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LACUNARY $\mathcal{I}$-CONVERGENT AND LACUNARY $\mathcal{I}$-BOUNDED SEQUENCE SPACES DEFINED BY A MUSIELAK-ORLICZ FUNCTION OVER n-NORMED SPACES

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Abstract. In the present paper we defined $\mathcal{I}$-convergent and $\mathcal{I}$-bounded sequence spaces defined by a Musielak-Orlicz function $\mathcal{M} = (M_k)$ over n-normed spaces. We also make an effort to study some topological properties and prove some inclusion relation between these spaces.

1. Introduction and preliminaries

The notion of ideal convergence was first introduced by P. Kostyrko [8] as a generalization of statistical convergence which was further studied in topological spaces by Das, Kostyrko, Wilczynski and Malik see [1]. More applications of ideals can be seen in ([1], [2]). We continue in this direction and introduced I-convergence of generalized sequences with respect to Musielak-Orlicz function in [19].

A family $\mathcal{I} \subset 2^X$ of subsets of a non empty set X is said to be an ideal in X if

(1) $\phi \in \mathcal{I}$,
(2) $A, B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$,
(3) $A \in \mathcal{I}$, $B \subset A$ imply $B \in \mathcal{I}$,

while an admissible ideal $\mathcal{I}$ of X further satisfies $\{x\} \in \mathcal{I}$ for each $x \in X$ see [8].

A sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be $\mathcal{I}$-convergent to $x \in X$, if for each $\epsilon > 0$ the set $A(\epsilon) = \Big\{n \in \mathbb{N} : ||x_n - x|| \geq \epsilon\Big\}$ belongs to $\mathcal{I}$.

A sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be $\mathcal{I}$-bounded to $x \in X$ if there exists an $K > 0$ such that $\{n \in \mathbb{N} : |x_n| > K\} \in \mathcal{I}$. For more details about ideal convergence sequence spaces (see [7], [9], [13], [15], [16], [17], [18], [21], [25], [26], [29], [31], [32]) and references therein.

Mursaleen and Noman [14] introduced the notion of λ-convergent and λ-bounded sequences as follows :

Let $\lambda = (\lambda_k)_{k=1}^{\infty}$ be a strictly increasing sequence of positive real numbers tending to infinity i.e.

$$0 < \lambda_0 < \lambda_1 < \cdots \quad \text{and} \quad \lambda_k \to \infty \quad \text{as} \quad k \to \infty$$

and said that a sequence $x = (x_k) \in w$ is λ-convergent to the number L, called the λ-limit of x if $\Lambda_m(x) \longrightarrow L$ as $m \to \infty$, where

$$\lambda_m(x) = \frac{1}{\lambda_m} \sum_{k=1}^{m} (\lambda_k - \lambda_{k-1}) x_k.$$

The sequence $x = (x_k) \in w$ is λ-bounded if $\sup_m |\Lambda_m(x)| < \infty$.

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It is well known n that (see for instant [14]) if $\lim_m x_m = a$ in the ordinary sense of convergence, then

$$\lim_m \left(\frac{1}{\lambda_m} \left(\sum_{k=1}^{m} (\lambda_k - \lambda_{k-1}) |x_k - a| \right) = 0.$$

This implies that

$$\lim_m |\Lambda_m(x) - a| = \lim_m | \frac{1}{\lambda_m} \sum_{k=1}^{m} (\lambda_k - \lambda_{k-1})(x_k - a)| = 0,$$

which yields that $\lim_m \Lambda_m(x) = a$ and hence $x = (x_k) \in w$ is λ-convergent to a. Let $A = (a_{jk})$ be an infinite matrix of real or complex numbers a_{jk}, where $jk \in \mathbb{N}$. We write $Ax = (A_k(x))$ if $A_k(x) = \sum_{k=1}^{\infty} a_{jk} x_k$ converges for each $k \in \mathbb{N}$.

An Orlicz function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.

Lindenstrauss and Tzafriri [10] used the idea of Orlicz function to define the following sequence space. Let w be the space of all real or complex sequences $x = (x_k)$, then

$$\ell_M = \Big\{ x \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty \Big\},$$

which is called as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$||x|| = \inf \Big\{ \rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1 \Big\}.$$

It is shown in [10] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. The Δ_2-condition is equivalent to $M(Lx) \leq kLM(x)$ for all values of $x \geq 0$, and for $L > 1$.

A sequence $\mathcal{M} = (M_k)$ of Orlicz function is called a Musielak-Orlicz function see ([11], [20]). A sequence $\mathcal{N} = (N_k)$ defined by

$$N_k(v) = \sup\{|v|u - (M_k) : u \geq 0\}, \quad k = 1, 2, \ldots$$

is called the complementary function of a Musielak-Orlicz function $\mathcal{M}$. For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows

$$t_{\mathcal{M}} = \Big\{ x \in w : I_{\mathcal{M}}(cx) < \infty \ \text{ for some } \ c > 0 \Big\},$$

$$h_{\mathcal{M}} = \Big\{ x \in w : I_{\mathcal{M}}(cx) < \infty \ \text{ for all } \ c > 0 \Big\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$||x|| = \inf \Big\{ k > 0 : I_{\mathcal{M}}\Big(\frac{x}{k}\Big) \leq 1 \Big\}$$

or equipped with the Orlicz norm

$$||x||^0 = \inf \Big\{ \frac{1}{k} \Big(1 + I_{\mathcal{M}}(kx)\Big) : k > 0 \Big\}.$$

For more details about sequence spaces defined by Orlicz function see ([22], [23], [24], [27], [28], [30]) and reference therein.

The concept of 2-normed spaces was initially developed by Gähler [3] in the mid of 1960's, while that of n-normed spaces one can see in Misiak [11]. Since then, many others have studied this concept and obtained various results, see Gunawan ([4], [5]) and Gunawan and Mashadi [6]. Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is field of real or complex numbers of dimension d, where $d \geq n \geq 2$. A real valued function $||\cdot,\cdot,\ldots,\cdot||$ on X^n satisfying the following four conditions:

(1) $||x_1, x_2, \ldots, x_n|| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent in X;
(2) $||x_1, x_2, \ldots, x_n||$ is invariant under permutation;
(3) $||\alpha x_1, x_2, \ldots, x_n|| = |\alpha| \; ||x_1, x_2, \ldots, x_n||$ for any $\alpha \in \mathbb{K}$, and
(4) $||x + x', x_2, \ldots, x_n|| \leq ||x, x_2, \ldots, x_n|| + ||x', x_2, \ldots, x_n||$

is called a n-norm on X, and the pair $(X, ||\cdot,\cdot,\ldots,\cdot||$ is called a n-normed space over the field $\mathbb{K}$.

For example, we may take $X = \mathbb{R}^n$ being equipped with the Euclidean n-norm $||x_1, x_2, \ldots, x_n||_E =$ the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \ldots, x_n$ which may be given explicitly by the formula

$$||x_1, x_2, \ldots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \ldots, n$. Let $(X, ||\cdot,\cdot,\ldots,\cdot||$ be a n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \ldots, a_n\}$ be linearly independent set in X. Then the following function $||\cdot,\cdot,\ldots,\cdot||_\infty$ on X^{n-1} defined by

$$||x_1, x_2, \ldots, x_{n-1}||_\infty = \max\{||x_1, x_2, \ldots, x_{n-1}, a_i|| : i = 1, 2, \ldots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \ldots, a_n\}$.

A sequence (x_k) in a n-normed space $(X, ||\cdot,\cdot,\ldots,\cdot||)$ is said to converge to some $L \in X$ if

$$\lim_{k\to\infty} ||x_k - L, z_1, \ldots, z_{n-1}|| = 0 \quad \text{for every} \quad z_1, \ldots, z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X, ||\cdot,\cdot,\ldots,\cdot||)$ is said to be Cauchy if

$$\lim_{k,p\to\infty} ||x_k - x_p, z_1, \ldots, z_{n-1}|| = 0 \quad \text{for every} \quad z_1, \ldots, z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.

Mursaleen and Sharma [19] introduced the following sequence spaces:

$$c^I(\mathcal{M}, \Lambda, p) = \Big\{x = (x_k) \in w : I - \lim_k M_k\Big(\frac{|\Lambda_k(x) - L|}{\rho}\Big)^{p_k} = 0, \text{ for some } L \text{ and } \rho > 0\Big\},$$

$$c_0^I(\mathcal{M}, \Lambda, p) = \Big\{x = (x_k) \in w : I - \lim_k M_k\Big(\frac{|\Lambda_k(x)|}{\rho}\Big)^{p_k} = 0, \text{ for some } \rho > 0\Big\}$$

and

$$l_\infty(\mathcal{M}, \Lambda, p) = \Big\{x = (x_k) \in w : \sup_k M_k\Big(\frac{|\Lambda_k(x)|}{\rho}\Big)^{p_k} < \infty, \text{ for some } \rho > 0\Big\}.$$

We can write

$$m^I(\mathcal{M}, \Lambda, p) = c^I(\mathcal{M}, \Lambda, p) \cap l_\infty(\mathcal{M}, \Lambda, p)$$

and

$$m_0^I(\mathcal{M}, \Lambda, p) = c_0^I(\mathcal{M}, \Lambda, p) \cap l_\infty(\mathcal{M}, \Lambda, p).$$

In the present paper, we define some new sequence spaces by using the concept of ideal convergence, lacunary sequence, Musielak-Orlicz function, n-normed and A transform as follows:

$\mathcal{I} - N_\theta^0(A, \mathcal{M}, p, ||\cdot,\cdot,\ldots,\cdot||) =$

$$\Big\{x \in w : \mathcal{I} - \lim_r \frac{1}{h_r} \sum_{k \in I_r} \Big[M_k\Big(||\frac{A_k(x)}{\rho}, z_1, \ldots, z_{n-1}||\Big)\Big]^{p_k} = 0, \text{ for some } \rho > 0\Big\},$$

$\mathcal{I}-N_\theta(A,\mathcal{M},p,||\cdot,\cdot,\ldots,\cdot||)=$

$$\Big\{x\in w:\mathcal{I}-\lim_r \frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{A_k(x)-L}{\rho},z_1,\ldots,z_{n-1}||\Big)\Big]^{p_k}=0,$$

$$\text{for some } L \text{ and } \rho>0\Big\},$$

$\mathcal{I}-N_\theta^\infty(A,\mathcal{M},p,||\cdot,\cdot,\ldots,\cdot||)=$

$$\Big\{x\in w:\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{A_k(x)}{\rho},z_1,\ldots,z_{n-1}||\Big)\Big]^{p_k}=0,\ \text{ is } \mathcal{I}\text{- bounded}$$

$$\text{for some }\ \rho>0\Big\}.$$

If we take $\mathcal{M}(x)=x$, we get the spaces $\mathcal{I}-N_\theta^0(A,p,||\cdot,\cdot,\ldots,\cdot||)$, $\mathcal{I}-N_\theta(A,p,||\cdot,\cdot,\ldots,\cdot||)$ and $\mathcal{I}-N_\theta^\infty(A,p,||\cdot,\cdot,\ldots,\cdot||)$.

If we take $p=(p_k)=1$, we get the spaces $\mathcal{I}-N_\theta^0(A,\mathcal{M},||\cdot,\cdot,\ldots,\cdot||)$, $\mathcal{I}-N_\theta(A,\mathcal{M},||\cdot,\cdot,\ldots,\cdot||)$ and $\mathcal{I}-N_\theta^\infty(A,\mathcal{M},||\cdot,\cdot,\ldots,\cdot||)$.

The following inequality will be used throughout the paper. If $0\le p_k\le \sup p_k=H$, $D=\max(1,2^{H-1})$ then

$$|a_k+b_k|^{p_k}\le D\{|a_k|^{p_k}+|b_k|^{p_k}\} \tag{1.1}$$

for all k and $a_k,b_k\in\mathbb{C}$. Also $|a|^{p_k}\le\max(1,|a|^H)$ for all $a\in\mathbb{C}$.

The main aim of this paper is to introduce the sequence spaces $\mathcal{I}–N_\theta^0(A,\mathcal{M},p,||\cdot,\cdot,\ldots,\cdot||)$, $\mathcal{I}–N_\theta(A,\mathcal{M},p,||\cdot,\cdot,\ldots,\cdot||)$ and $\mathcal{I}–N_\theta^\infty(A,\mathcal{M},p,||\cdot,\cdot,\ldots,\cdot||)$ defined by a Musielak-Orlicz function $\mathcal{M}=(M_k)$ over n-normed spaces. We also make an effort to study some topological properties and prove some inclusion relation between these spaces.

2. Main results

Theorem 2.1. *Let $\mathcal{M}=(M_k)$ be a Musielak-Orlicz function, $p=(p_k)$ be a bounded sequence of positive real numbers. Then the spaces $\mathcal{I}-N_\theta^0(A,\mathcal{M},p,||\cdot,\cdot,\ldots,\cdot||)$, $\mathcal{I}-N_\theta(A,\mathcal{M},p,||\cdot,\cdot,\ldots,\cdot||)$ and $\mathcal{I}-N_\theta^\infty(A,\mathcal{M},p,||\cdot,\cdot,\ldots,\cdot||)$ are linear.*

Proof. Let $x,y\in\mathcal{I}-N_\theta^0(A,\mathcal{M},p,||\cdot,\cdots,\cdot||)$ and let α,β be scalars. Then there exist positive numbers ρ_1 and ρ_2 such that for every $\epsilon>0$

$$D_1=\Big\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{A_k(x)}{\rho_1},z_1,\ldots,z_{n-1}||\Big)\Big]^{p_k}\ge\frac{\epsilon}{2D}\Big\}\in\mathcal{I}, \tag{2.1}$$

$$D_1=\Big\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{A_k(y)}{\rho_2},z_1,\ldots,z_{n-1}||\Big)\Big]^{p_k}\ge\frac{\epsilon}{2D}\Big\}\in\mathcal{I}. \tag{2.2}$$

Let $\rho_3=\max\big\{2|\alpha|\rho_1,2|\beta|\rho_2\big\}$. Since $\mathcal{M}=(M_k)$ is non-decreasing, convex function and so by using inequality (1.1), we have

$$\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{A_k(\alpha x+\beta y)}{\rho_3},z_1,\ldots,z_{n-1}||\Big)\Big]^{p_k}$$
$$\le\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\alpha A_k(x)}{\rho_3},z_1,\ldots,z_{n-1}||\Big)\Big]^{p_k}+\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\beta A_k(y)}{\rho_3},z_1,\ldots,z_{n-1}||\Big)\Big]^{p_k}$$
$$\le\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{A_k(x)}{\rho_1},z_1,\ldots,z_{n-1}||\Big)\Big]^{p_k}+\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{A_k(y)}{\rho_1},z_1,\ldots,z_{n-1}||\Big)\Big]^{p_k}.$$

Now by (2.1) and (2.2), we have

$$\Big\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \Big[M_k\Big(||\frac{A_k(\alpha x + \beta y)}{\rho_3}, z_1, \ldots, z_{n-1}||\Big)\Big]^{p_k} > \epsilon\Big\} \subset D_1 \cup D_2.$$

Therefore $\alpha x + \beta y \in \mathcal{I} - N_\theta^0(A, \mathcal{M}, p, ||\cdot, \cdot, \ldots, \cdot||)$. Hence $\mathcal{I} - N_\theta^0(A, \mathcal{M}, p, ||\cdot, \cdot, \ldots, \cdot||)$ is a linear space. Similarly we can establish that $\mathcal{I} - N_\theta(A, \mathcal{M}, p, ||\cdot, \cdot, \ldots, \cdot||)$ and $\mathcal{I} - N_\theta^\infty(A, \mathcal{M}, p, ||\cdot, \cdot, \ldots, \cdot||)$ are linear spaces. □

Theorem 2.2. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function. Then*

$$\mathcal{I} - N_\theta^0(A, \mathcal{M}, p, ||\cdot, \cdot, \ldots, \cdot||) \subset \mathcal{I} - N_\theta(A, \mathcal{M}, p, ||\cdot, \cdot, \ldots, \cdot||) \subset \mathcal{I} - N_\theta^\infty(A, \mathcal{M}, p, ||\cdot, \cdot, \ldots, \cdot||).$$

Proof. The first inclusion is obvious. For second inclusion, let

$$x \in \mathcal{I} - N_\theta(A, \mathcal{M}, p, ||\cdot, \cdot, \ldots, \cdot||).$$

Then there exists $\rho_1 > 0$ such that for every $\epsilon > 0$

$$A_1 = \Big\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \Big[M_k\Big(||\frac{A_k(x) - L}{\rho_1}, z_1, \ldots, z_{n-1}||\Big)\Big]^{p_k} \geq \epsilon\Big\} \in \mathcal{I}.$$

Let $\rho = 2\rho_1$. Since $\mathcal{M} = (M_k)$ is non-decreasing and convex, we have

$$\begin{aligned} M_k\Big(||\frac{A_k(x)}{\rho}, z_1, \ldots, z_{n-1}||\Big) &\leq M_k\Big(||\frac{A_k(x) - L}{\rho_1}, z_1, \ldots, z_{n-1}||\Big) \\ &\quad + M_k\Big(||\frac{L}{\rho_1}, z_1, \ldots, z_{n-1}||\Big). \end{aligned}$$

Suppose that $r \notin A_1$. Hence by above inequality and (1.1), we have

$$\begin{aligned} &\frac{1}{h_r} \sum_{k \in I_r} \Big[M_k\Big(||\frac{A_k(x)}{\rho}, z_1, \ldots, z_{n-1}||\Big)\Big]^{p_k} \\ &\leq D\Big\{\frac{1}{h_r} \sum_{k \in I_r} \Big[M_k\Big(||\frac{A_k(x) - L}{\rho_1}, z_1, \ldots, z_{n-1}||\Big)\Big]^{p_k} \\ &+ \frac{1}{h_r} \sum_{k \in I_r} \Big[M_k\Big(||\frac{L}{\rho_1}, z_1, \ldots, z_{n-1}||\Big)\Big]^{p_k}\Big\} \\ &< D\Big\{\epsilon + \frac{1}{h_r} \sum_{k \in I_r} \Big[M_k\Big(||\frac{L}{\rho}, z_1, \ldots, z_{n-1}||\Big)\Big]^{p_k}\Big\}. \end{aligned}$$

Because of the fact that

$$\Big[M_k\Big(||\frac{L}{\rho_1}, z_1, \ldots, z_{n-1}||\Big)\Big]^{p_k} \leq \max\Big\{1, \Big[M_k\Big(||\frac{L}{\rho_1}, z_1, \ldots, z_{n-1}||\Big)\Big]^H\Big\},$$

we have

$$\frac{1}{h_r} \sum_{k \in I_r} \Big[M_k\Big(||\frac{L}{\rho}, z_1, \ldots, z_{n-1}||\Big)\Big]^{p_k} < \infty.$$

Put $K = D\Big\{\epsilon + \frac{1}{h_r} \sum_{k \in I_r} \Big[M_k\Big(||\frac{L}{\rho}, z_1, \ldots, z_{n-1}||\Big)\Big]^{p_k}\Big\}$. It follows that

$$\Big\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \Big[M_k\Big(||\frac{A_k(x)}{\rho}, z_1, \ldots, z_{n-1}||\Big)\Big]^{p_k} > K\Big\} \in \mathcal{I},$$

which means $x \in \mathcal{I} - N_\theta^\infty(A, \mathcal{M}, p, ||\cdot, \cdot, \ldots, \cdot||)$. This completes the proof of the theorem. □

Theorem 2.3. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers. Then $\mathcal{I} - N_\theta^\infty(A, \mathcal{M}, p, ||\cdot, \cdot, \ldots, \cdot||)$ is a paranormed space with paranorm defined by*

$$g(x) = \inf\Big\{\rho > 0 : \frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{A_k(x)}{\rho}, z_1, \ldots, z_{n-1}||\Big)^{p_k} \le 1\Big\}.$$

Proof. It is clear that $g(x) = g(-x)$. Since $M_k(0) = 0$, we get $g(0) = 0$. Let us take $x, y \in \mathcal{I} - N_\theta^\infty(A, \mathcal{M}, p, ||\cdot, \cdot, \ldots, \cdot||)$. Let

$$B(x) = \Big\{\rho > 0 : \frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{A_k(x)}{\rho}, z_1, \ldots, z_{n-1}||\Big)^{p_k} \le 1\Big\},$$

$$B(y) = \Big\{\rho > 0 : \frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{A_k(y)}{\rho}, z_1, \ldots, z_{n-1}||\Big)^{p_k} \le 1\Big\}.$$

Let $\rho_1 \in B(x)$ and $\rho_2 \in B(y)$ and $\rho = \rho_1 + \rho_2$, then we have

$$\begin{aligned}
&\frac{1}{h_r}\sum_{k\in I_r}\Big[M\Big(||\frac{A_k(x+y)}{\rho}, z_1, \ldots, z_{n-1}||\Big)\\
&\le \Big(\frac{\rho_1}{\rho_1+\rho_2}\Big)\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{A_k(x)}{\rho_1}, z_1, \ldots, z_{n-1}||\Big)\\
&+ \frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{A_k(x)}{\rho_2}, z_1, \ldots, z_{n-1}||\Big).
\end{aligned}$$

Thus $\frac{1}{h_r}\sum_{k\in I_r}\Big[M\Big(||\frac{A_k(x+y)}{\rho_1+\rho_2}, z_1, \ldots, z_{n-1}||\Big)^{p_k} \le 1$ and

$$\begin{aligned}
g(x+y) &\le \inf\Big\{(\rho_1+\rho_2) > 0 : \rho_1 \in B(x),\ \rho_2 \in B(y)\Big\}\\
&\le \inf\Big\{\rho_1 > 0 : \rho_1 \in B(x)\Big\} + \inf\Big\{\rho_2 > 0 : \rho_2 \in B(y)\Big\}\\
&= g(x) + g(y).
\end{aligned}$$

Let $\sigma^s \to \sigma$ where $\sigma, \sigma^s \in \mathbb{C}$ and let $g(x^s - x) \to 0$ as $s \to \infty$. We have to show that $g(\sigma^s x^s - \sigma x) \to 0$ as $s \to \infty$. Let

$$B(x^s) = \Big\{\rho_s > 0 : \frac{1}{h_r}\sum_{k\in I_r}\Big[M\Big(||\frac{A_k(x^s)}{\rho_s}, z_1, \ldots, z_{n-1}||\Big)^{p_k} \le 1\Big\},$$

$$B(x^s - x) = \Big\{\rho_s' > 0 : \frac{1}{h_r}\sum_{k\in I_r}\Big[M\Big(||\frac{A_k(x^s - x)}{\rho_s'}, z_1, \ldots, z_{n-1}||\Big)^{p_k} \le 1\Big\}.$$

If $\rho_s \in B(x^s)$ and $\rho_s' \in B(x^s - x)$ then we observe that

$$\begin{aligned}
&\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{A_k(\sigma^s x^s - \sigma x)}{\rho_s|\sigma^s - \sigma| + \rho_s'|\sigma|}, z_1, \ldots, z_{n-1}||\Big)\\
&\le \frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{A_k(\sigma^s x^s - \sigma x^s)}{\rho_s|\sigma^s - \sigma| + \rho_s'|\sigma|} + \frac{|(\sigma x^s - \sigma x)|}{\rho_s|\sigma^s - \sigma| + \rho_s'|\sigma|}, z_1, \ldots, z_{n-1}||\Big)\\
&\le \frac{|\sigma^s - \sigma|\rho_s}{\rho_s|\sigma^s - \sigma| + \rho_s'|\sigma|}\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{(A_k(x^s))|}{\rho_s}, z_1, \ldots, z_{n-1}||\Big)\\
&+ \frac{|\sigma|\rho_s^{'}}{\rho_s|\sigma^s - \sigma| + \rho_s'|\sigma|}\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{A_k(x^s - x)}{\rho_s'}, z_1, \ldots, z_{n-1}||\Big).
\end{aligned}$$

From the above inequality, it follows that

$$\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\ \Big(||\frac{A_k(\sigma^s x^s-\sigma x)}{\rho_s|\sigma^s-\sigma|+\rho_s^{'}|\sigma|},z_1,\dots,z_{n-1}||\Big)\Big]^{p_k}\le 1$$

and consequently,

$$\begin{aligned} g(\sigma^s x^s-\sigma x) &\le \inf\Big\{\Big(\rho_s|\sigma^s-\sigma|+\rho_s^{'}|\sigma|\Big)>0:\rho_s\in B(x^s),\rho_s^{'}\in B(x^s-x)\Big\}\\ &\le (|\sigma^s-\sigma|)>0\inf\Big\{\rho>0:\rho_s\in B(x^s)\Big\}\\ &+ (|\sigma|)>0\inf\Big\{(\rho_s^{'})^{\frac{p_n}{H}}:\rho_s^{'}\in B(x^s-x)\Big\}\\ &\longrightarrow 0\ \text{ as }\ s\longrightarrow\infty. \end{aligned}$$

This completes the proof of the theorem. □

Theorem 2.4. *Let $\mathcal{M}'=(M_k')$ and $\mathcal{M}''=(M_k'')$ are Musielak-Orlicz functions those satisfy the Δ_2-condition. Then*

(i) $\mathcal{I}-N_\theta^0(A,\mathcal{M},p,||\cdot,\cdot,\dots,\cdot||)\subseteq\mathcal{I}-N_\theta^0(A,\mathcal{M}'\circ\mathcal{M},p,||\cdot,\cdot,\dots,\cdot||)$,

(ii) $\mathcal{I}-N_\theta(A,\mathcal{M},p,||\cdot,\cdot,\dots,\cdot||)\subseteq\mathcal{I}-N_\theta(A,\mathcal{M}'\circ\mathcal{M},p,||\cdot,\cdot,\dots,\cdot||)$,

(iii) $\mathcal{I}-N_\theta^\infty(A,\mathcal{M},p,||\cdot,\cdot,\dots,\cdot||)\subseteq\mathcal{I}-N_\theta^\infty(A,\mathcal{M}'\circ\mathcal{M},p,||\cdot,\cdot,\dots,\cdot||)$.

Proof. (i) Let $x\in\mathcal{I}-N_\theta^0(A,\mathcal{M},p,||\cdot,\cdot,\dots,\cdot||)$. Then there exists $K_1>0$ such that

$$A_1=\Big\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{A_k(x)}{\rho_1},z_1,\dots,z_{n-1}||\Big)\Big]^{p_k}\ge K_1\Big\}\in\mathcal{I}$$

for $\rho>0$. Since $\mathcal{M}'$ is nondecreasing, convex and satisfies Δ_2-condition, we have

$$\frac{1}{h_r}\sum_{\substack{k\in I_r\\ M_k(||\frac{A_k(x)}{\rho},z_1,\dots,z_{n-1}||)>\delta}}\Big[M_k'\Big(M_k\Big(||\frac{A_k(x)}{\rho},z_1,\dots,z_{n-1}||\Big)\Big)\Big]^{p_k}$$

(2.3)
$$\le\max\Big\{1,\Big(K\frac{1}{\delta}M_k'(2)\Big)^H\Big\}\frac{1}{h_r}\sum_{\substack{k\in I_r\\ M_k(||\frac{A_k(x)}{\rho},z_1,\dots,z_{n-1}||)>\delta}}\Big(M_k\Big(||\frac{A_k(x)}{\rho},z_1,\dots,z_{n-1}||\Big)\Big)^{p_k}$$

for $K\ge1$. By continuity of $\mathcal{M}'$, we have

$$\frac{1}{h_r}\sum_{\substack{k\in I_r\\ M_k(||\frac{A_k(x)}{\rho},z_1,\dots,z_{n-1}||)\le\delta}}\Big[M_k'\Big(M_k\Big(||\frac{A_k(x)}{\rho},z_1,\dots,z_{n-1}||\Big)\Big)\Big]^{p_k}$$

$$\le\frac{1}{h_r}\sum_{\substack{k\in I_r\\ M_k(||\frac{A_k(x)}{\rho},z_1,\dots,z_{n-1}||)\le\delta}}\epsilon^{p_k}\le\frac{1}{h_r}\sum_{\substack{k\in I_r\\ M_k(||\frac{A_k(x)}{\rho},z_1,\dots,z_{n-1}||)\le\delta}}\max\{\epsilon^h,\epsilon^H\}. \tag{2.4}$$

Suppose $r\notin A_1$. Then by using (2.3) and (2.4), we have

$$\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k'\Big(M_k\Big(||\frac{A_k(x)}{\rho},z_1,\dots,z_{n-1}\Big)\Big)\Big]^{p_k}$$

$$=\frac{1}{h_r}\sum_{\substack{k\in I_r\\ M_k(||\frac{A_k(x)}{\rho},z_1,\dots,z_{n-1}||)>\delta}}\Big(M_k\Big(||\frac{A_k(x)}{\rho},z_1,\dots,z_{n-1}||\Big)\Big)^{p_k}$$

$$+ \quad \frac{1}{h_r} \sum_{\substack{k \in I_r \\ M_k(||\frac{A_k(x)}{\rho}, z_1, \ldots, z_{n-1}||) \leq \delta}} \Big(M_k\Big(||\frac{A_k(x)}{\rho}, z_1, \ldots, z_{n-1}||\Big)\Big)^{p_k}$$

$$\leq \quad \max\Big\{1, \Big(K\frac{1}{\delta}M'_k(2)\Big)^H\Big\}K_1 + \max\{\epsilon^h, \epsilon^H\} = K_2,$$

hence $r \notin B_1 = \Big\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k \in I_r}\Big[M'_k\Big(M_k\Big(||\frac{A_k(x)}{\rho}, z_1, \ldots, z_{n-1}||\Big)\Big)\Big]^{p_k} > K_2\}$ and so $B_1 \subset A_1$ which implies $B_1 \in \mathcal{I}$. This means that $x \in \mathcal{I} - N_\theta^0(A, \mathcal{M}' \circ \mathcal{M}, p, ||\cdot, \cdot, \ldots, \cdot||)$. This completes the proof of (i) part of the theorem. Similarly, we can prove (ii) and (iii) part. □

We state the following results without proof in view of the above theorem.

Corollary 2.5. *Let $\mathcal{M} = (M_k)$ be a Musielak -Orlicz function which satisfies Δ_2-condition. Then*

(i) $\mathcal{I} - N_\theta^0(A, p, ||\cdot, \cdot, \ldots, \cdot||) \subseteq \mathcal{I} - N_\theta^0(A, \mathcal{M}, p, ||\cdot, \cdot, \ldots, \cdot||)$,

(ii) $\mathcal{I} - N_\theta(A, p, ||\cdot, \cdot, \ldots, \cdot||) \subseteq \mathcal{I} - N_\theta(A, \mathcal{M}, p, ||\cdot, \cdot, \ldots, \cdot||)$,

(iii) $\mathcal{I} - N_\theta^\infty(A, p, ||\cdot, \cdot, \ldots, \cdot||) \subseteq \mathcal{I} - N_\theta^\infty(A, \mathcal{M}, p, ||\cdot, \cdot, \ldots, \cdot||)$.

Theorem 2.6. *Let $\mathcal{M}' = (M'_k)$ and $\mathcal{M}'' = (M''_k)$ are Musielak-Orlicz functions that satisfies the Δ_2-condition. Then*

(i) $\mathcal{I} - N_\theta^0(A, \mathcal{M}, p, ||\cdot, \cdot, \ldots, \cdot||) \cap \mathcal{I} - N_\theta^0(A, \mathcal{M}', p, ||\cdot, \cdot, \ldots, \cdot||)$
$\subseteq \mathcal{I} - N_\theta^0(A, \mathcal{M}' + \mathcal{M}, p, ||\cdot, \cdot, \ldots, \cdot||)$,

(ii) $\mathcal{I} - N_\theta(A, \mathcal{M}, p, ||\cdot, \cdot, \ldots, \cdot||) \cap \mathcal{I} - N_\theta(A, \mathcal{M}', p, ||\cdot, \cdot, \ldots, \cdot||)$
$\subseteq \mathcal{I} - N_\theta(A, \mathcal{M}' + \mathcal{M}, p, ||\cdot, \cdot, \ldots, \cdot||)$,

(iii) $\mathcal{I} - N_\theta^\infty(A, \mathcal{M}, p, ||\cdot, \cdot, \ldots, \cdot||) \cap \mathcal{I} - N_\theta^\infty(A, \mathcal{M}', p, ||\cdot, \cdot, \ldots, \cdot||)$
$\subseteq \mathcal{I} - N_\theta^\infty(A, \mathcal{M}' + \mathcal{M}, p, ||\cdot, \cdot, \ldots, \cdot||)$.

Proof. (i) Let $x \in \mathcal{I} - N_\theta^0(A, \mathcal{M}, p, ||\cdot, \cdot, \ldots, \cdot||) \cap \mathcal{I} - N_\theta^0(A, \mathcal{M}', p, ||\cdot, \cdot, \ldots, \cdot||)$. Then there exist $K_1 > 0$ and $K_2 > 0$ such that

$$A_1 = \Big\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k \in I_r}\Big[M_k\Big(||\frac{A_k(x)}{\rho_1}, z_1, \ldots, z_{n-1}||\Big)\Big]^{p_k} \geq K_1\Big\} \in \mathcal{I}$$

and

$$A_2 = \Big\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k \in I_r}\Big[M'_k\Big(||\frac{A_k(x)}{\rho_1}, z_1, \ldots, z_{n-1}||\Big)\Big]^{p_k} \geq K_2\Big\} \in \mathcal{I}$$

for some $\rho > 0$. Let $r \notin A_1 \cup A_2$. Then we have

$$\frac{1}{h_r}\sum_{k \in I_r}\Big[M_k + M'_k\Big(||\frac{A_k(x)}{\rho}, z_1, \ldots, z_{n-1}||\Big)\Big]^{p_k}$$

$$\leq \quad D\Big\{\frac{1}{h_r}\sum_{k \in I_r}\Big(M_k\Big(||\frac{A_k(x)}{\rho}, z_1, \ldots, z_{n-1}||\Big)\Big)^{p_k}$$

$$+ \quad \frac{1}{h_r}\sum_{k \in I_r}\Big(M'_k\Big(||\frac{A_k(x)}{\rho}, z_1, \ldots, z_{n-1}||\Big)\Big)^{p_k}\Big\}$$

$$< \quad \{K_1 + K_2\}.$$

$r \notin B = \Big\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r}\Big[(M_k'+M_k)\Big(||\frac{A_k(x)}{\rho}, z_1, \dots, z_{n-1}||\Big)\Big)^{p_k} > K\Big\}$. We have $A_1 \cup A_2 \in \mathcal{I}$ and so $B \subset A_1 \cup A_2$ which implies $B \in \mathcal{I}$. This means that $x \in \mathcal{I} - N_\theta^0(A, \mathcal{M}' + \mathcal{M}, p, ||\cdot, \cdot, \dots, \cdot||)$. This completes the proof of (i) part of the theorem.

Similarly, we can prove (ii) and (iii) part. □

Theorem 2.7. *Let* $0 < p_k \le q_k$ *and* $\left(\frac{q_k}{p_k}\right)$ *be bounded. Then following inclusions hold:*

(i) $\mathcal{I} - N_\theta^0(A, \mathcal{M}, q, ||\cdot, \cdot, \dots, \cdot||) \subseteq \mathcal{I} - N_\theta^0(A, \mathcal{M}, p, ||\cdot, \cdot, \dots, \cdot||)$,

(ii) $\mathcal{I} - N_\theta(A, \mathcal{M}, q, ||\cdot, \cdot, \dots, \cdot||) \subseteq \mathcal{I} - N_\theta(A, \mathcal{M}, p, ||\cdot, \cdot, \dots, \cdot||)$.

Proof. (i) Let $x \in \mathcal{I} - N_\theta^0(A, \mathcal{M}, q, ||\cdot, \cdot, \dots, \cdot||)$.

Write $t_k = \Big[M_k\Big(||\frac{A_k(x)}{\rho}, z_1, \dots, z_{n-1}||\Big)\Big]^{q_k}$ and $\lambda_k = \frac{p_k}{q_k}$, so that $0 < \lambda < \lambda_k \le 1$. By using Hölder inequality, we have

$$\begin{aligned}
\frac{1}{h_r}\sum_{k\in I_r} &= \frac{1}{h_r}\sum_{\substack{k\in I_r\\ t_k\ge 1}}(t_k)^{\lambda_k} + \frac{1}{h_r}\sum_{\substack{k\in I_r\\ t_k<1}}(t_k)^{\lambda_k}\\
&\le \frac{1}{h_r}\sum_{\substack{k\in I_r\\ t_k\ge 1}}(t_k) + \frac{1}{h_r}\sum_{\substack{k\in I_r\\ t_k<1}}(t_k)^{\lambda}\\
&= \frac{1}{h_r}\sum_{\substack{k\in I_r\\ t_k\ge 1}}(t_k) + \sum_{\substack{k\in I_r\\ t_k<1}}\Big(\frac{1}{h_r}t_k\Big)^{\lambda}\Big(\frac{1}{h_r}\Big)^{1-\lambda}\\
&\le \frac{1}{h_r}\sum_{\substack{k\in I_r\\ t_k\ge 1}}(t_k) + \Big(\sum_{\substack{k\in I_r\\ t_k<1}}\Big[\Big(\frac{1}{h_r}t_k\Big)^{\lambda}\Big]^{\frac{1}{\lambda}}\Big)^{\lambda}\Big(\sum_{\substack{k\in I_r\\ t_k<1}}\Big[(\frac{1}{h_r}\Big)^{1-\lambda}\Big]^{\frac{1}{\lambda-1}}\Big)^{1-\lambda}\\
&\le \frac{1}{h_r}\sum_{\substack{k\in I_r\\ t_k\ge 1}}(t_k) + \Big(\frac{1}{h_r}\sum_{\substack{k\in I_r\\ t_k<1}}t_k\Big)^{\lambda}.
\end{aligned}$$

Hence for every $\epsilon > 0$, we have

$$\begin{aligned}
&\Big\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{A_k(x)}{\rho}, z_1, \dots, z_{n-1}||\Big)\Big]^{p_k} \ge \epsilon\Big\}\\
&\subset \Big\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{\substack{k\in I_r\\ t_k\ge 1}}\Big[M_k\Big(||\frac{A_k(x)}{\rho}, z_1, \dots, z_{n-1}||\Big)\Big]^{q_k} \ge \frac{\epsilon}{2}\Big\}\\
&\cup \Big\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{\substack{k\in I_r\\ t_k<1}}\Big[M_k\Big(||\frac{A_k(x)}{\rho}, z_1, \dots, z_{n-1}||\Big)\Big]^{q_k} \ge \Big(\frac{\epsilon}{2}\Big)^{\frac{1}{\lambda}}\Big\}.
\end{aligned}$$

This implies that $\Big\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{A_k(x)}{\rho}, z_1, \dots, z_{n-1}||\Big)\Big]^{p_k} \ge \epsilon\Big\} \in \mathcal{I}$ and so $x \in \mathcal{I} - N_\theta^0(A, \mathcal{M}, p, ||\cdot, \cdot, \dots, \cdot||)$. This completes the proof of (i) part. Similarly, we can prove (ii) part. □

Corollary 2.8. *If* $0 < \inf p_k \le 1$. *Then the following inclusions hold*:

(i) $\mathcal{I} - N_\theta^0(A, \mathcal{M}, ||\cdot, \cdot, \dots, \cdot||) \subseteq \mathcal{I} - N_\theta^0(A, \mathcal{M}, p, ||\cdot, \cdot, \dots, \cdot||)$,

(ii) $\mathcal{I} - N_\theta(A, \mathcal{M}, ||\cdot, \cdot, \dots, \cdot||) \subseteq \mathcal{I} - N_\theta(A, \mathcal{M}, p, ||\cdot, \cdot, \dots, \cdot||)$.

Proof. The proof follows from Theorem 2.7. □

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