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ON THE GEVREY ULTRADIFFERENTIABILITY OF WEAK SOLUTIONS OF AN ABSTRACT EVOLUTION EQUATION WITH A SCALAR TYPE SPECTRAL OPERATOR

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To Dr. Valentina I. Gorbachuk, a remarkable person and mathematician, in honor of her jubilee

Abstract. Found are conditions on a scalar type spectral operator A in a complex Banach space necessary and sufficient for all weak solutions of the evolution equation

$$y'(t) = Ay(t), \quad t \geq 0,$$

to be strongly Gevrey ultradifferentiable of order $\beta \geq 1$, in particular analytic or entire, on $[0, \infty)$. Certain inherent smoothness improvement effects are analyzed.

1. Introduction

We find conditions on a scalar type spectral operator A in a complex Banach space X necessary and sufficient for all *weak solutions* of the evolution equation

$$y'(t) = Ay(t), \quad t \geq 0, \tag{1.1}$$

to be strongly *Gevrey ultradifferentiable* of order $\beta \geq 1$, in particular *analytic* or *entire*, on $[0, \infty)$ and analyze certain inherent smoothness improvement effects to generalize the corresponding results for equation (1.1) with a *normal operator* A in a complex Hilbert space [21].

The results of the present paper develop those of [27], where similar consideration is given to the strong differentiability of the weak solutions of (1.1) on $[0, \infty)$ and $(0, \infty)$.

Definition 1.1 (Weak solution)**.**
Let A be a closed densely defined linear operator in a Banach space X. A strongly continuous vector function $y : [0, \infty) \to X$ is called a *weak solution* of equation (1.1) if, for any $g^* \in D(A^*)$,

$$\frac{d}{dt}\langle y(t), g^* \rangle = \langle y(t), A^* g^* \rangle, \quad t \geq 0,$$

where $D(\cdot)$ is the *domain* of an operator, A^* is the operator *adjoint* to A, and $\langle \cdot, \cdot \rangle$ is the *pairing* between the space X and its dual X^* (see [1]).

Due to the *closedness* of A, the weak solution of (1.1) can be equivalently defined to be a strongly continuous vector function $y : [0, \infty) \mapsto X$ such that, for all $t \geq 0$,

$$\int_0^t y(s)\, ds \in D(A) \quad \text{and} \quad y(t) = y(0) + A \int_0^t y(s)\, ds$$

and is also called a *mild solution* (cf. [6, Ch. II, Definition 6.3], see also [29, Preliminaries]).

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Such a notion of *weak solution*, which need not be differentiable in the strong sense, generalizes that of *classical* one, strongly differentiable on $[0,\infty)$ and satisfying the equation in the traditional plug-in sense, the classical solutions being precisely the weak ones strongly differentiable on $[0,\infty)$.

When a closed densely defined linear operator A in a complex Banach space X generates a C_0-semigroup $\{T(t)\}_{t\ge 0}$ of bounded linear operators (see, e.g., [6, 13]), i.e., the associated *abstract Cauchy problem* (ACP)

$$\begin{cases} y'(t) = Ay(t),\ t \ge 0, \\ y(0) = f \end{cases} \tag{1.2}$$

is *well-posed* (cf. [6, Ch. II, Definition 6.8]), the weak solutions of equation (1.1) are the orbits

$$y(t) = T(t)f, \quad t \ge 0, \tag{1.3}$$

with $f \in X$ [6, Ch. II, Proposition 6.4] (see also [1, Theorem]), whereas the classical ones are those with $f \in D(A)$ (see, e.g., [6, Ch. II, Proposition 6.3]).

Observe that, in our discourse, the associated ACP may be *ill-posed*, i.e., the scalar type spectral operator A need not generate a C_0-semigroup (cf. [24]).

2. Preliminaries

For the reader's convenience, we outline in this section certain essential preliminaries.

2.1. **Scalar type spectral operators.**

Henceforth, unless specified otherwise, A is supposed to be a *scalar type spectral operator* in a complex Banach space $(X, \|\cdot\|)$ and $E_A(\cdot)$ to be its strongly σ-additive *spectral measure* (the *resolution of the identity*) assigning to each *Borel set* δ of the *complex plane* $\mathbb{C}$ a *projection operator* $E_A(\delta)$ on X and having the operator's *spectrum* $\sigma(A)$ as its *support* [2, 5].

Observe that, in a complex finite-dimensional space, the scalar type spectral operators are those linear operators on the space, for which there is an *eigenbasis* (see, e.g., [2, 5]) and, in a complex Hilbert space, the scalar type spectral operators are precisely those that are similar to the *normal* ones [33].

Associated with a scalar type spectral operator in a complex Banach space is the *Borel operational calculus* analogous to that for a *normal operator* in a complex Hilbert space [2, 4, 5, 31], which assigns to any Borel measurable function $F : \sigma(A) \to \mathbb{C}$ a scalar type spectral operator

$$F(A) := \int\limits_{\sigma(A)} F(\lambda)\, dE_A(\lambda)$$

defined as follows:

$$F(A)f := \lim_{n\to\infty} F_n(A)f,\ f \in D(F(A)),\ D(F(A)) := \left\{ f \in X \,\middle|\, \lim_{n\to\infty} F_n(A)f \text{ exists} \right\},$$

where

$$F_n(\cdot) := F(\cdot)\chi_{\{\lambda\in\sigma(A)\,|\,|F(\lambda)|\le n\}}(\cdot), \quad n \in \mathbb{N},$$

($\chi_\delta(\cdot)$ is the *characteristic function* of a set $\delta \subseteq \mathbb{C}$, $\mathbb{N} := \{1, 2, 3, \dots\}$ is the set of *natural numbers*) and

$$F_n(A) := \int\limits_{\sigma(A)} F_n(\lambda)\, dE_A(\lambda), \quad n \in \mathbb{N},$$

are *bounded* scalar type spectral operators on X defined in the same manner as for a *normal operator* (see, e.g., [4, 31]).

In particular,

$$A^n = \int\limits_{\sigma(A)} \lambda^n \, dE_A(\lambda), \quad n \in \mathbb{Z}_+, \tag{2.4}$$

($\mathbb{Z}_+ := \{0, 1, 2, \dots\}$ is the set of *nonnegative integers*, $A^0 := I$, I is the *identity operator* on X) and

$$e^{zA} := \int\limits_{\sigma(A)} e^{z\lambda} \, dE_A(\lambda), \quad z \in \mathbb{C}. \tag{2.5}$$

The properties of the *spectral measure* and *operational calculus*, exhaustively delineated in [2, 5], underlie the entire subsequent discourse. Here, we touch upon a few facts of special importance.

Due to its *strong countable additivity*, the spectral measure $E_A(\cdot)$ is *bounded* [3, 5], i.e., there is such an $M > 0$ that, for any Borel set $\delta \subseteq \mathbb{C}$,

$$\|E_A(\delta)\| \le M. \tag{2.6}$$

Observe that the notation $\|\cdot\|$ is recycled here to designate the norm in the space $L(X)$ of all bounded linear operators on X. We shall adhere to this rather conventional economy of symbols in what follows adopting the same notation for the norm in the dual space X^* as well (cf. [6, 24]).

For any $f \in X$ and $g^* \in X^*$, the *total variation* $v(f, g^*, \cdot)$ of the complex-valued Borel measure $\langle E_A(\cdot)f, g^*\rangle$ is a *finite* positive Borel measure with

$$v(f, g^*, \mathbb{C}) = v(f, g^*, \sigma(A)) \le 4M\|f\|\|g^*\| \tag{2.7}$$

(see, e.g., [25, 26]).

Also (Ibid.), for a Borel measurable function $F : \mathbb{C} \to \mathbb{C}$, $f \in D(F(A))$, $g^* \in X^*$, and a Borel set $\delta \subseteq \mathbb{C}$,

$$\int\limits_{\delta} |F(\lambda)| \, dv(f, g^*, \lambda) \le 4M\|E_A(\delta)F(A)f\|\|g^*\|. \tag{2.8}$$

In particular, for $\delta = \sigma(A)$,

$$\int\limits_{\sigma(A)} |F(\lambda)| \, dv(f, g^*, \lambda) \le 4M\|F(A)f\|\|g^*\|. \tag{2.9}$$

Observe that the constant $M > 0$ in (2.7)–(2.9) is from (2.6).

Further, for a Borel measurable function $F : \mathbb{C} \to [0, \infty)$, a Borel set $\delta \subseteq \mathbb{C}$, a sequence $\{\Delta_n\}_{n=1}^\infty$ of pairwise disjoint Borel sets in $\mathbb{C}$, and $f \in X$, $g^* \in X^*$,

$$\int\limits_{\delta} F(\lambda) \, dv(E_A(\cup_{n=1}^\infty \Delta_n)f, g^*, \lambda) = \sum_{n=1}^\infty \int\limits_{\delta\cap\Delta_n} F(\lambda) \, dv(E_A(\Delta_n)f, g^*, \lambda). \tag{2.10}$$

Indeed, since, for any Borel sets $\delta, \sigma \subseteq \mathbb{C}$,

$$E_A(\delta)E_A(\sigma) = E_A(\delta \cap \sigma)$$

[2, 5], for the total variation,

$$v(E_A(\delta)f, g^*, \sigma) = v(f, g^*, \delta \cap \sigma).$$

Whence, due to the *nonnegativity* of $F(\cdot)$ (see, e.g., [12]),

$$\int\limits_{\delta} F(\lambda) \, dv(E_A(\cup_{n=1}^\infty \Delta_n)f, g^*, \lambda) = \int\limits_{\delta\cap\cup_{n=1}^\infty \Delta_n} F(\lambda) \, dv(f, g^*, \lambda)$$

$$= \sum_{n=1}^{\infty} \int_{\delta\cap\Delta_n} F(\lambda)\, dv(f, g^*, \lambda) = \sum_{n=1}^{\infty} \int_{\delta\cap\Delta_n} F(\lambda)\, dv(E_A(\Delta_n)f, g^*, \lambda).$$

The following statement, allowing to characterize the domains of Borel measurable functions of a scalar type spectral operator in terms of positive Borel measures, is fundamental for our discourse.

Proposition 2.1 ([23, Proposition 3.1]).
Let A be a scalar type spectral operator in a complex Banach space $(X, \|\cdot\|)$ with spectral measure $E_A(\cdot)$ and $F : \mathbb{C} \to \mathbb{C}$ (or $F : \sigma(A) \to \mathbb{C}$) be Borel measurable function. Then $f \in D(F(A))$ iff

(i) *for each $g^* \in X^*$,* $\displaystyle\int_{\sigma(A)} |F(\lambda)|\, dv(f, g^*, \lambda) < \infty$ *and*

(ii) $$\sup_{\{g^*\in X^* \,|\, \|g^*\|=1\}} \int_{\{\lambda\in\sigma(A)\,|\,|F(\lambda)|>n\}} |F(\lambda)|\, dv(f, g^*, \lambda) \to 0,\ n \to \infty,$$

where $v(f, g^*, \cdot)$ is the total variation of $\langle E_A(\cdot)f, g^*\rangle$.

The succeeding key theorem provides a full description of the weak solutions of equation (1.1) with a scalar type spectral operator A in a complex Banach space.

Theorem 2.1 ([23, Theorem 4.2]).
Let A be a scalar type spectral operator in a complex Banach space $(X, \|\cdot\|)$. A vector function $y : [0, \infty) \to X$ is a weak solution of equation (1.1) iff there is an $f \in \bigcap_{t\geq 0} D(e^{tA})$ such that

$$y(t) = e^{tA} f, \quad t \geq 0, \tag{2.11}$$

the operator exponentials understood in the sense of the Borel operational calculus (see (2.5)).

Theorem 2.1 generalizes [20, Theorem 3.1], its counterpart for a normal operator A in a complex Hilbert space, and implies, in particular,

- that the subspace $\bigcap_{t\geq 0} D(e^{tA})$ of all possible initial values of the weak solutions of equation (1.1) is the largest permissible for the exponential form given by (2.11), which highlights the contextual naturalness of the notion of weak solution, and
- that associated ACP (1.2), whenever solvable, is solvable *uniquely*.

Observe that the initial-value subspace $\bigcap_{t\geq 0} D(e^{tA})$ of equation (1.1), containing the dense in X subspace $\bigcup_{\alpha>0} E_A(\Delta_\alpha)X$, where

$$\Delta_\alpha := \{\lambda \in \mathbb{C} \,|\, |\lambda| \leq \alpha\}, \quad \alpha > 0,$$

which coincides with the class $\mathscr{E}^{\{0\}}(A)$ of *entire* vectors of A of *exponential type* (see below), is *dense* in X as well.

When a scalar type spectral operator A in a complex Banach space generates a C_0-semigroup $\{T(t)\}_{t\geq 0}$,

$$T(t) = e^{tA} \text{ and } D(e^{tA}) = X, \quad t \geq 0,$$

[24], and hence, Theorem 2.1 is consistent with the well-known description of the weak solutions for this setup (see (1.3)).

We also need the following characterization of a particular weak solution's of equation (1.1) with a scalar type spectral operator A in a complex Banach space being strongly infinite differentiable on a subinterval I of $[0, \infty)$.

Proposition 2.2 ([27, Corollary 3.2] with $T=\infty$).
Let A be a scalar type spectral operator in a complex Banach space $(X, \|\cdot\|)$ and I be a subinterval of $[0,\infty)$. A weak solution $y(\cdot)$ of equation (1.1) is strongly infinite differentiable on I iff, for each $t \in I$,

$$y(t) \in C^\infty(A),$$

in which case

$$y^{(n)}(t) = A^n y(t), \quad n \in \mathbb{N}, \quad t \in I.$$

Subsequently, the frequent terms *"spectral measure"* and *"operational calculus"* are abbreviated to *s.m.* and *o.c.*, respectively.

2.2. **Gevrey classes of functions.**

Definition 2.1 (Gevrey classes of functions).
Let $(X, \|\cdot\|)$ be a (real or complex) Banach space, $C^\infty(I,X)$ be the space of all X-valued functions strongly infinite differentiable on an interval $I \subseteq (-\infty,\infty)$, and $0 \le \beta < \infty$.

The following subspaces of $C^\infty(I,X)$

$$\mathscr{E}^{\{\beta\}}(I,X) := \big\{g(\cdot) \in C^\infty(I,X) \,\big|\forall [a,b] \subseteq I \ \exists \alpha > 0 \ \exists c > 0 : \max_{a \le t \le b} \|g^{(n)}(t)\| \le c\alpha^n [n!]^\beta, \ n \in \mathbb{Z}_+\big\},$$

$$\mathscr{E}^{(\beta)}(I,X) := \big\{g(\cdot) \in C^\infty(I,X) \,\big|\forall [a,b] \subseteq I \ \forall \alpha > 0 \ \exists c > 0 : \max_{a \le t \le b} \|g^{(n)}(t)\| \le c\alpha^n [n!]^\beta, \ n \in \mathbb{Z}_+\big\}$$

are called the *βth-order Gevrey classes* of strongly ultradifferentiable vector functions on I of *Roumieu* and *Beurling type*, respectively (see, e.g., [7, 14–16]).

In view of *Stirling's formula*, the sequence $\big\{[n!]^\beta\big\}_{n=0}^\infty$ can be replaced with $\big\{n^{\beta n}\big\}_{n=0}^\infty$.

For $0 \le \beta < \beta' < \infty$, the inclusions

$$\mathscr{E}^{(\beta)}(I,X) \subseteq \mathscr{E}^{\{\beta\}}(I,X) \subseteq \mathscr{E}^{(\beta')}(I,X) \subseteq \mathscr{E}^{\{\beta'\}}(I,X) \subseteq C^\infty(I,X)$$

hold.

- For $1 < \beta < \infty$, the Gevrey classes are *non-quasianalytic* (see, e.g., [15]).
- For $\beta = 1$, $\mathscr{E}^{\{1\}}(I,X)$ is the class of all *analytic* on I, i.e., *analytically continuable* into complex neighborhoods of I, vector functions and $\mathscr{E}^{(1)}(I,X)$ is the class of all *entire*, i.e., allowing *entire* continuations, vector functions [19].
- For $0 \le \beta < 1$, the Gevrey class $\mathscr{E}^{\{\beta\}}(I,X)$ ($\mathscr{E}^{(\beta)}(I,X)$) consists of all functions $g(\cdot) \in \mathscr{E}^{(1)}(I,X)$ such that, for some (any) $\gamma > 0$, there is an $M > 0$ for which

$$\|g(z)\| \le Me^{\gamma |z|^{1/(1-\beta)}}, \quad z \in \mathbb{C}, \tag{2.12}$$

  [22]. In particular, for $\beta = 0$, $\mathscr{E}^{\{0\}}(I,X)$ and $\mathscr{E}^{(0)}(I,X)$ are the classes of entire vector functions of *exponential* and *minimal exponential type*, respectively (see, e.g., [17]).

2.3. **Gevrey classes of vectors.**

One can consider the Gevrey classes in a more general sense.

Definition 2.2 (Gevrey classes of vectors).
Let A be a densely defined closed linear operator in a (real or complex) Banach space $(X, \|\cdot\|)$, $0 \le \beta < \infty$, and

$$C^\infty(A) := \bigcap_{n=0}^\infty D(A^n)$$

be the subspace of infinite differentiable vectors of A.

The following subspaces of $C^\infty(A)$

$$\mathscr{E}^{\{\beta\}}(A) := \left\{x \in C^\infty(A) \,\middle|\, \exists \alpha > 0\ \exists c > 0 : \|A^n x\| \le c\alpha^n [n!]^\beta,\ n \in \mathbb{Z}_+\right\},$$
$$\mathscr{E}^{(\beta)}(A) := \left\{x \in C^\infty(A) \,\middle|\, \forall \alpha > 0\ \exists c > 0 : \|A^n x\| \le c\alpha^n [n!]^\beta,\ n \in \mathbb{Z}_+\right\}$$

are called the *βth-order Gevrey classes* of ultradifferentiable vectors of A of *Roumieu* and *Beurling type*, respectively (see, e.g., [9–11]).

In view of *Stirling's formula*, the sequence $\left\{[n!]^\beta\right\}_{n=0}^\infty$ can be replaced with $\left\{n^{\beta n}\right\}_{n=0}^\infty$. For $0 \le \beta < \beta' < \infty$, the inclusions

$$\mathscr{E}^{(\beta)}(A) \subseteq \mathscr{E}^{\{\beta\}}(A) \subseteq \mathscr{E}^{(\beta')}(A) \subseteq \mathscr{E}^{\{\beta'\}}(A) \subseteq C^\infty(A)$$

hold.

In particular, $\mathscr{E}^{\{1\}}(A)$ and $\mathscr{E}^{(1)}(A)$ are the classes of *analytic* and *entire* vectors of A, respectively [8, 30] and $\mathscr{E}^{\{0\}}(A)$ and $\mathscr{E}^{(0)}(A)$ are the classes of *entire* vectors of A of *exponential* and *minimal exponential type*, respectively (see, e.g., [11, 32]).

In view of the *closedness* of A, it is easily seen that the class $\mathscr{E}^{(1)}(A)$ forms the subspace of initial values in X generating the (classical) solutions of (1.1), which are entire vector functions represented by the power series

$$\sum_{n=0}^\infty \frac{t^n}{n!} A^n f, \quad t \ge 0, \quad f \in \mathscr{E}^{(1)}(A),$$

the classes $\mathscr{E}^{\{\beta\}}(A)$ and $\mathscr{E}^{(\beta)}(A)$ with $0 \le \beta < 1$ being the subspaces of such initial values for which the solutions satisfy growth condition (2.12) with some (any) $\gamma > 0$ and some $M > 0$, respectively (cf. [17]).

As is shown in [9] (see also [10, 11]), if $0 < \beta < \infty$, for a *normal operator* A in a complex Hilbert space,

$$\mathscr{E}^{\{\beta\}}(A) = \bigcup_{t>0} D(e^{t|A|^{1/\beta}}) \quad \text{and} \quad \mathscr{E}^{(\beta)}(A) = \bigcap_{t>0} D(e^{t|A|^{1/\beta}}), \tag{2.13}$$

the operator exponentials $e^{t|A|^{1/\beta}}$, $t > 0$, understood in the sense of the Borel operational calculus (see, e.g., [4, 31]).

In [26, 28], descriptions (2.13) are extended to *scalar type spectral operators* in a complex Banach space, in which form they are basic for our discourse. In [28], similar nature descriptions of the classes $\mathscr{E}^{\{0\}}(A)$ and $\mathscr{E}^{(0)}(A)$ ($\beta = 0$), known for a normal operator A in a complex Hilbert space (see, e.g., [11]), are also generalized to scalar type spectral operators in a complex Banach space. In particular [28, Theorem 5.1],

$$\mathscr{E}^{\{0\}}(A) = \bigcup_{\alpha>0} E_A(\Delta_\alpha)X,$$

where

$$\Delta_\alpha := \{\lambda \in \mathbb{C} \,|\, |\lambda| \le \alpha\}, \quad \alpha > 0.$$

3. Gevrey ultradifferentiability of a particular weak solution

Here, we characterize a particular weak solution's of equation (1.1) with a scalar type spectral operator A in a complex Banach space being strongly Gevrey ultradifferentiable on a subinterval I of $[0, \infty)$.

Proposition 3.1. *Let A be a scalar type spectral operator in a complex Banach space $(X, \|\cdot\|)$ with spectral measure $E_A(\cdot)$, $0 \le \beta < \infty$, and I be a subinterval of $[0, \infty)$. Then the restriction of a weak solution $y(\cdot)$ of equation* (1.1) *to I belongs to the Gevrey class $\mathscr{E}^{\{\beta\}}(I, X)$ ($\mathscr{E}^{(\beta)}(I, X)$) iff, for each $t \in I$,*

$$y(t) \in \mathscr{E}^{\{\beta\}}(A) \ (\mathscr{E}^{(\beta)}(A), \text{ respectively}),$$

in which case

$$y^{(n)}(t) = A^n y(t), \quad n \in \mathbb{N}, \quad t \in I.$$

Proof.

"Only if" part. Assume that a weak solution $y(\cdot)$ of (1.1) restricted to I belongs to $\mathscr{E}^{\{\beta\}}(I,X)$ $(\mathscr{E}^{(\beta)}(I,X))$.

This immediately implies that $y(\cdot) \in C^\infty(I,X)$. Whence, by Proposition 2.2,

$$y(t) \in C^\infty(A), \quad t \in I,$$

and

$$y^{(n)}(t) = A^n y(t), \quad n \in \mathbb{N}, \quad t \in I.$$

Furthermore, the fact that the restriction of $y(\cdot)$ to I belongs to $\mathscr{E}^{\{\beta\}}(I,X)$ $(\mathscr{E}^{(\beta)}(I,X))$ implies that, for an arbitrary $t \in I$, some (any) $\alpha > 0$, and some $c > 0$:

$$\|A^n y(t)\| = \|y^{(n)}(t)\| \le c\alpha^n [n!]^\beta, \quad n = Z_+.$$

Therefore, for each $t \in I$,

$$y(t) \in \mathscr{E}^{\{\beta\}}(A) \ (\mathscr{E}^{(\beta)}(A)).$$

"If" part. Let $y(\cdot)$ be a weak solution of equation (1.1) such that, for each $t \in I$,

$$y(t) \in \mathscr{E}^{\{\beta\}}(A) \ (\mathscr{E}^{(\beta)}(A)).$$

Hence, for an arbitrary $t \in I$ and some (any) $\alpha > 0$, there is a $c(t,\alpha) > 0$ such that

$$\|A^n y(t)\| \le c(t,\alpha)\alpha^n [n!]^\beta, \quad n \in \mathbb{Z}_+. \tag{3.14}$$

The inclusions

$$\mathscr{E}^{(\beta)}(A) \subseteq \mathscr{E}^{\{\beta\}}(A) \subseteq C^\infty(A)$$

imply by Proposition 2.2 that

$$y(\cdot) \in C^\infty(I,X)$$

and

$$y^{(n)}(t) = A^n y(t), \quad n \in \mathbb{N}, \quad t \in I.$$

By Theorem 2.1,

$$y(t) = e^{tA} f, \ t \ge 0, \quad \text{with some} \quad f \in \bigcap_{t \ge 0} D(e^{tA}).$$

Fixing an arbitrary $[a,b] \subseteq I$, for every $n \in Z_+$, we have

$$\max_{a \le t \le b} \|y^{(n)}(t)\| = \max_{a \le t \le b} \|A^n y(t)\| = \max_{a \le t \le b} \|A^n e^{tA} f\|$$

by the properties of the *o.c*;

$$= \max_{a \le t \le b} \left\| \int_{\sigma(A)} \lambda^n e^{t\lambda} \, dE_A(\lambda) f \right\|$$

as follows form the *Hahn-Banach Theorem*;

$$= \max_{a \le t \le b} \sup_{g^* \in X^*, \, \|g^*\|=1} \left| \left\langle \int_{\sigma(A)} \lambda^n e^{t\lambda} \, dE_A(\lambda) f, g^* \right\rangle \right|$$

by the properties of the *o.c*;

$$= \max_{a \le t \le b} \sup_{g^* \in X^*, \, \|g^*\|=1} \left| \int_{\sigma(A)} \lambda^n e^{t\lambda} \, d\langle E_A(\lambda) f, g^* \rangle \right|$$

$$\le \max_{a \le t \le b} \sup_{g^* \in X^*, \, \|g^*\|=1} \int_{\sigma(A)} |\lambda|^n e^{t \operatorname{Re} \lambda} \, dv(f, g^*, \lambda)$$

$$= \sup_{g^*\in X^*,\,\|g^*\|=1}\ \sup_{a\le t\le b}\left[\ \int\limits_{\{\lambda\in\sigma(A)\,|\,\mathrm{Re}\,\lambda\le 0\}} |\lambda|^n e^{t\,\mathrm{Re}\,\lambda}\,dv(f,g^*,\lambda)\right.$$

$$\left. + \int\limits_{\{\lambda\in\sigma(A)\,|\,\mathrm{Re}\,\lambda> 0\}} |\lambda|^n e^{t\,\mathrm{Re}\,\lambda}\,dv(f,g^*,\lambda)\right]$$

$$\le \sup_{g^*\in X^*,\,\|g^*\|=1}\left[\ \int\limits_{\{\lambda\in\sigma(A)\,|\,\mathrm{Re}\,\lambda\le 0\}} |\lambda|^n e^{a\,\mathrm{Re}\,\lambda}\,dv(f,g^*,\lambda)\right.$$

$$\left. + \int\limits_{\{\lambda\in\sigma(A)\,|\,\mathrm{Re}\,\lambda> 0\}} |\lambda|^n e^{b\,\mathrm{Re}\,\lambda}\,dv(f,g^*,\lambda)\right]$$

$$\le \sup_{g^*\in X^*,\,\|g^*\|=1}\left[\int\limits_{\sigma(A)} |\lambda|^n e^{a\,\mathrm{Re}\,\lambda}\,dv(f,g^*,\lambda) + \int\limits_{\sigma(A)} |\lambda|^n e^{b\,\mathrm{Re}\,\lambda}\,dv(f,g^*,\lambda)\right]$$

by the properties of the $o.c$ (see (2.4) and [5, Theorem XVIII.2.11 (f)]) and (2.9);

$$\le \sup_{g^*\in X^*,\,\|g^*\|=1} 4M\left[\|A^n e^{aA}f\| + \|A^n e^{bA}f\|\right]\|g^*\| \le 4M\left[\|A^n e^{aA}f\| + \|A^n e^{bA}f\|\right]$$

$$= 4M\left[\|A^n y(a)\| + \|A^n y(b)\|\right] = 4M\left[\|y^{(n)}(a)\| + \|y^{(n)}(b)\|\right].$$

Hence, in view of (3.14),

$$\max_{a\le t\le b}\|y^{(n)}(t)\| \le 4M[c(a,\alpha)+c(b,\alpha)]\max\left[\alpha(a),\alpha(b)\right]^n [n!]^\beta, \quad n\in\mathbb{Z}_+,$$

which implies that $y(\cdot)$ restricted to I belongs to the Gevrey class $\mathscr{E}^{\{\beta\}}(I,X)$ ($\mathscr{E}^{(\beta)}(I,X)$ completing the proof. □

Thus, we have obtained a generalization of [21, Proposition 3.1], the counterpart for a normal operator A in a complex Hilbert space.

4. Gevrey ultradifferentiability of weak solutions

In this section, we characterize the strong Gevrey ultradifferentiability of order $\beta\ge 1$ on $[0,\infty)$ of all weak solutions of equation (1.1) with a scalar type spectral operator A in a complex Banach space.

Theorem 4.1. *Let A be a scalar type spectral operator in a complex Banach space $(X,\|\cdot\|)$ with spectral measure $E_A(\cdot)$ and $1\le\beta<\infty$. Then the following statements are equivalent.*

(i) *Every weak solution of equation* (1.1) *belongs to the βth-order Gevrey class $\mathscr{E}^{(\beta)}([0,+\infty),X)$ of Beurling type.*
(ii) *Every weak solution of equation* (1.1) *belongs to the βth-order Gevrey class $\mathscr{E}^{\{\beta\}}([0,+\infty),X)$ of Roumieu type.*
(iii) *There is a $b_+>0$ such that the set $\sigma(A)\setminus\mathscr{P}^\beta_{b_+}$, where*

$$\mathscr{P}^\beta_{b_+} := \left\{\lambda\in\mathbb{C}\,\middle|\,\mathrm{Re}\,\lambda\ge b_+|\,\mathrm{Im}\,\lambda|^{1/\beta}\right\},$$

is bounded (see Fig. 1).

Proof. We are to prove the closed chain of implications

$$\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)} \Rightarrow \text{(i)},$$

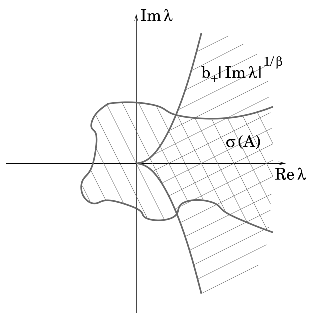

FIGURE 1

the implication (i) ⇒ (ii) following directly from the inclusion

$$\mathscr{E}^{(\beta)}([0,+\infty),X) \subseteq \mathscr{E}^{\{\beta\}}([0,+\infty),X)$$

(see Sec. 2.2).

To prove the implication (iii) ⇒ (i), suppose that there is a $b_+ > 0$ such that the set $\sigma(A) \setminus \mathscr{P}^\beta_{b_+}$ is *bounded* and let $y(\cdot)$ be an arbitrary weak solution of equation (1.1).

By Theorem 2.1,

$$y(t) = e^{tA}f, \quad t \geq 0, \quad \text{with some} \quad f \in \bigcap_{t\geq 0} D(e^{tA}).$$

Our purpose is to show that $y(\cdot) \in \mathscr{E}^{(\beta)}([0,+\infty),X)$, which, by Proposition 3.1 and (2.13), is accomplished by showing that, for each $t \geq 0$,

$$y(t) \in \mathscr{E}^{(\beta)}(A) = \bigcap_{s>0} D(e^{s|A|^{1/\beta}}).$$

Let us proceed by proving that, for any $t \geq 0$ and $s > 0$,

$$y(t) \in D(e^{s|A|^{1/\beta}})$$

via Proposition 2.1.

For any $s > 0$, $t \geq 0$ and an arbitrary $g^* \in X^*$,

$$\begin{aligned}(4.15) \quad \int_{\sigma(A)} e^{s|\lambda|^{1/\beta}} e^{t\,\mathrm{Re}\,\lambda}\, dv(f,g^*,\lambda) &= \int_{\sigma(A)\setminus\mathscr{P}^\beta_{b_+}} e^{s|\lambda|^{1/\beta}} e^{t\,\mathrm{Re}\,\lambda}\, dv(f,g^*,\lambda) \\ &+ \int_{\left\{\lambda\in\sigma(A)\cap\mathscr{P}^\beta_{b_+} \,\middle|\, \mathrm{Re}\,\lambda<1\right\}} e^{s|\lambda|^{1/\beta}} e^{t\,\mathrm{Re}\,\lambda}\, dv(f,g^*,\lambda) \\ &+ \int_{\left\{\lambda\in\sigma(A)\cap\mathscr{P}^\beta_{b_+} \,\middle|\, \mathrm{Re}\,\lambda\geq 1\right\}} e^{s|\lambda|^{1/\beta}} e^{t\,\mathrm{Re}\,\lambda}\, dv(f,g^*,\lambda) < \infty.\end{aligned}$$

Indeed,

$$\int_{\sigma(A)\setminus\mathscr{P}^\beta_{b_+}} e^{s|\lambda|^{1/\beta}} e^{t\,\mathrm{Re}\,\lambda}\, dv(f,g^*,\lambda) < \infty$$

and

$$\int_{\left\{\lambda\in\sigma(A)\cap\mathscr{P}^\beta_{b_+} \,\middle|\, \mathrm{Re}\,\lambda<1\right\}} e^{s|\lambda|^{1/\beta}} e^{t\,\mathrm{Re}\,\lambda}\, dv(f,g^*,\lambda) < \infty$$

due to the boundedness of the sets

$$\sigma(A)\setminus \mathscr{P}^\beta_{b_+} \quad \text{and} \quad \left\{\lambda\in\sigma(A)\cap\mathscr{P}^\beta_{b_+}\,\middle|\,\operatorname{Re}\lambda<1\right\},$$

the continuity of the integrated function on $\mathbb{C}$, and the finiteness of the measure $v(f,g^*,\cdot)$.

Further, for any $s>0$, $t\ge 0$ and an arbitrary $g^*\in X^*$,

$$\tag{4.16} \begin{aligned} &\int\limits_{\left\{\lambda\in\sigma(A)\cap\mathscr{P}^\beta_{b_+}\,\middle|\,\operatorname{Re}\lambda\ge 1\right\}} e^{s|\lambda|^{1/\beta}}e^{t\operatorname{Re}\lambda}\,dv(f,g^*,\lambda)\\ &\le \int\limits_{\left\{\lambda\in\sigma(A)\cap\mathscr{P}^\beta_{b_+}\,\middle|\,\operatorname{Re}\lambda\ge 1\right\}} e^{s[|\operatorname{Re}\lambda|+|\operatorname{Im}\lambda|]^{1/\beta}}e^{t\operatorname{Re}\lambda}\,dv(f,g^*,\lambda)\\ &\qquad\qquad \text{since, for } \lambda\in\sigma(A)\cap\mathscr{P}^\beta_{b_+},\ b_+^{-\beta}\operatorname{Re}\lambda^\beta\ge|\operatorname{Im}\lambda|;\\ &\le \int\limits_{\left\{\lambda\in\sigma(A)\cap\mathscr{P}^\beta_{b_+}\,\middle|\,\operatorname{Re}\lambda\ge 1\right\}} e^{s\left[\operatorname{Re}\lambda+b_+^{-\beta}\operatorname{Re}\lambda^\beta\right]^{1/\beta}}e^{t\operatorname{Re}\lambda}\,dv(f,g^*,\lambda)\\ &\qquad\qquad \text{since, in view of } \operatorname{Re}\lambda\ge 1 \text{ and } \beta\ge 1,\ \operatorname{Re}\lambda^\beta\ge\operatorname{Re}\lambda;\\ &\le \int\limits_{\left\{\lambda\in\sigma(A)\cap\mathscr{P}^\beta_{b_+}\,\middle|\,\operatorname{Re}\lambda\ge 1\right\}} e^{s\left(1+b_+^{-\beta}\right)^{1/\beta}\operatorname{Re}\lambda}e^{t\operatorname{Re}\lambda}\,dv(f,g^*,\lambda)\\ &= \int\limits_{\left\{\lambda\in\sigma(A)\cap\mathscr{P}^\beta_{b_+}\,\middle|\,\operatorname{Re}\lambda\ge 1\right\}} e^{\left[s\left(1+b_+^{-\beta}\right)^{1/\beta}+t\right]\operatorname{Re}\lambda}\,dv(f,g^*,\lambda)\\ &\qquad\qquad \text{since } f\in\bigcap_{t\ge 0}D(e^{tA}), \text{ by Proposition 2.1;}\\ &<\infty. \end{aligned}$$

Also, for any $s>0$, $t\ge 0$ and an arbitrary $n\in\mathbb{N}$,

$$\tag{4.17} \begin{aligned} &\sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}} \int\limits_{\left\{\lambda\in\sigma(A)\,\middle|\,e^{s|\lambda|^{1/\beta}}e^{t\operatorname{Re}\lambda}>n\right\}} e^{s|\lambda|^{1/\beta}}e^{t\operatorname{Re}\lambda}\,dv(f,g^*,\lambda)\\ &\le \sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}} \int\limits_{\left\{\lambda\in\sigma(A)\setminus\mathscr{P}^\beta_{b_+}\,\middle|\,e^{s|\lambda|^{1/\beta}}e^{t\operatorname{Re}\lambda}>n\right\}} e^{s|\lambda|^{1/\beta}}e^{t\operatorname{Re}\lambda}\,dv(f,g^*,\lambda)\\ &+ \sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}} \int\limits_{\left\{\lambda\in\sigma(A)\cap\mathscr{P}^\beta_{b_+}\,\middle|\,\operatorname{Re}\lambda<1,\,e^{s|\lambda|^{1/\beta}}e^{t\operatorname{Re}\lambda}>n\right\}} e^{s|\lambda|^{1/\beta}}e^{t\operatorname{Re}\lambda}\,dv(f,g^*,\lambda)\\ &+ \sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}} \int\limits_{\left\{\lambda\in\sigma(A)\cap\mathscr{P}^\beta_{b_+}\,\middle|\,\operatorname{Re}\lambda\ge1,\,e^{s|\lambda|^{1/\beta}}e^{t\operatorname{Re}\lambda}>n\right\}} e^{s|\lambda|^{1/\beta}}e^{t\operatorname{Re}\lambda}\,dv(f,g^*,\lambda)\\ &\to 0,\quad n\to\infty. \end{aligned}$$

Indeed, since, due to the boundedness of the sets

$$\sigma(A)\setminus \mathscr{P}^\beta_{b_+} \text{ and } \left\{\lambda\in\sigma(A)\cap\mathscr{P}^\beta_{b_+}\,\middle|\,\operatorname{Re}\lambda<1\right\}$$

and the continuity of the integrated function on $\mathbb{C}$, the sets

$$\left\{\lambda \in \sigma(A) \setminus \mathscr{P}^{\beta}_{b_+} \,\middle|\, e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda} > n\right\}$$

and

$$\left\{\lambda \in \sigma(A) \cap \mathscr{P}^{\beta}_{b_+} \,\middle|\, \operatorname{Re}\lambda < 1,\, e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda} > n\right\}$$

are *empty* for all sufficiently large $n \in \mathbb{N}$, we immediately infer that, for any $s > 0$ and $t \geq 0$,

$$\lim_{n\to\infty} \sup_{\{g^*\in X^* \,|\, \|g^*\|=1\}} \int\limits_{\left\{\lambda\in\sigma(A)\setminus\mathscr{P}^{\beta}_{b_+} \,\middle|\, e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda}>n\right\}} e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda}\, dv(f,g^*,\lambda) = 0$$

and

$$\lim_{n\to\infty} \sup_{\{g^*\in X^* \,|\, \|g^*\|=1\}} \int\limits_{\left\{\lambda\in\sigma(A)\cap\mathscr{P}^{\beta}_{b_+} \,\middle|\, \operatorname{Re}\lambda<1,\, e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda}>n\right\}} e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda}\, dv(f,g^*,\lambda) = 0.$$

Further, for any $s > 0$, $t \geq 0$, and an arbitrary $n \in \mathbb{N}$,

$$\begin{aligned}
&\sup_{\{g^*\in X^* \,|\, \|g^*\|=1\}} \int\limits_{\left\{\lambda\in\sigma(A)\cap\mathscr{P}^{\beta}_{b_+} \,\middle|\, \operatorname{Re}\lambda\geq 1,\, e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda}>n\right\}} e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda}\, dv(f,g^*,\lambda) \\
&\hspace{20em}\text{as in (4.16);} \\
&\leq \sup_{\{g^*\in X^* \,|\, \|g^*\|=1\}} \int\limits_{\left\{\lambda\in\sigma(A)\cap\mathscr{P}^{\beta}_{b_+} \,\middle|\, \operatorname{Re}\lambda\geq 1,\, e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda}>n\right\}} e^{\left[s\left(1+b_+^{-\beta}\right)^{1/\beta}+t\right]\operatorname{Re}\lambda}\, dv(f,g^*,\lambda) \\
&\hspace{20em}\text{since } f \in \bigcap_{t\geq 0} D(e^{tA}),\ \text{by (2.8);} \\
&\leq \sup_{\{g^*\in X^* \,|\, \|g^*\|=1\}} \\
&4M\left\|E_A\left(\left\{\lambda\in\sigma(A)\cap\mathscr{P}^{\beta}_{b_+} \,\middle|\, \operatorname{Re}\lambda\geq 1,\, e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda}>n\right\}\right) e^{\left[s\left(1+b_+^{-\beta}\right)^{1/\beta}+t\right]A} f\right\| \|g^*\| \\
&\leq 4M\left\|E_A\left(\left\{\lambda\in\sigma(A)\cap\mathscr{P}^{\beta}_{b_+} \,\middle|\, \operatorname{Re}\lambda\geq 1,\, e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda}>n\right\}\right) e^{\left[s\left(1+b_+^{-\beta}\right)^{1/\beta}+t\right]A} f\right\| \\
&\hspace{20em}\text{by the strong continuity of the } s.m.; \\
&\to 4M\left\|E_A(\emptyset)\, e^{\left[s\left(1+b_+^{-\beta}\right)^{1/\beta}+t\right]A} f\right\| = 0, \quad n\to\infty.
\end{aligned}$$

By Proposition 2.1 and the properties of the *o.c.* (see [5, Theorem XVIII.2.11 (f)]), (4.15) and (4.17) jointly imply that, for any $t \geq 0$ and $s > 0$,

$$f \in D(e^{s|A|^{1/\beta}} e^{tA}).$$

In view of (2.13), the latter implies that, for each $t \geq 0$,

$$y(t) = e^{tA} f \in \bigcap_{s>0} D(e^{s|A|^{1/\beta}}) = \mathscr{E}^{(\beta)}(A).$$

Whence, by Proposition 3.1, we infer that

$$y(\cdot) \in \mathscr{E}^{(\beta)}([0,\infty), X),$$

which completes the proof of the implication (iii) $\Rightarrow$ (i).

Let us prove the remaining implication (ii) $\Rightarrow$ (iii) *by contrapositive* assuming that, for any $b_+ > 0$, the set $\sigma(A) \setminus \mathscr{P}^{\beta}_{b_+}$ is *unbounded*. In particular, this means that, for any $n \in \mathbb{N}$, unbounded is the set

$$\sigma(A) \setminus \mathscr{P}^{\beta}_{n^{-2}} = \left\{ \lambda \in \sigma(A) \,\middle|\, \operatorname{Re}\lambda < n^{-2}|\operatorname{Im}\lambda|^{1/\beta} \right\}.$$

Hence, we can choose a sequence of points $\{\lambda_n\}_{n=1}^{\infty}$ in the complex plane as follows:

$$\begin{aligned} &\lambda_n \in \sigma(A), \quad n \in \mathbb{N}, \\ &\operatorname{Re}\lambda_n < n^{-2}|\operatorname{Im}\lambda_n|^{1/\beta}, \quad n \in \mathbb{N}, \\ &\lambda_0 := 0, \ |\lambda_n| > \max\left[n, |\lambda_{n-1}|\right], \quad n \in \mathbb{N}. \end{aligned}$$

The latter implies, in particular, that the points λ_n, $n \in \mathbb{N}$, are *distinct* ($\lambda_i \neq \lambda_j$, $i \neq j$).

Since, for each $n \in \mathbb{N}$, the set

$$\left\{ \lambda \in \mathbb{C} \,\middle|\, \operatorname{Re}\lambda < n^{-2}|\operatorname{Im}\lambda|^{1/\beta}, \ |\lambda| > \max\left[n, |\lambda_{n-1}|\right] \right\}$$

is *open* in $\mathbb{C}$, along with the point λ_n, it contains the *open disk*

$$\Delta_n := \{\lambda \in \mathbb{C} \,|\, |\lambda - \lambda_n| < \varepsilon_n\}$$

of some radius $\varepsilon_n > 0$, i.e., for each $\lambda \in \Delta_n$,

$$\operatorname{Re}\lambda < n^{-2}|\operatorname{Im}\lambda|^{1/\beta} \quad \text{and} \quad |\lambda| > \max\left[n, |\lambda_{n-1}|\right]. \tag{4.18}$$

Furthermore, under the circumstances, we can regard the radii of the disks to be small enough so that

$$\begin{aligned} &0 < \varepsilon_n < \frac{1}{n}, \quad n \in \mathbb{N}, \quad \text{and} \\ &\Delta_i \cap \Delta_j = \emptyset, \quad i \neq j \quad \text{(i.e., the disks are \textit{pairwise disjoint}).} \end{aligned} \tag{4.19}$$

Whence, by the properties of the *s.m.*,

$$E_A(\Delta_i)E_A(\Delta_j) = 0, \quad i \neq j,$$

where 0 stands for the *zero operator* on X.

Observe also, that the subspaces $E_A(\Delta_n)X$, $n \in \mathbb{N}$, are *nontrivial* since

$$\Delta_n \cap \sigma(A) \neq \emptyset, \quad n \in \mathbb{N},$$

with Δ_n being an *open set* in $\mathbb{C}$.

By choosing a unit vector $e_n \in E_A(\Delta_n)X$, $n \in \mathbb{N}$, we obtain a vector sequence $\{e_n\}_{n=1}^{\infty}$ such that

$$\|e_n\| = 1, \quad n \in \mathbb{N}, \quad \text{and} \quad E_A(\Delta_i)e_j = \delta_{ij}e_j, \quad i, j \in \mathbb{N}, \tag{4.20}$$

where δ_{ij} is the *Kronecker delta*.

As is easily seen, (4.20) implies that the vectors e_n, $n \in \mathbb{N}$, are *linearly independent*.

Furthermore, there is an $\varepsilon > 0$ such that

$$d_n := \operatorname{dist}\left(e_n, \operatorname{span}\left(\{e_i \,|\, i \in \mathbb{N}, \ i \neq n\}\right)\right) \geq \varepsilon, \quad n \in \mathbb{N}. \tag{4.21}$$

Indeed, the opposite implies the existence of a subsequence $\left\{d_{n(k)}\right\}_{k=1}^{\infty}$ such that

$$d_{n(k)} \to 0, \quad k \to \infty.$$

Then, by selecting a vector

$$f_{n(k)} \in \operatorname{span}\left(\{e_i \,|\, i \in \mathbb{N}, \ i \neq n(k)\}\right), \quad k \in \mathbb{N},$$

such that

$$\|e_{n(k)} - f_{n(k)}\| < d_{n(k)} + 1/k, \quad k \in \mathbb{N},$$

we arrive at

$$
\begin{aligned}
1 &= \|e_{n(k)}\| && \text{since, by (4.20), } E_A(\Delta_{n(k)})f_{n(k)} = 0;\\
&= \|E_A(\Delta_{n(k)})(e_{n(k)} - f_{n(k)})\| \le \|E_A(\Delta_{n(k)})\|\|e_{n(k)} - f_{n(k)}\| && \text{by (2.6)};\\
&\le M\|e_{n(k)} - f_{n(k)}\| \le M\left[d_{n(k)} + 1/k\right] \to 0, \quad k \to \infty,
\end{aligned}
$$

which is a *contradiction* proving (4.21).

As follows from the *Hahn-Banach Theorem*, for any $n \in \mathbb{N}$, there is an $e_n^* \in X^*$ such that

$$\|e_n^*\| = 1, \quad n \in \mathbb{N}, \quad \text{and} \quad \langle e_i, e_j^* \rangle = \delta_{ij} d_i, \quad i, j \in \mathbb{N}. \tag{4.22}$$

Let us consider separately the two possibilities concerning the sequence of the real parts $\{\mathrm{Re}\ \lambda_n\}_{n=1}^{\infty}$: its being *bounded above* or *unbounded above*.

First, suppose that the sequence $\{\mathrm{Re}\ \lambda_n\}_{n=1}^{\infty}$ is *bounded above*, i.e., there is such an $\omega > 0$ that

$$\mathrm{Re}\ \lambda_n \le \omega, \quad n \in \mathbb{N}, \tag{4.23}$$

and consider the element

$$f := \sum_{k=1}^{\infty} k^{-2} e_k \in X,$$

which is well defined since $\left\{k^{-2}\right\}_{k=1}^{\infty} \in l_1$ (l_1 is the space of absolutely summable sequences) and $\|e_k\| = 1$, $k \in \mathbb{N}$ (see (4.20)).

In view of (4.20), by the properties of the *s.m.*,

$$E_A(\cup_{k=1}^{\infty}\Delta_k)f = f \quad \text{and} \quad E_A(\Delta_k)f = k^{-2}e_k, \quad k \in \mathbb{N}. \tag{4.24}$$

For any $t \ge 0$ and an arbitrary $g^* \in X^*$,

$$
\begin{aligned}
&\int_{\sigma(A)} e^{t\,\mathrm{Re}\,\lambda}\, dv(f, g^*, \lambda) && \text{by (4.24)};\\
&= \int_{\sigma(A)} e^{t\,\mathrm{Re}\,\lambda}\, dv(E_A(\cup_{k=1}^{\infty}\Delta_k)f, g^*, \lambda) && \text{by (2.10)};\\
&= \sum_{k=1}^{\infty} \int_{\sigma(A)\cap\Delta_k} e^{t\,\mathrm{Re}\,\lambda}\, dv(E_A(\Delta_k)f, g^*, \lambda) && \text{by (4.24)};\\
&= \sum_{k=1}^{\infty} k^{-2} \int_{\sigma(A)\cap\Delta_k} e^{t\,\mathrm{Re}\,\lambda}\, dv(e_k, g^*, \lambda)
\end{aligned}
\tag{4.25}
$$

since, for $\lambda \in \Delta_k$, by (4.23) and (4.19), $\mathrm{Re}\ \lambda = \mathrm{Re}\ \lambda_k + (\mathrm{Re}\ \lambda - \mathrm{Re}\ \lambda_k)$
$\le \mathrm{Re}\ \lambda_k + |\lambda - \lambda_k| \le \omega + \varepsilon_k \le \omega + 1$;

$$\le e^{t(\omega+1)} \sum_{k=1}^{\infty} k^{-2} \int_{\sigma(A)\cap\Delta_k} 1\, dv(e_k, g^*, \lambda) = e^{t(\omega+1)} \sum_{k=1}^{\infty} k^{-2} v(e_k, g^*, \Delta_k)$$

by (2.7);

$$\le e^{t(\omega+1)} \sum_{k=1}^{\infty} k^{-2} 4M\|e_k\|\|g^*\| = 4Me^{t(\omega+1)}\|g^*\| \sum_{k=1}^{\infty} k^{-2} < \infty.$$

Similarly, for any $t \ge 0$ and an arbitrary $n \in \mathbb{N}$,

$$\sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}} \int\limits_{\{\lambda\in\sigma(A)\,|\,e^{t\operatorname{Re}\lambda}>n\}} e^{t\operatorname{Re}\lambda}\,dv(f,g^*,\lambda) \tag{4.26}$$

$$\le \sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}} e^{t(\omega+1)}\sum_{k=1}^{\infty}k^{-2}\int\limits_{\{\lambda\in\sigma(A)\,|\,e^{t\operatorname{Re}\lambda}>n\}\cap\Delta_k} 1\,dv(e_k,g^*,\lambda)$$

by (4.24);

$$= e^{t(\omega+1)}\sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}}\sum_{k=1}^{\infty}\int\limits_{\{\lambda\in\sigma(A)\,|\,e^{t\operatorname{Re}\lambda}>n\}\cap\Delta_k} 1\,dv(E_A(\Delta_k)f,g^*,\lambda)$$

by (2.10);

$$= e^{t(\omega+1)}\sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}}\int\limits_{\{\lambda\in\sigma(A)\,|\,e^{t\operatorname{Re}\lambda}>n\}} 1\,dv(E_A(\cup_{k=1}^{\infty}\Delta_k)f,g^*,\lambda)$$

by (4.24);

$$= e^{t(\omega+1)}\sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}}\int\limits_{\{\lambda\in\sigma(A)\,|\,e^{t\operatorname{Re}\lambda}>n\}} 1\,dv(f,g^*,\lambda) \qquad \text{by (2.8);}$$

$$\le e^{t(\omega+1)}\sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}} 4M\left\|E_A\left(\left\{\lambda\in\sigma(A)\,\middle|\,e^{t\operatorname{Re}\lambda}>n\right\}\right)f\right\|\|g^*\|$$

$$\le 4Me^{t(\omega+1)}\left\|E_A\left(\left\{\lambda\in\sigma(A)\,\middle|\,e^{t\operatorname{Re}\lambda}>n\right\}\right)f\right\|$$

by the strong continuity of the *s.m.*;

$$\to 4Me^{t(\omega+1)}\left\|E_A(\emptyset)f\right\| = 0, \quad n\to\infty.$$

By Proposition 2.1, (4.25) and (4.26) jointly imply that

$$f\in\bigcap_{t\ge 0}D(e^{tA}),$$

and hence, by Theorem 2.1,

$$y(t) := e^{tA}f, \quad t\ge 0,$$

is a weak solution of equation (1.1).

Let

$$h^* := \sum_{k=1}^{\infty}k^{-2}e_k^*\in X^*, \tag{4.27}$$

the functional being well defined since $\{k^{-2}\}_{k=1}^{\infty}\in l_1$ and $\|e_k^*\|=1$, $k\in\mathbb{N}$ (see (4.22)).

In view of (4.22) and (4.21), we have

$$\langle e_n,h^*\rangle = \langle e_k,k^{-2}e_k^*\rangle = d_k k^{-2}\ge\varepsilon k^{-2}, \quad k\in\mathbb{N}. \tag{4.28}$$

For any $s>0$,

$$\int\limits_{\sigma(A)} e^{s|\lambda|^{1/\beta}}\,dv(f,h^*,\lambda) \qquad \text{by (2.10) as in (4.25);}$$

$$= \sum_{k=1}^{\infty}k^{-2}\int\limits_{\sigma(A)\cap\Delta_k} e^{s|\lambda|^{1/\beta}}\,dv(e_k,h^*,\lambda) \qquad \text{since, for } \lambda\in\Delta_k \text{, by (4.18), } |\lambda|\ge k;$$

$$\ge \sum_{k=1}^{\infty}k^{-2}e^{sk^{1/\beta}}\int\limits_{\sigma(A)\cap\Delta_k} 1\,dv(e_k,h^*,\lambda) = \sum_{k=1}^{\infty}k^{-2}e^{sk^{1/\beta}}v(e_k,h^*,\Delta_k)$$

$$\geq \sum_{k=1}^{\infty} k^{-2} e^{sk^{1/\beta}} |\langle E_A(\Delta_k)e_k, h^*\rangle| \qquad \text{by (4.20) and (4.28);}$$

$$\geq \sum_{k=1}^{\infty} \varepsilon k^{-4} e^{sk^{1/\beta}} = \infty.$$

Whence, by Proposition 2.1 and (2.13), we infer that

$$y(0) = f \notin \bigcup_{s>0} D(e^{s|A|^{1/\beta}}) = \mathscr{E}^{\{\beta\}}(A)$$

which, by Proposition 3.1, implies that the weak solution $y(t) = e^{tA}f$, $t \geq 0$, of equation (1.1) does not belong to the Gevrey class $\mathscr{E}^{\{\beta\}}([0, +\infty), X)$ of Roumieu type and completes our consideration of the case of the sequence's $\{\text{Re } \lambda_n\}_{n=1}^{\infty}$ being *bounded above.*

Now, suppose that the sequence $\{\text{Re } \lambda_n\}_{n=1}^{\infty}$ is *unbounded above.*

Therefore, there is a subsequence $\{\text{Re } \lambda_{n(k)}\}_{k=1}^{\infty}$ such that

$$\text{Re } \lambda_{n(k)} \geq k, \quad k \in \mathbb{N}. \tag{4.29}$$

Consider the elements

$$f := \sum_{k=1}^{\infty} e^{-n(k) \text{ Re } \lambda_{n(k)}} e_{n(k)} \in X \quad \text{and} \quad h := \sum_{k=1}^{\infty} e^{-\frac{n(k)}{2} \text{ Re } \lambda_{n(k)}} e_{n(k)} \in X,$$

well defined since, by (4.29),

$$\left\{e^{-n(k) \text{ Re } \lambda_{n(k)}}\right\}_{k=1}^{\infty}, \ \left\{e^{-\frac{n(k)}{2} \text{ Re } \lambda_{n(k)}}\right\}_{k=1}^{\infty} \in l_1$$

and $\|e_{n(k)}\| = 1$, $k \in \mathbb{N}$ (see (4.20)).

By (4.20),

$$E_A(\cup_{k=1}^{\infty} \Delta_{n(k)})f = f \quad \text{and} \quad E_A(\Delta_{n(k)})f = e^{-n(k) \text{ Re } \lambda_{n(k)}} e_{n(k)}, \quad k \in \mathbb{N}, \tag{4.30}$$

and

$$E_A(\cup_{k=1}^{\infty} \Delta_{n(k)})h = h \quad \text{and} \quad E_A(\Delta_{n(k)})h = e^{-\frac{n(k)}{2} \text{ Re } \lambda_{n(k)}} e_{n(k)}, \quad k \in \mathbb{N}. \tag{4.31}$$

For any $t \geq 0$ and an arbitrary $g^* \in X^*$,

$$\int_{\sigma(A)} e^{t \text{ Re } \lambda} \, dv(f, g^*, \lambda) \qquad \text{by (2.10) as in (4.25);} \tag{4.32}$$

$$= \sum_{k=1}^{\infty} e^{-n(k) \text{ Re } \lambda_{n(k)}} \int_{\sigma(A) \cap \Delta_{n(k)}} e^{t \text{ Re } \lambda} \, dv(e_{n(k)}, g^*, \lambda)$$

since, for $\lambda \in \Delta_{n(k)}$, by (4.19), $\text{Re } \lambda = \text{Re } \lambda_{n(k)} + (\text{Re } \lambda - \text{Re } \lambda_{n(k)})$
$\leq \text{Re } \lambda_{n(k)} + |\lambda - \lambda_{n(k)}| \leq \text{Re } \lambda_{n(k)} + 1;$

$$\leq \sum_{k=1}^{\infty} e^{-n(k) \text{ Re } \lambda_{n(k)}} e^{t(\text{Re } \lambda_{n(k)} + 1)} \int_{\sigma(A) \cap \Delta_{n(k)}} 1 \, dv(e_{n(k)}, g^*, \lambda)$$

$$= e^t \sum_{k=1}^{\infty} e^{-[n(k) - t] \text{ Re } \lambda_{n(k)}} v(e_{n(k)}, g^*, \Delta_{n(k)}) \qquad \text{by (2.7);}$$

$$\leq e^t \sum_{k=1}^{\infty} e^{-[n(k) - t] \text{ Re } \lambda_{n(k)}} 4M \|e_{n(k)}\| \|g^*\| = 4Me^t \|g^*\| \sum_{k=1}^{\infty} e^{-[n(k) - t] \text{ Re } \lambda_{n(k)}}$$

$$< \infty.$$

Indeed, for all $k \in \mathbb{N}$ sufficiently large so that

$$n(k) \geq t+1,$$

in view of (4.29),

$$e^{-[n(k)-t]\,\mathrm{Re}\,\lambda_{n(k)}} \leq e^{-k}.$$

Similarly, for any $t \geq 0$ and an arbitrary,

$$\begin{aligned}
(4.33)\quad & \sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}} \int\limits_{\{\lambda\in\sigma(A)\,|\,e^{t\,\mathrm{Re}\,\lambda}>n\}} e^{t\,\mathrm{Re}\,\lambda}\,dv(f,g^*,\lambda) \\
& \leq \sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}} e^t \sum_{k=1}^{\infty} e^{-[n(k)-t]\,\mathrm{Re}\,\lambda_{n(k)}} \int\limits_{\{\lambda\in\sigma(A)\,|\,e^{t\,\mathrm{Re}\,\lambda}>n\}\cap\Delta_{n(k)}} 1\,dv(e_{n(k)},g^*,\lambda) \\
& = e^t \sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}} \sum_{k=1}^{\infty} e^{-\left[\frac{n(k)}{2}-t\right]\,\mathrm{Re}\,\lambda_{n(k)}} e^{-\frac{n(k)}{2}\,\mathrm{Re}\,\lambda_{(k)}} \\
& \int\limits_{\{\lambda\in\sigma(A)\,|\,e^{t\,\mathrm{Re}\,\lambda}>n\}\cap\Delta_{n(k)}} 1\,dv(e_{n(k)},g^*,\lambda) \\
& \qquad \text{since, by (4.29), there is an } L>0 \text{ such that } e^{-\left[\frac{n(k)}{2}-t\right]\,\mathrm{Re}\,\lambda_{n(k)}} \leq L,\ k\in\mathbb{N}; \\
& \leq Le^t \sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}} \sum_{k=1}^{\infty} e^{-\frac{n(k)}{2}\,\mathrm{Re}\,\lambda_{n(k)}} \int\limits_{\{\lambda\in\sigma(A)\,|\,e^{t\,\mathrm{Re}\,\lambda}>n\}\cap\Delta_{n(k)}} 1\,dv(e_{n(k)},g^*,\lambda) \\
& \qquad\qquad \text{by (4.31);} \\
& = Le^t \sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}} \sum_{k=1}^{\infty} \int\limits_{\{\lambda\in\sigma(A)\,|\,e^{t\,\mathrm{Re}\,\lambda}>n\}\cap\Delta_{n(k)}} 1\,dv(E_A(\Delta_{n(k)})h,g^*,\lambda) \\
& \qquad\qquad \text{by (2.10);} \\
& = Le^t \sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}} \int\limits_{\{\lambda\in\sigma(A)\,|\,e^{t\,\mathrm{Re}\,\lambda}>n\}} 1\,dv(E_A(\cup_{k=1}^{\infty}\Delta_{n(k)})h,g^*,\lambda) \\
& \qquad\qquad \text{by (4.31);} \\
& = Le^t \sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}} \int\limits_{\{\lambda\in\sigma(A)\,|\,e^{t\,\mathrm{Re}\,\lambda}>n\}} 1\,dv(h,g^*,\lambda) \qquad \text{by (2.8);} \\
& \leq Le^t \sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}} 4M \left\|E_A\left(\left\{\lambda\in\sigma(A)\,\middle|\,e^{t\,\mathrm{Re}\,\lambda}>n\right\}\right)h\right\| \|g^*\| \\
& \leq 4LMe^t\|E_A(\{\lambda\in\sigma(A)\,|\,e^{t\,\mathrm{Re}\,\lambda}>n\})h\| \\
& \qquad\qquad \text{by the strong continuity of the } \textit{s.m.}; \\
& \to 4LMe^t\left\|E_A(\emptyset)\,h\right\| = 0, \quad n\to\infty.
\end{aligned}$$

By Proposition 2.1, (4.32) and (4.33) jointly imply that

$$f \in \bigcap_{t\geq 0} D(e^{tA}),$$

and hence, by Theorem 2.1,

$$y(t) := e^{tA}f, \quad t\geq 0,$$

is a weak solution of equation (1.1).

Since, for any $\lambda\in\Delta_{n(k)}$, $k\in\mathbb{N}$, by (4.19), (4.29),

$$\begin{aligned}&\operatorname{Re}\lambda = \operatorname{Re}\lambda_{n(k)} - (\operatorname{Re}\lambda_{n(k)} - \operatorname{Re}\lambda) \ge \operatorname{Re}\lambda_{n(k)} - |\operatorname{Re}\lambda_{n(k)} - \operatorname{Re}\lambda|\\ &\ge \operatorname{Re}\lambda_{n(k)} - \varepsilon_{n(k)} \ge \operatorname{Re}\lambda_{n(k)} - 1/n(k) \ge k-1 \ge 0\end{aligned}$$

and, by (4.18),

$$\operatorname{Re}\lambda < n(k)^{-2}|\operatorname{Im}\lambda|^{1/\beta},$$

we infer that, for any $\lambda \in \Delta_{n(k)}$, $k \in \mathbb{N}$,

$$|\lambda| \ge |\operatorname{Im}\lambda| \ge \left[n(k)^2 \operatorname{Re}\lambda\right]^\beta \ge \left[n(k)^2(\operatorname{Re}\lambda_{n(k)} - 1/n(k))\right]^\beta .$$

Using this estimate, for an arbitrary $s > 0$ and the functional $h^* \in X^*$ defined by (4.27), we have

$$\begin{aligned}(4.34)\quad &\int\limits_{\sigma(A)} e^{s|\lambda|^{1/\beta}}\, dv(f, h^*, \lambda) \qquad \text{by (2.10) as in (4.25);}\\ &= \sum_{k=1}^\infty e^{-n(k)\operatorname{Re}\lambda_{n(k)}} \int\limits_{\sigma(A)\cap\Delta_{n(k)}} e^{s|\lambda|^{1/\beta}}\, dv(e_{n(k)}, h^*, \lambda)\\ &\ge \sum_{k=1}^\infty e^{-n(k)\operatorname{Re}\lambda_{n(k)}} e^{sn(k)^2(\operatorname{Re}\lambda_{n(k)}-1/n(k))} v(e_{n(k)}, h^*, \Delta_{n(k)})\\ &\ge \sum_{k=1}^\infty e^{-n(k)\operatorname{Re}\lambda_{n(k)}} e^{sn(k)^2(\operatorname{Re}\lambda_{n(k)}-1/n(k))} |\langle E_A(\Delta_{n(k)})e_{n(k)}, h^*\rangle|\\ &\qquad \text{by (4.20) and (4.28);}\\ &\ge \sum_{k=1}^\infty \varepsilon e^{(sn(k)-1)n(k)\operatorname{Re}\lambda_{n(k)} - sn(k)} n(k)^{-2} = \infty.\end{aligned}$$

Indeed, for all $k \in \mathbb{N}$ sufficiently large so that

$$sn(k) \ge s+2,$$

in view of (4.29),

$$\begin{aligned}&e^{(sn(k)-1)n(k)\operatorname{Re}\lambda_{n(k)} - sn(k)} n(k)^{-2} \ge e^{(s+1)n(k)-sn(k)} n(k)^{-2} = e^{n(k)} n(k)^{-2}\\ &\to \infty, \quad k \to \infty.\end{aligned}$$

By Proposition 2.1 and (2.13), (4.34) implies that

$$y(0) = f \notin \bigcup_{s>0} D(e^{s|A|^{1/\beta}}) = \mathscr{E}^{\{\beta\}}(A)$$

which, by Proposition 3.1, further implies that the weak solution $y(t) = e^{tA}f$, $t \ge 0$, of equation (1.1) does not belong to the Gevrey class $\mathscr{E}^{\{\beta\}}([0, +\infty), X)$ of Roumieu type and completes our consideration of the case of the sequence's $\{\operatorname{Re}\lambda_n\}_{n=1}^\infty$ being *unbounded above.*

With every possibility concerning $\{\operatorname{Re}\lambda_n\}_{n=1}^\infty$ considered, the proof by contrapositive of the implication (ii) $\Rightarrow$ (iii) is complete and so is the proof of the entire statement. □

For $\beta = 1$, we obtain the following important particular case.

Corollary 4.1 (Characterization of the entireness of weak solutions)**.** *Let A be a scalar type spectral operator in a complex Banach space $(X, \|\cdot\|)$. Every weak solution of equation* (1.1) *is an entire vector function iff there is a $b_+ > 0$ such that the set $\sigma(A) \setminus \mathscr{P}^1_{b_+}$, where*

$$\mathscr{P}^1_{b_+} := \{\lambda \in \mathbb{C} \,|\, \operatorname{Re}\lambda \ge b_+|\operatorname{Im}\lambda|\},$$

is bounded (see Fig. 2).

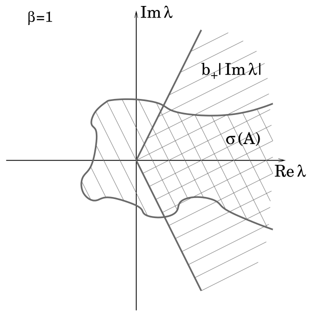

FIGURE 2

Observe that the region $\mathscr{P}^1_{b_+}$ is an angular sector with the vertex at the origin, bisected by the positive x-semi-axis (see Fig. 2).

Thus, we have obtained generalizations of Theorem 4.1 and the entireness characterization contained in [21], the counterparts for a normal operator A in a complex Hilbert space.

As follows from the prior characterization, all weak solutions of equation (1.1) with a scalar type spectral operator A in a complex Banach space can attain the level of strong smoothness as high as *entireness* while the operator A remains *unbounded*, e.g., when A is a semibounded below *self-adjoint* unbounded operator in a complex Hilbert space (see [21, Corollary 4.1] and, for symmetric operators, [21, Theorem 6.1]). This fact contrasts the situation when a closed densely defined linear operator A in a complex Banach space generates a C_0-semigroup, in which case the strong differentiability of all weak solutions of (1.1) at 0 immediately implies *boundedness* for A (cf. [6], see also [24]).

5. Certain inherent smoothness improvement effects

Theorem 4.1 implies, in particular, that

> *if, for some* $1 \le \beta < \infty$*, every weak solution of equation* (1.1) *with a scalar type spectral operator* A *in a complex Banach space* X *belongs to the Gevrey class* $\mathscr{E}^{\{\beta\}}([0,\infty), X)$ *of Roumieu type, then all of them belong to the narrower Gevrey class* $\mathscr{E}^{(\beta)}([0,\infty), X)$ *of Beurling type,*

which is a jump-like effect of the weak solutions' smoothness improvement.

Notably, for $\beta = 1$, we obtain the following statement:

> *if every weak solution of equation* (1.1) *with a scalar type spectral operator* A *in a complex Banach space* X *is analytically continuable into a complex neighborhood* $[0,\infty)$ *(each one into its own), then all of them are entire vector functions,*

which can be further strengthened as follows.

Proposition 5.1. *Let A be a scalar type spectral operator in a complex Banach space $(X, \|\cdot\|)$. If every weak solution of equation* (1.1) *is analytically continuable into a complex neighborhood of* 0 *(each one into its own), then all of them are entire vector functions.*

Proof. Let us show first that, if a weak solution $y(\cdot)$ of equation (1.1) is analytically continuable into a complex neighborhood of 0, then $y(0)$ is an *analytic vector* of the operator A, i.e.,

$$y(0) \in \mathscr{E}^{\{1\}}(A).$$

Let a weak solution $y(\cdot)$ of equation (1.1) be analytically continuable into a complex neighborhood of 0. This implies that there is a $\delta > 0$ such that

$$y(t) = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} t^n, \quad t \in [0, \delta].$$

The power series converging at $t = \delta$, there is a $c > 0$ such that

$$\left\| \frac{y^{(n)}(0)}{n!} \delta^n \right\| \le c, \quad n \in Z_+.$$

Whence, considering that, by Proposition 2.2 with $I = [0, \delta]$,

$$y(0) \in C^\infty(A) \quad \text{and} \quad y^{(n)}(0) = A^n y(0), \quad n \in \mathbb{Z}_+,$$

we infer that

$$\|A^n y(0)\| = \|y^{(n)}(0)\| \le c \left[\delta^{-1}\right]^n n!, \quad n \in \mathbb{Z}_+,$$

which implies

$$y(0) \in \mathscr{E}^{\{1\}}(A).$$

Now, let us prove the statement *by contrapositive* assuming that there is a weak solution of equation (1.1), which is not an entire vector function. This, by Theorem 4.1 with $\beta = 1$, implies that there is a weak solution $y(\cdot)$ of equation (1.1), which is not analytically continuable into a complex neighborhood of $[0, \infty)$. Then, by Proposition 3.1, for some $t_0 \ge 0$,

$$y(t_0) \notin \mathscr{E}^{\{1\}}(A).$$

Therefore, for the weak solution

$$y_{t_0}(t) := y(t + t_0), \quad t \ge 0,$$

of equation (1.1),

$$y_{t_0}(0) = y(t_0) \notin \mathscr{E}^{\{1\}}(A),$$

which, as is shown above, implies that $y_{t_0}(\cdot)$ is not analytically continuable into a complex neighborhood of 0, and hence, completes the proof by contrapositive. □

Thus, we have obtained a generalization of [21, Proposition 5.1], the counterpart for a normal operator A in a complex Hilbert space.

6. Concluding remark

Due to the *scalar type spectrality* of the operator A, Theorem 4.1 is stated exclusively in terms of the location of its *spectrum* in the complex plane as well as the celebrated *Lyapunov stability theorem* [18] (cf. [6, Ch. I, Theorem 2.10]), and thus, is intrinsically qualitative (cf. [27]).

7. Dedication and Acknowledgments

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