite polynomials He_n over the family H_n because of its relation to the standard normal distribution [1].

2.2. **Hilbert spaces of power series and their projective limit.** In this section, we consider for the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ an increasing countable system of Hilbert norms with parameter $p = 0, 1, 2, \dots$ as follows.

Definition 2.2. For $p = 0, 1, 2, \dots$, we define Hilbert spaces

$$\mathcal{H}_p := \Big\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \Big| \|f\|_p^2 := \sum_{n=0}^{\infty} |a_n|^2 e^{pn} (n!)^2 < \infty \Big\}.$$

Remark 2.3.

(i) The monomials

$$e_n^{(p)}(z) = c_n^{(p)} z^n \quad \text{with} \quad c_n^{(p)} = e^{\frac{-pn}{2}} (n!)^{-1} \quad \text{and} \quad n = 0, 1, 2, \dots$$

are an orthonormal base in $\mathcal{H}_p$.

(ii) We have a chain of dense continuous embeddings

$$\dots \subset \mathcal{H}_{p+1} \subset \mathcal{H}_p \subset \mathcal{H}_{p-1} \dots .$$

(iii) For $p > q$ the embedding $\mathcal{H}_p \subset \mathcal{H}_q$ is injective, hence the corresponding scalar products are compatible.

Definition 2.4. We define the space $\mathcal{E}$ as the projective limit of the spaces $\mathcal{H}_p$, i.e.,

$$\mathcal{E} := \operatorname*{proj\,lim}_{p \to \infty} \mathcal{H}_p = \bigcap_{p \geq 0} \mathcal{H}_p.$$

In the projective limit topology a neighborhood basis for $\mathcal{E}$ is given by

$$U_{p,\epsilon} = \{ f \in \mathcal{E} : \|f\|_p < \epsilon \}, \quad p \geq 0, \quad \epsilon > 0. \tag{9}$$

For more detail on such topological vector spaces see [8, 9], and for a related construction see Chapter 3 and Appendix A5 of [10]. From the compatibility of the above norms it follows that $\mathcal{E}$ is a complete, countably Hilbert, metrizable space, hence a locally convex Fréchet space. It is easy to verify that it is nuclear, i.e, embeddings are Hilbert-Schmidt.

2.3. **Order and type of entire functions.** In this section, we first recall some facts and notations on order of growth and type of the entire functions that are essential in this paper (see e.g., [4, 15]).

Theorem 2.5. [4] *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of order*

$$\rho := \inf \Big\{ K : \max_{|z|=r} |f(z)| \overset{as}{<} \exp(r^K) \Big\},$$

where $\overset{as}{<}$ means "for sufficiently large argument." Then

$$\rho = \limsup_{n \to \infty} \frac{n \ln n}{\ln(1/|a_n|)}. \tag{10}$$

Moreover, if $f(z)$ is an entire function of order of growth ρ and type

$$\tau := \inf \Big\{ A : \max_{|z|=r} |f(z)| \overset{as}{<} \exp(A r^\rho) \Big\},$$

then

$$\tau = \frac{1}{\rho e} \limsup_{n \to \infty} \Big(n \sqrt[n]{|a_n|^\rho} \Big). \tag{11}$$

Lemma 2.6. [15] *If the asymptotic inequality*

$$\max_{|z|=r} |f(z)| \overset{as}{<} \exp(Ar^{\rho})$$

is fulfilled, then

$$|a_n| \overset{as}{<} \left(\frac{eA\rho}{n}\right)^{\frac{n}{\rho}}. \tag{12}$$

Furthermore, if the asymptotic inequality (12) is fulfilled, then

$$\max_{|z|=r} |f(z)| \overset{as}{<} \exp((A+\epsilon)r^{\rho}), \quad \forall \epsilon > 0.$$

Lemma 2.7. *The functions f in $\mathcal{H}_p$ are of at most first order of growth and type $\tau \leq e^{\frac{-p}{2}}$.*

Proof. For any $f \in \mathcal{H}_p$,

$$\|f\|_p^2 = \sum_{n=0}^{\infty} |a_n|^2 e^{pn} (n!)^2 < \infty$$

implies that

$$|a_n|^2 \leq \frac{C}{(n!)^2 e^{pn}}$$

for some constant $C > 0$. Insertion of this estimate into (10) and (11) proves the statements of the lemma. $hfill\square$

This result leads to

Proposition 2.8. *The space $\mathcal{E} = \bigcap_{p\geq 0} \mathcal{H}_p$ is the space $\mathcal{E}_0^1$ of entire functions of order at most $\rho = 1$ and minimal type ($\tau = 0$).*

Proof. The inclusion $\mathcal{E} = \bigcap_{p\geq 0} \mathcal{H}_p \subset \mathcal{E}_0^1$ follows from the previous lemma. Conversely, consider now $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{E}_0^1$. The first part of Lemma 2.6 implies that for any $A > 0$

$$|a_n| \overset{as}{<} \left(\frac{eA}{n}\right)^n.$$

Hence, using Stirling's formula

$$e^{np}(n!)^2 |a_n|^2 \overset{as}{<} 2\pi n (e^p A^2)^n$$

so that $f \in \mathcal{H}_p$ with $e^p A^2 < 1$. Choosing A sufficiently small this implies that $f \in \mathcal{H}_p$ for all p, hence in their intersection

$$\mathcal{E} = \bigcap_{p\geq 0} \mathcal{H}_p \supset \mathcal{E}_0^1.$$

$\square$

3. Main results and proofs

In what follows, we show that the Sheffer operator for polynomial sequences such as the rising factorials and falling factorials, the Charlier or the Hermite polynomials, is a continuous automorphism on $\mathcal{E}_0^1$. To this end we shall use the following

Remark 3.1. *To show that a linear map*

$$L : \mathcal{H}_p \to \mathcal{H}_{p'}$$
$$L : f \in \mathcal{H}_p \to Lf \in \mathcal{H}_{p'}$$

is bounded, it suffices to show that the Hilbert-Schmidt estimate is finite

$$\sum_{n,l}\left|\left(Le_n^{(p)}, e_l^{(p')}\right)_{\mathcal{H}_{p'}}\right|^2 < \infty.$$

Theorem 3.2. *The Sheffer operator*

$$\lambda : z^n \to (z)_n$$

for falling factorials acts as a continuous automorphism on $\mathcal{E}_0^1$.

Proof. For the Sheffer map

$$\lambda : f(z) = \sum_{n=0}^{\infty} a_n z^n \to \sum_{n=0}^{\infty} a_n (z)_n$$

we invoke Remark 3.1 with $L = \lambda$. In this case

$$Le_n^{(p)}(z) = c_n^{(p)} \lambda z^n = c_n^{(p)} (z)_n.$$

We use (1) to obtain

$$Le_n^{(p)}(z) = c_n^{(p)} \sum_{k=0}^{n} s(n,k) z^k = c_n^{(p)} \sum_{k=0}^{n} s(n,k) \left(c_k^{(p')}\right)^{-1} e_k^{(p')}(z).$$

As a consequence

$$\left(Le_n^{(p)}, e_l^{(p')}\right)_{\mathcal{H}_{p'}} = c_n^{(p)} s(n,l) \left(c_l^{(p')}\right)^{-1}$$

for $0 \le l \le n$ and zero otherwise. Using (2) we have

$$\left|\left(Le_n^{(p)}, e_l^{(p')}\right)_{\mathcal{H}_{p'}}\right| \le c_n^{(p)} \left(c_l^{(p')}\right)^{-1} \frac{n!}{(1-e^{-1})^n l!} = \frac{e^{\frac{-pn+p'l}{2}}}{(1-e^{-1})^n},$$

so that

$$\sum_{l=0}^{n}\left|\left(Le_n^{(p)}, e_l^{(p')}\right)_{\mathcal{H}_{p'}}\right|^2 \le \frac{e^{-pn}}{(1-e^{-1})^{2n}} \sum_{l=0}^{n} e^{p'l} = \frac{e^{-pn}}{(1-e^{-1})^{2n}} \frac{e^{p'(1+n)}-1}{e^{p'}-1}$$

and

$$\sum_{n,l}\left|\left(Le_n^{(p)}, e_l^{(p')}\right)_{\mathcal{H}_{p'}}\right|^2 \le \sum_{n=0}^{\infty} \frac{e^{-(p-p')n}}{(1-e^{-1})^{2n}} \frac{e^{p'}}{e^{p'}-1} < \infty$$

if $p' = p-1$. Hence the map λ is bounded from $\mathcal{H}_p$ into $\mathcal{H}_{p-1}$, uniformly as $p \to \infty$, and so in the limit $\lambda : \mathcal{E}_0^1 \to \mathcal{E}_0^1$ is also bounded. It is continuous there since $U_{p,\epsilon}$ of equation (9) are a neighborhood basis on $\mathcal{E}_0^1$.

For the inverse mapping we now have

$$L = \lambda^{-1} : (z)_n \in \mathcal{H}_p \to z^n \in \mathcal{H}_{p'}$$

and applying (3) to get

$$\begin{aligned}\lambda^{-1} e_n^{(p)}(z) &= e^{\frac{-pn}{2}} (n!)^{-1} \lambda^{-1} z^n = c_n^{(p)} \sum_{k=0}^{n} S(n,k) \lambda^{-1} (z)_k \\ &= c_n^{(p)} \sum_{k=0}^{n} S(n,k) z^k = c_n^{(p)} \sum_{k=0}^{n} S(n,k) \left(c_k^{(p')}\right)^{-1} e_k^{(p')}(z),\end{aligned}$$

so that

$$\left(Le_n^{(p)}, e_l^{(p')}\right)_{\mathcal{H}_{p'}} = c_n^{(p)} S(n,l) \left(c_l^{(p')}\right)^{-1}$$

for $0 \le l \le n$ and zero otherwise. Using the estimate (13)

$$\left|\left(Le_n^{(p)}, e_l^{(p')}\right)_{\mathcal{H}_{p'}}\right| \le \frac{1}{2} c_n^{(p)} \left(c_l^{(p')}\right)^{-1} \frac{n!}{l!} \frac{1}{\sqrt{2\pi}} e^{(n-l)} d^n = \frac{1}{2\sqrt{2\pi}} e^{\frac{-pn+p'l}{2}} e^{(n-l)} d^n$$

and hence

$$\sum_{l=0}^{n} \left|\left(Le_n^{(p)}, e_l^{(p')}\right)_{\mathcal{H}_{p'}}\right|^2 \le \frac{1}{8\pi} e^{-pn} e^{2n} d^{2n} \sum_{l=0}^{n} e^{(p'-2)l} \le \frac{n}{8\pi} e^{-(p-p')n} d^{2n}$$

for $p' \ge 2$. As a result

$$\sum_{n,l} \left|\left(Le_n^{(p)}, e_l^{(p')}\right)_{\mathcal{H}_{p'}}\right|^2 \le \frac{1}{8\pi} \sum_{n=0}^{\infty} n e^{-(p-p')n} d^{2n}$$

is uniformly bounded for $p - 3 \ge p' \ge 2$, i.e., the map $L = \lambda^{-1} : \mathcal{H}_p \to \mathcal{H}_{p'}$ is bounded and so thus $\lambda : \mathcal{E}_0^1 \to \mathcal{E}_0^1$. Continuity follows as above. □

Theorem 3.3. *The Sheffer operator for the system of rising factorials $\{(z)^n : n = 0, 1, 2, \dots\}$ is a continuous automorphism on $\mathcal{E}_0^1$.*

Proof. Since the rising factorials are obtained from the falling one's by the relation (5), the proof is a straightforward adaptation of the previous one. □

Theorem 3.4. *The Sheffer operator for the system of Charlier functions $\{C_n(z) : n = 0, 1, 2, \dots\}$ is a continuous automorphism on $\mathcal{E}_0^1$.*

Proof. In this case we use equalities (6) and (1) for the Sheffer map

$$\lambda : z^n \to C_n(z) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (z)_k = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \sum_{j=0}^{k} s(k,j) z^j,$$

so that, with $L = \lambda$

$$Le_n^{(p)}(z) = c_n^{(p)} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \sum_{j=0}^{k} s(k,j) \left(c_j^{(p')}\right)^{-1} e_j^{(p')}(z)$$

and

$$\left|\left(Le_n^{(p)}, e_l^{(p')}\right)_{\mathcal{H}_{p'}}\right| \le c_n^{(p)} \sum_{k=0}^{n} \binom{n}{k} |s(k,l)| \left(c_l^{(p')}\right)^{-1}.$$

We estimate this further using (2)

$$\begin{aligned}\left|\left(Le_n^{(p)}, e_l^{(p')}\right)_{\mathcal{H}_{p'}}\right| &\le c_n^{(p)} \sum_{k=0}^{n} \binom{n}{k} \frac{k!}{(1-e^{-1})^k l!} \left(c_l^{(p')}\right)^{-1} \\ &= e^{\frac{-pn}{2}} \sum_{k=0}^{n} \frac{1}{(n-k)!(1-e^{-1})^k} e^{\frac{p'l}{2}} \\ &= e^{\frac{-pn+p'l}{2}} \sum_{r=0}^{n} \frac{(1-e^{-1})^{r-n}}{r!} \\ &\le e^{\frac{-pn+p'l}{2}} (1-e^{-1})^{-n} \exp(1-e^{-1}),\end{aligned}$$

so that

$$\sum_{l=0}^{n}\left|\left(Le_n^{(p)},e_l^{(p')}\right)_{\mathcal{H}_{p'}}\right|^2 \leq e^{-pn}(1-e^{-1})^{-2n}\exp(2-2e^{-1})\sum_{l=0}^{n}e^{p'l}$$
$$= e^{-pn}(1-e^{-1})^{-2n}\exp(2-2e^{-1})\frac{e^{p'(1+n)}-1}{e^{p'}-1}$$

and

$$\sum_{n,l}\left|\left(Le_n^{(p)},e_l^{(p')}\right)_{\mathcal{H}_{p'}}\right|^2 \leq \exp(2-2e^{-1})\frac{e^{p'}}{e^{p'}-1}\sum_{n=0}^{\infty}\frac{e^{-(p-p')n}}{(1-e^{-1})^{2n}},$$

which is uniformly bounded for all $p-p'\geq 1$, and $\lambda:\mathcal{E}_0^1\to\mathcal{E}_0^1$, so that Remark 3.1 is satisfied.

For the inverse map use equations (3) and (7)

$$\lambda^{-1}: z^n=\sum_{k=0}^{n}S(n,k)\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}C_j(z)\to\sum_{k=0}^{n}S(n,k)\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}z^j$$

or in terms of the orthonormal base $e_n^{(p)}(z)$

$$\lambda^{-1}e_n^{(p)}(z)\to Le_n^{(p)}(z)=c_n^{(p)}\sum_{k=0}^{n}S(n,k)\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\left(c_j^{(p')}\right)^{-1}e_j^{(p')}(z),$$

so that

$$\left|\left(Le_n^{(p)},e_l^{(p')}\right)_{\mathcal{H}_{p'}}\right|\leq c_n^{(p)}\sum_{k=l}^{n}S(n,k)\binom{k}{l}\left(c_l^{(p')}\right)^{-1}.$$

Now we use (13)

$$\left|\left(Le_n^{(p)},e_l^{(p')}\right)_{\mathcal{H}_{p'}}\right|\leq c_n^{(p)}\left(c_l^{(p')}\right)^{-1}\sum_{k=l}^{n}\binom{k}{l}\frac{1}{2}\frac{n!}{k!}\frac{1}{\sqrt{2\pi}}e^{(n-k)}d^n$$
$$=\frac{1}{2\sqrt{2\pi}}e^{\frac{-pn+p'l}{2}}e^nd^n\sum_{k=l}^{n}\frac{1}{(k-l)!e^k}$$

as in the previous proof to conclude the argument. □

Theorem 3.5. *The Sheffer operator for the Hermite polynomials*

$$\lambda: z^n\to He_n(z)$$

acts as a continuous automorphism on $\mathcal{E}_0^1$.

Remark 3.6. *For this result see also* [6].

Proof. To apply Remark 3.1 for λ one uses

$$L=\lambda: z^n\to He_n(z)=n!\sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(-1)^m}{2^m m!(n-2m)!}z^{n-2m}$$

and proceeds as in the previous proofs.

To control the inverse map one uses (8)

$$L: z^n=n!\sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\frac{1}{2^m m!(n-2m)!}He_{n-2m}(z)\to n!\sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\frac{1}{2^m m!(n-2m)!}z^{n-2m}$$

with identical estimates as in the first half of the proof. □

4. Appendix: An estimate for $S(n,l)$

Starting from (4)

$$0 < S(n,l) < \frac{1}{2}\binom{n}{l} l^{n-l} = \frac{1}{2}\frac{n!}{l!}\frac{l^{n-l}}{(n-l)!}$$

we now use the Stirling approximation to obtain

$$\frac{l^{n-l}}{(n-l)!} \leq \frac{1}{\sqrt{2\pi}} e^{n-l}\left(\frac{l}{n-l}\right)^{n-l}.$$

Now we bound

$$\left(\frac{l}{n-l}\right)^{n-l} = e^{(n-l)(\ln l - \ln(n-l))} = e^{g(l)}.$$

For the maximum of g one has

$$\frac{d}{dl} g(l) = \frac{1}{l}(n + l\ln(n-l) - l\ln l) = 0$$

or equivalently

$$\frac{n}{l} = -\ln\left(\frac{n}{l} - 1)\right),$$

which has the solution

$$\frac{n}{l} \approx 1.2785\ldots$$

respectively

$$l = cn \quad \text{with} \quad c \approx 0.782$$

for which $\left(\frac{l}{n-l}\right)^{n-l}$ is maximal, so that we have bound

$$\left(\frac{l}{n-l}\right)^{n-l} \leq \left(\frac{c}{1-c}\right)^{n}$$

and

$$0 < S(n,l) < \frac{1}{2}\frac{n!}{l!}\frac{1}{\sqrt{2\pi}} e^{(n-l)} d^n \tag{13}$$

with

$$d = \frac{c}{1-c} \approx 3.5872.$$

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