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CONTINUOUS SYMMETRIC 3-HOMOGENEOUS POLYNOMIALS ON SPACES OF LEBESGUE MEASURABLE ESSENTIALLY BOUNDED FUNCTIONS

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Abstract. Vector spaces of all homogeneous continuous polynomials on infinite dimensional Banach spaces are infinite dimensional. But spaces of homogeneous continuous polynomials with some additional natural properties can be finite dimensional. The so-called symmetry of polynomials on some classes of Banach spaces is one of such properties.

In this paper we consider continuous symmetric 3-homogeneous polynomials on the complex Banach space L_∞ of all Lebesgue measurable essentially bounded complex-valued functions on $[0,1]$ and on the Cartesian square of this space. We construct Hamel bases of spaces of such polynomials and prove formulas for representing of polynomials as linear combinations of base polynomials.

Results of the paper can be used for investigations of algebras of symmetric continuous polynomials and of symmetric analytic functions on L_∞ and on its Cartesian square. In particular, in order to describe appropriate topologies on the spectrum (the set of complex valued homomorphisms) of a given algebra of analytic functions, it is useful to have representations for polynomials, obtained in this paper.

1. Introduction

Polynomials and analytic functions on a Banach space X, which are invariant (symmetric) with respect to a group of operators $G(X)$ acting on X, were studied by a number of authors [1–10, 12–14, 18–23]. If X has a symmetric structure, like has a symmetric basis (a countable basis such that for every element of a unit ball, every element, formed by a permutation of its coordinates with respect to this basis, belongs to the unit ball; see, e.g., [15, Definition 3.a.1, p. 113]) or is rearrangement invariant, then it is natural to consider the case when $G(X)$ is a group of operators which preserve this structure (see, e.g., [11]). In particular, if X is a rearrangement invariant Banach space of Lebesgue measurable functions on $[0,1]$ (for every function, which belongs to the space, a composition of this function with any bijection of $[0,1]$, which preserves the measure, also belongs to the space; see, e.g., [16, Definition 2.a.1, p. 117]), then $G(X)$ is used to be the group of all bijections of $[0,1]$, which preserve the measure. Firstly symmetric polynomials on the real Banach space of Lebesgue measurable integrable in a power p functions on $[0,1]$, where $1 \le p < \infty$, were studied by Nemirovski and Semenov in [18]. Some of their results were generalized to real separable rearrangement invariant Banach spaces of Lebesgue measurable functions by González, Gonzalo and Jaramillo in [9].

In [7] the authors together with Galindo investigated the algebra of all symmetric continuous polynomials and the algebra of all symmetric entire functions of bounded type on the complex Banach space L_∞ of all Lebesgue measurable essentially bounded

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complex-valued functions on $[0, 1]$. In particular, it was constructed the so-called algebraic basis of the algebra of all symmetric continuous polynomials on L_∞, i.e., a sequence of some "elementary" symmetric continuous polynomials on L_∞ such that every symmetric continuous polynomial on L_∞ can be uniquely represented as a linear combination of products of powers of these "elementary" polynomials. Consequently, such products form Hamel bases in the spaces of homogeneous symmetric continuous polynomials on L_∞. Although the existence and the uniqueness of the above-mentioned representation are proved in [7, Theorem 4.3], a constructive method of the evaluation of its coefficients is not known in the general case. But the information about these coefficients is important for the investigation of symmetric analytic functions on L_∞. In particular, it might help to answer the following open question: whether every entire symmetric function on L_∞ is bounded on bounded subsets of L_∞?

Formulas for the evaluation of the coefficients of the above-mentioned representations for 1-homogeneous (linear functionals) and 2-homogeneous symmetric continuous polynomials on L_∞ were constructed in [19] and [20] respectively. Also in [20] analogical formulas were constructed for 1-homogeneous and 2-homogeneous symmetric ("block-symmetric" in the terminology of [20]) continuous polynomials on the Cartesian square of the space L_∞.

In this paper we consider the spaces of 3-homogeneous symmetric continuous polynomials on L_∞ and on the Cartesian square of L_∞. In Section 3 we establish some properties of bilinear forms on L_∞. Also we construct formulas for the evaluation of coefficients of the representation of a symmetric continuous 3-homogeneous polynomial on L_∞ in the form of a linear combination of elements of the respective Hamel basis. In Section 4 we construct a Hamel basis of the space of 3-homogeneous symmetric continuous polynomials on the Cartesian square of L_∞ and construct formulas for the evaluation of coefficients in the respective representation.

Results of the paper can be used for investigations of algebras of symmetric continuous polynomials and of symmetric analytic functions on L_∞ and on its Cartesian square. In particular, in order to describe appropriate topologies on the spectrum (the set of complex valued homomorphisms) of a given algebra of analytic functions, it is useful to have representations for polynomials, obtained in this paper.

2. Preliminaries

A mapping $P : X \to \mathbb{C}$, where X is a complex Banach space, is called an *n-homogeneous polynomial* if there exists an n-linear symmetric (with respect to the permutations of its arguments) form $A_P : X^n \to \mathbb{C}$ such that P is the restriction to the diagonal of A_P, i.e. $P(x) = A_P(\underbrace{x, \ldots, x}_{n})$ for every $x \in X$. The form A_P is called the *n-linear symmetric form associated with* P. It is known (see, e.g., [17, Theorem 1.10]) that A_P can be recovered from P by means of the so-called Polarization Formula:

$$A_P(x_1, \ldots, x_n) = \frac{1}{n!2^n} \sum_{\varepsilon_1,\ldots,\varepsilon_n=\pm 1} \varepsilon_1 \ldots \varepsilon_n P(\varepsilon_1 x_1 + \cdots + \varepsilon_n x_n). \tag{1}$$

The Polarization Formula implies that A_P is continuous if and only if P is continuous.

Note that in the definition of an n-homogeneous polynomial it is sufficient to claim the existence of an n-linear (not necessarily symmetric) form $A : X^n \to \mathbb{C}$ such that $P(x) = A(\underbrace{x, \ldots, x}_{n})$ because every n-linear form can be symmetrized by means of the symmetrization operator

$$A^s(x_1, \ldots, x_n) = \frac{1}{n!} \sum_{\tau \in S_n} A(x_{\tau(1)}, \ldots, x_{\tau(n)}),$$

where S_n is the group of all the permutations on the set $\{1, \dots, n\}$, and the restriction of A^s to the diagonal is equal to the restriction of A to the diagonal.

Let μ be the Lebesgue measure on $[0,1]$ and L_∞ be the complex Banach space of all Lebesgue measurable essentially bounded complex-valued functions x on $[0,1]$ with norm

$$\|x\|_\infty = \operatorname{ess\,sup}_{t\in[0,1]}|x(t)|.$$

Let Ξ be the set of all bijections $\sigma : [0,1] \to [0,1]$ such that both σ and σ^{-1} are Lebesgue measurable and both σ and σ^{-1} preserve the measure. A function $F : L_\infty \to \mathbb{C}$ is called Ξ-*symmetric* (or just *symmetric* when the context is clear) if for every $x \in L_\infty$ and $\sigma \in \Xi$

$$F(x \circ \sigma) = F(x). \tag{2}$$

By [7, Remark 4.1], for every n-homogeneous Ξ-symmetric polynomial Q its associated symmetric n-linear form A_Q has the property that

$$A_Q(x_1 \circ \sigma, \dots, x_n \circ \sigma) = A_Q(x_1, \dots, x_n) \tag{3}$$

for every $x_1, \dots, x_n \in L_\infty$ and $\sigma \in \Xi$.

For every $n \in \mathbb{N} \cup \{0\}$ we define $R_n : L_\infty \to \mathbb{C}$ by $R_n(x) = \int_0^1 x^n(t)\,dt$. The functions R_n are called *the elementary symmetric polynomials*. In [7, Theorem 4.3] it is shown that every continuous Ξ-symmetric n-homogeneous polynomial $P : L_\infty \to \mathbb{C}$ can be uniquely represented as

$$P = \sum_{k_1+2k_2+\cdots+nk_n=n} \alpha_{k_1,k_2,\dots,k_n} R_1^{k_1} R_2^{k_2} \dots R_n^{k_n}, \tag{4}$$

where $k_1, k_2, \dots, k_n \in \mathbb{N} \cup \{0\}$ and $\alpha_{k_1,k_2,\dots,k_n} \in \mathbb{C}$. In particular, for every continuous linear Ξ-symmetric functional $f : L_\infty \to \mathbb{C}$ and for every $x \in L_\infty$

$$f(x) = f(1)R_1(x). \tag{5}$$

Note that formula (5) was firstly proved in [19]. By (4), the set of polynomials

$$\left\{R_1^{k_1} R_2^{k_2} \dots R_n^{k_n} : \quad k_1, k_2, \dots, k_n \in \mathbb{N} \cup \{0\},\ k_1 + 2k_2 + \cdots + nk_n = n\right\}$$

is a Hamel basis of the vector space of all continuous Ξ-symmetric n-homogeneous polynomials.

Let $(L_\infty)^2$ be the Cartesian square of the space L_∞, endowed with norm $\|(x,y)\| = \max\{\|x\|_\infty, \|y\|_\infty\}$. We say that a function $F : (L_\infty)^2 \to \mathbb{C}$ is Ξ-*symmetric* if for every $(x,y) \in (L_\infty)^2$ and $\sigma \in \Xi$

$$F((x \circ \sigma, y \circ \sigma)) = F((x,y)).$$

Note that such functions are also called block-symmetric (see [20]).

For every $E \subset [0,1]$ let

$$1_E(t) = \begin{cases} 1, & \text{if } t \in E \\ 0, & \text{if } t \in [0,1] \setminus E. \end{cases}$$

Let

$$1 = 1_{[0,1]} \quad \text{and} \quad r = 1_{[0,\frac{1}{2}]} - 1_{[\frac{1}{2},1]}.$$

3. Symmetric 3-homogeneous polynomials on L_∞

By (4), every continuous Ξ-symmetric 3-homogeneous polynomial on L_∞ can be uniquely represented in the form $\alpha R_1^3 + \beta R_1 R_2 + \gamma R_3$, where $\alpha, \beta, \gamma \in \mathbb{C}$. Consequently, the set of polynomials $\{R_1^3, R_1 R_2, R_3\}$ is a Hamel basis of the space of continuous Ξ-symmetric 3-homogeneous polynomials on L_∞. In this section we shall construct formulas for the evaluation of coefficients α, β, γ. First, we prove some properties of bilinear forms.

Proposition 3.1. *Let $B : (L_\infty)^2 \to \mathbb{C}$ be a continuous bilinear symmetric form such that*

$$B(x \circ \sigma, y \circ \sigma) = B(x, y) \tag{6}$$

for every $x, y \in L_\infty$ and $\sigma \in \Xi$. Then

$$B(x, y) = \alpha R_1(x) R_1(y) + \beta \int_0^1 x(t) y(t)\, dt,$$

where $\alpha = B(1,1) - B(r,r)$ and $\beta = B(r,r)$.

Proof. Let $\widehat{B}$ be the restriction of B to the diagonal. Since B is a continuous bilinear symmetric form, $\widehat{B}$ is a continuous 2-homogeneous polynomial. By (6), $\widehat{B}$ is Ξ-symmetric. By (4), there exist coefficients α and β such that

$$\widehat{B} = \alpha R_1^2 + \beta R_2. \tag{7}$$

By (7), $\widehat{B}(1) = \alpha + \beta$ and $\widehat{B}(r) = \beta$. Therefore, $\alpha = B(1,1) - B(r,r)$ and $\beta = B(r,r)$.

By the Polarization Formula,

$$B(x, y) = \frac{1}{4}\Big(\widehat{B}(x + y) - \widehat{B}(x - y)\Big).$$

Therefore,

$$B(x, y) = \frac{1}{4}\Big(\alpha R_1^2(x + y) + \beta R_2(x + y) - \alpha R_1^2(x - y) - \beta R_2(x - y)\Big).$$

It can be checked that

$$R_1^2(x + y) - R_1^2(x - y) = 4 R_1(x) R_1(y)$$

and

$$R_2(x + y) - R_2(x - y) = 4 \int_0^1 x(t) y(t)\, dt.$$

Therefore,

$$B(x, y) = \alpha R_1(x) R_1(y) + \beta \int_0^1 x(t) y(t)\, dt.$$

□

We define linear continuous operators $\varkappa_1, \varkappa_2 : L_\infty \to L_\infty$ and mappings $v_1, v_2 : \Xi \to \Xi$ by

$$\varkappa_1(x)(t) = \begin{cases} x(2t), & \text{if } t \in [0, \frac{1}{2}] \\ 0, & \text{if } t \in (\frac{1}{2}, 1], \end{cases}$$

$$\varkappa_2(x)(t) = \begin{cases} 0, & \text{if } t \in [0, \frac{1}{2}) \\ x(2t - 1), & \text{if } t \in [\frac{1}{2}, 1], \end{cases}$$

$$v_1(\sigma)(t) = \begin{cases} \frac{1}{2}\sigma(2t), & \text{if } t \in [0, \frac{1}{2}] \\ t, & \text{if } t \in (\frac{1}{2}, 1], \end{cases}$$

$$v_2(\sigma)(t) = \begin{cases} t, & \text{if } t \in [0, \frac{1}{2}) \\ \frac{1}{2}\sigma(2t - 1) + \frac{1}{2}, & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

Lemma 3.1. *For every $x \in L_\infty$ and $\sigma \in \Xi$ we have*

$$\begin{aligned} \varkappa_1(x \circ \sigma) &= \varkappa_1(x) \circ v_1(\sigma), \\ \varkappa_2(x \circ \sigma) &= \varkappa_2(x) \circ v_2(\sigma), \\ \varkappa_1(x) &= \varkappa_1(x) \circ v_2(\sigma), \\ \varkappa_2(x) &= \varkappa_2(x) \circ v_1(\sigma). \end{aligned}$$

Proof. Let us prove the first equality. For $t \in [0, \frac{1}{2}]$ we have $v_1(\sigma)(t) = \frac{1}{2}\sigma(2t)$. Since $\sigma(2t) \in [0,1]$, it follows that $\frac{1}{2}\sigma(2t) \in [0, \frac{1}{2}]$. Therefore,

$$\varkappa_1(x)(v_1(\sigma)(t)) = x(2v_1(\sigma)(t)) = x(\sigma(2t)) = \varkappa_1(x \circ \sigma)(t).$$

For $t \in (\frac{1}{2}, 1]$ we have $v_1(\sigma)(t) = t$. Therefore,

$$\varkappa_1(x)(v_1(\sigma)(t)) = 0 = \varkappa_1(x \circ \sigma)(t).$$

Thus, $\varkappa_1(x) \circ v_1(\sigma) = \varkappa_1(x \circ \sigma)$.

The second equality can be proved analogically to the first one.

Let us prove the third equality. For $t \in [0, \frac{1}{2})$ we have $v_2(\sigma)(t) = t$ and so

$$\varkappa_1(x)(v_2(\sigma)(t)) = \varkappa_1(x)(t).$$

Since $v_2(\sigma)(t) \in [\frac{1}{2}, 1]$ for $t \in [\frac{1}{2}, 1]$, it follows that

$$\varkappa_1(x)(v_2(\sigma)(t)) = 0$$

in this case. Thus, $\varkappa_1(x) = \varkappa_1(x) \circ v_2(\sigma)$.

The fourth equality can be proved analogically to the third one. □

Proposition 3.2. *Let $B : (L_\infty)^2 \to \mathbb{C}$ be a continuous bilinear form such that*

$$B(x \circ \sigma, y \circ \sigma) = B(x, y) \tag{8}$$

for every $x, y \in L_\infty$ and $\sigma \in \Xi$. Then B is symmetric.

Proof. The bilinear form B can be represented as the sum of symmetric and antisymmetric forms

$$B^s(x,y) = \frac{1}{2}\big(B(x,y) + B(y,x)\big)$$

and

$$B^a(x,y) = \frac{1}{2}\big(B(x,y) - B(y,x)\big) \tag{9}$$

respectively.

By (8) and (9),

$$B^a(x \circ \sigma, y \circ \sigma) = B^a(x, y) \tag{10}$$

for every $x, y \in L_\infty$ and $\sigma \in \Xi$.

Let us show that $B^a(x,y) = 0$ for every $x, y \in L_\infty$.

We prove that $B^a\big(1_{[0,\frac{1}{2}]}, 1_{[\frac{1}{2},1]}\big) = 0$. By the antisymmetry of B^a,

$$B^a\big(1_{[0,\frac{1}{2}]}, 1_{[\frac{1}{2},1]}\big) = -B^a\big(1_{[\frac{1}{2},1]}, 1_{[0,\frac{1}{2}]}\big).$$

On the other hand, on substituting $\sigma(t) = 1 - t$ into (10) we obtain

$$B^a\big(1_{[0,\frac{1}{2}]}, 1_{[\frac{1}{2},1]}\big) = B^a\big(1_{[0,\frac{1}{2}]} \circ \sigma, 1_{[\frac{1}{2},1]} \circ \sigma\big) = B^a\big(1_{[\frac{1}{2},1]}, 1_{[0,\frac{1}{2}]}\big).$$

Hence, $B^a\big(1_{[0,\frac{1}{2}]}, 1_{[\frac{1}{2},1]}\big) = 0$.

For a fixed $y \in L_\infty$ the mapping

$$f : x \mapsto B^a(\varkappa_1(x), \varkappa_2(y)) \tag{11}$$

is a continuous linear functional. For $x \in L_\infty$ and $\sigma \in \Xi$

$$f(x \circ \sigma) = B^a(\varkappa_1(x \circ \sigma), \varkappa_2(y)).$$

By Lemma 3.1, $\varkappa_1(x \circ \sigma) = \varkappa_1(x) \circ v_1(\sigma)$ and $\varkappa_2(y) = \varkappa_2(y) \circ v_1(\sigma)$. Therefore, by (10),

$$B^a(\varkappa_1(x \circ \sigma), \varkappa_2(y)) = B^a(\varkappa_1(x) \circ v_1(\sigma), \varkappa_2(y) \circ v_1(\sigma)) = B^a(\varkappa_1(x), \varkappa_2(y)).$$

Thus, f is Ξ-symmetric. By (5),

$$B^a(\varkappa_1(x), \varkappa_2(y)) = B^a(\varkappa_1(1), \varkappa_2(y))R_1(x).$$

Analogically, for the fixed $x \in L_\infty$ the mapping

$$g : y \mapsto B^a(\varkappa_1(x), \varkappa_2(y))$$

is a continuous linear Ξ-symmetric functional. Therefore,

$$B^a(\varkappa_1(1), \varkappa_2(y)) = B^a(\varkappa_1(1), \varkappa_2(1))R_1(y).$$

Thus, for every $x, y \in L_\infty$

$$B^a(\varkappa_1(x), \varkappa_2(y)) = B^a(\varkappa_1(1), \varkappa_2(1))R_1(x)R_1(y).$$

Taking into account $\varkappa_1(1) = 1_{[0,\frac{1}{2}]}$, $\varkappa_2(1) = 1_{[\frac{1}{2},1]}$ and $B^a(1_{[0,\frac{1}{2}]}, 1_{[\frac{1}{2},1]}) = 0$, we have

$$B^a(\varkappa_1(x), \varkappa_2(y)) = 0 \tag{12}$$

for every $x, y \in L_\infty$.

Let $E \subset [0, \frac{1}{2}]$ and $F \subset [\frac{1}{2}, 1]$ be measurable sets. On substituting $x = \varkappa_1^{-1}(1_E)$ and $y = \varkappa_2^{-1}(1_F)$ into (12) we obtain

$$B^a(1_E, 1_F) = 0.$$

Let G and H be disjoint measurable subsets of $[0, 1]$ such that $\mu(G), \mu(H) \le \frac{1}{4}$. By [7, Proposition 2.2], there exists the mapping $\sigma_{G,H} \in \Xi$ such that $1_G = 1_{[0,a]} \circ \sigma_{G,H}$ and $1_H = 1_{[a,a+b]} \circ \sigma_{G,H}$, where $a = \mu(G), b = \mu(H)$. Let

$$\sigma_1(t) = \begin{cases} t - a + \frac{1}{2}, & \text{if } t \in [a, a+b) \\ t - \frac{1}{2} + a, & \text{if } t \in [\frac{1}{2}, \frac{1}{2} + b) \\ t, & \text{otherwise.} \end{cases}$$

Since $1_{[0,a]} = 1_{[0,a]} \circ \sigma_1$ and $1_{[a,a+b]} = 1_{[\frac{1}{2},\frac{1}{2}+b]} \circ \sigma_1$, it follows that $1_G = 1_{[0,a]} \circ \sigma_1 \circ \sigma_{G,H}$ and $1_H = 1_{[\frac{1}{2},\frac{1}{2}+b]} \circ \sigma_1 \circ \sigma_{G,H}$. Therefore,

$$B^a(1_G, 1_H) = B^a(1_{[0,a]} \circ \sigma_1 \circ \sigma_{G,H}, 1_{[\frac{1}{2},\frac{1}{2}+b]} \circ \sigma_1 \circ \sigma_{G,H}) = B^a(1_{[0,a]}, 1_{[\frac{1}{2},\frac{1}{2}+b]}) = 0.$$

Thus, $B^a(1_G, 1_H) = 0$ for every disjoint measurable subsets $G, H \subset [0, 1]$, such that $\mu(G), \mu(H) \le \frac{1}{4}$.

Let G and H be arbitrary disjoint measurable subsets of $[0, 1]$. Note that

$$1_G = \sum_{j=1}^4 1_{G_j} \quad \text{and} \quad 1_H = \sum_{j=1}^4 1_{H_j},$$

where $G_j = G \cap \left[\frac{j-1}{4}, \frac{j}{4}\right]$ and $H_j = H \cap \left[\frac{j-1}{4}, \frac{j}{4}\right]$ for $j \in \{1, \ldots, 4\}$. Therefore,

$$B^a(1_G, 1_H) = \sum_{j=1}^4 \sum_{k=1}^4 B^a(1_{G_j}, 1_{H_k}).$$

Since G and H are disjoint, it follows that G_j and H_k are disjoint for every $j, k \in \{1, \ldots, 4\}$. Also note that $\mu(G_j), \mu(H_j) \le \frac{1}{4}$ for every $j \in \{1, \ldots, 4\}$. Therefore, $B^a(G_j, H_k) = 0$ for every $j, k \in \{1, \ldots, 4\}$. Consequently, $B^a(1_G, 1_H) = 0$.

Let G and H be arbitrary measurable subsets of $[0, 1]$ (not necessarily disjoint). Then

$$1_G = 1_{G\cap H} + 1_{G\setminus H} \quad \text{and} \quad 1_H = 1_{G\cap H} + 1_{H\setminus G}.$$

Therefore,

$$B^a(1_G, 1_H) = B^a(1_{G\cap H}, 1_{G\cap H}) + B^a(1_{G\cap H}, 1_{H\setminus G}) + B^a(1_{G\setminus H}, 1_{G\cap H}) + B^a(1_{G\setminus H}, 1_{H\setminus G}).$$

Since B^a is antisymmetric, $B^a(1_{G\cap H}, 1_{G\cap H}) = -B^a(1_{G\cap H}, 1_{G\cap H}) = 0$. Since $G \cap H$ and $H \setminus G$ are disjoint, $B^a(1_{G\cap H}, 1_{H\setminus G}) = 0$. Analogically, $B^a(1_{G\setminus H}, 1_{G\cap H}) = 0$ and $B^a(1_{G\setminus H}, 1_{H\setminus G}) = 0$. Thus, $B^a(1_G, 1_H) = 0$ for every measurable sets $G, H \subset [0, 1]$.

Note that the set of all simple measurable functions is dense in L_∞. Let x and y be simple measurable functions. Then there exist complex numbers $c_1, \ldots, c_M, d_1, \ldots, d_N$ and measurable sets $C_1, \ldots, C_M, D_1, \ldots, D_N \subset [0,1]$, where $M, N \in \mathbb{N}$, such that

$$x = \sum_{j=1}^{M} c_j 1_{C_j} \quad \text{and} \quad y = \sum_{j=1}^{N} d_j 1_{D_j}.$$

Therefore,

$$B^a(x,y) = \sum_{j=1}^{M} \sum_{k=1}^{N} c_j d_k B^a(1_{C_j}, 1_{D_k}) = 0.$$

Thus, $B^a(x,y) = 0$ for every simple measurable functions x and y. By the continuity of B^a we have that $B^a(x,y) = 0$ for every $x, y \in L_\infty$. Thus, the form B is symmetric. □

Let us find the coefficients in (4) for a continuous 3-homogeneous Ξ-symmetric polynomial on L_∞.

Proposition 3.3. *Let Q be a continuous 3-homogeneous Ξ-symmetric polynomial on L_∞. Then*

$$Q = \alpha R_1^3 + \beta R_1 R_2 + \gamma R_3, \tag{13}$$

where

$$\alpha = 64 A_Q(1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{4},\frac{1}{2}]}, 1_{[\frac{1}{2},\frac{3}{4}]}),$$
$$\beta = 48 A_Q(1_{[0,\frac{1}{4}]}, 1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{4},\frac{1}{2}]}) - 48 A_Q(1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{4},\frac{1}{2}]}, 1_{[\frac{1}{2},\frac{3}{4}]}),$$
$$\gamma = 4Q(1_{[0,\frac{1}{4}]}) - 12 A_Q(1_{[0,\frac{1}{4}]}, 1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{4},\frac{1}{2}]}) + 8 A_Q(1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{4},\frac{1}{2}]}, 1_{[\frac{1}{2},\frac{3}{4}]})$$

and A_Q is the 3-linear symmetric form associated with Q.

Proof. Note that the existence of decomposition (13) and the uniqueness of coefficients α, β and γ are proved in [7, Theorem 4.3].

By (13),

$$\begin{cases} Q(1) = \alpha R_1^3(1) + \beta R_1(1) R_2(1) + \gamma R_3(1) \\ Q(1_{[0,\frac{1}{2}]}) = \alpha R_1^3(1_{[0,\frac{1}{2}]}) + \beta R_1(1_{[0,\frac{1}{2}]}) R_2(1_{[0,\frac{1}{2}]}) + \gamma R_3(1_{[0,\frac{1}{2}]}) \\ Q(1_{[0,\frac{1}{4}]}) = \alpha R_1^3(1_{[0,\frac{1}{4}]}) + \beta R_1(1_{[0,\frac{1}{4}]}) R_2(1_{[0,\frac{1}{4}]}) + \gamma R_3(1_{[0,\frac{1}{4}]}). \end{cases} \tag{14}$$

Note that

$$R_j(1) = 1, \quad R_j(1_{[0,\frac{1}{2}]}) = \frac{1}{2}, \quad R_j(1_{[0,\frac{1}{4}]}) = \frac{1}{4} \quad (j = 1, 2, 3).$$

Therefore, the equations (14) can be rewritten as

$$M(\alpha, \beta, \gamma)^T = (Q(1), Q(1_{[0,\frac{1}{2}]}), Q(1_{[0,\frac{1}{4}]}))^T, \tag{15}$$

where

$$M = \begin{pmatrix} 1 & 1 & 1 \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{64} & \frac{1}{16} & \frac{1}{4} \end{pmatrix}.$$

Let $d_1, d_2, d_3, d_4 \in [0,1]$ be such that $d_1 < d_2 \le d_3 < d_4$ and $d_2 - d_1 = d_4 - d_3$. Let

$$\sigma_{[d_1,d_2],[d_3,d_4]}(t) = \begin{cases} t, & \text{if } t \in [0, d_1) \cup (d_2, d_3) \cup (d_4, 1] \\ t - d_1 + d_3, & \text{if } t \in [d_1, d_2] \\ t - d_3 + d_1, & \text{if } t \in [d_3, d_4]. \end{cases}$$

Then $\sigma_{[d_1,d_2],[d_3,d_4]} \in \Xi$. Also note that

$$1_{[d_1,d_2]} \circ \sigma_{[d_1,d_2],[d_3,d_4]} = 1_{[d_3,d_4]} \quad \text{and} \quad 1_{[d_3,d_4]} \circ \sigma_{[d_1,d_2],[d_3,d_4]} = 1_{[d_1,d_2]}.$$

Let $[a,b] \subset [0,1]$ and $c = \frac{a+b}{2}$. Then, by the Binomial Formula,

$$\begin{aligned} Q(1_{[a,b]}) &= A_Q(1_{[a,b]}, 1_{[a,b]}, 1_{[a,b]}) = A_Q(1_{[a,c]} + 1_{[c,b]}, 1_{[a,c]} + 1_{[c,b]}, 1_{[a,c]} + 1_{[c,b]}) \\ &= Q(1_{[a,c]}) + 3A_Q(1_{[a,c]}, 1_{[a,c]}, 1_{[c,b]}) + 3A_Q(1_{[a,c]}, 1_{[c,b]}, 1_{[c,b]}) + Q(1_{[c,b]}). \end{aligned}$$

By (2) and (3), where we set $\sigma = \sigma_{[a,c],[c,b]}$,

$$Q(1_{[c,b]}) = Q(1_{[a,c]})$$

and

$$A_Q(1_{[a,c]}, 1_{[c,b]}, 1_{[c,b]}) = A_Q(1_{[c,b]}, 1_{[a,c]}, 1_{[a,c]}).$$

By the symmetry of A_Q,

$$A_Q(1_{[c,b]}, 1_{[a,c]}, 1_{[a,c]}) = A_Q(1_{[a,c]}, 1_{[a,c]}, 1_{[c,b]}).$$

Thus,

$$A_Q(1_{[a,c]}, 1_{[c,b]}, 1_{[c,b]}) = A_Q(1_{[a,c]}, 1_{[a,c]}, 1_{[c,b]}).$$

Therefore,

$$Q(1_{[a,b]}) = 2Q(1_{[a,c]}) + 6A_Q(1_{[a,c]}, 1_{[a,c]}, 1_{[c,b]}). \tag{16}$$

By (16),

$$Q(1) = 2Q(1_{[0,\frac{1}{2}]}) + 6A_Q(1_{[0,\frac{1}{2}]}, 1_{[0,\frac{1}{2}]}, 1_{[\frac{1}{2},1]}) \tag{17}$$

and

$$Q(1_{[0,\frac{1}{2}]}) = 2Q(1_{[0,\frac{1}{4}]}) + 6A_Q(1_{[0,\frac{1}{4}]}, 1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{4},\frac{1}{2}]}). \tag{18}$$

By the 3-linearity of A_Q and by the symmetry of A_Q, taking into account $1_{[0,\frac{1}{2}]} = 1_{[0,\frac{1}{4}]} + 1_{[\frac{1}{4},\frac{1}{2}]}$ and $1_{[\frac{1}{2},1]} = 1_{[\frac{1}{2},\frac{3}{4}]} + 1_{[\frac{3}{4},1]}$,

$$\begin{aligned} A_Q(1_{[0,\frac{1}{2}]}, 1_{[0,\frac{1}{2}]}, 1_{[\frac{1}{2},1]}) &= A_Q(1_{[0,\frac{1}{4}]}, 1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{2},\frac{3}{4}]}) + A_Q(1_{[0,\frac{1}{4}]}, 1_{[0,\frac{1}{4}]}, 1_{[\frac{3}{4},1]}) \\ &+ 2A_Q(1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{4},\frac{1}{2}]}, 1_{[\frac{1}{2},\frac{3}{4}]}) + 2A_Q(1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{4},\frac{1}{2}]}, 1_{[\frac{3}{4},1]}) \\ &+ A_Q(1_{[\frac{1}{4},\frac{1}{2}]}, 1_{[\frac{1}{4},\frac{1}{2}]}, 1_{[\frac{1}{2},\frac{3}{4}]}) + A_Q(1_{[\frac{1}{4},\frac{1}{2}]}, 1_{[\frac{1}{4},\frac{1}{2}]}, 1_{[\frac{3}{4},1]}). \end{aligned}$$

By (3),

$$\begin{aligned} A_Q(1_{[0,\frac{1}{4}]}, 1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{2},\frac{3}{4}]}) &= A_Q(1_{[0,\frac{1}{4}]}, 1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{4},\frac{1}{2}]}), \\ A_Q(1_{[0,\frac{1}{4}]}, 1_{[0,\frac{1}{4}]}, 1_{[\frac{3}{4},1]}) &= A_Q(1_{[0,\frac{1}{4}]}, 1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{4},\frac{1}{2}]}), \\ A_Q(1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{4},\frac{1}{2}]}, 1_{[\frac{3}{4},1]}) &= A_Q(1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{4},\frac{1}{2}]}, 1_{[\frac{1}{2},\frac{3}{4}]}), \\ A_Q(1_{[\frac{1}{4},\frac{1}{2}]}, 1_{[\frac{1}{4},\frac{1}{2}]}, 1_{[\frac{1}{2},\frac{3}{4}]}) &= A_Q(1_{[0,\frac{1}{4}]}, 1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{4},\frac{1}{2}]}), \\ A_Q(1_{[\frac{1}{4},\frac{1}{2}]}, 1_{[\frac{1}{4},\frac{1}{2}]}, 1_{[\frac{3}{4},1]}) &= A_Q(1_{[0,\frac{1}{4}]}, 1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{4},\frac{1}{2}]}), \end{aligned}$$

where we set

$$\sigma = \sigma_{[\frac{1}{4},\frac{1}{2}],[\frac{1}{2},\frac{3}{4}]},$$
$$\sigma = \sigma_{[\frac{1}{4},\frac{1}{2}],[\frac{3}{4},1]},$$
$$\sigma = \sigma_{[\frac{1}{2},\frac{3}{4}],[\frac{3}{4},1]},$$
$$\sigma = \sigma_{[0,\frac{1}{4}],[\frac{1}{4},\frac{1}{2}]} \circ \sigma_{[\frac{1}{4},\frac{1}{2}],[\frac{1}{2},\frac{3}{4}]}$$

and

$$\sigma = \sigma_{[0,\frac{1}{4}],[\frac{1}{4},\frac{1}{2}]} \circ \sigma_{[\frac{1}{4},\frac{1}{2}],[\frac{3}{4},1]}$$

respectively. Therefore,

$$A_Q(1_{[0,\frac{1}{2}]}, 1_{[0,\frac{1}{2}]}, 1_{[\frac{1}{2},1]}) = 4A_Q(1_{[0,\frac{1}{4}]}, 1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{4},\frac{1}{2}]}) + 4A_Q(1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{4},\frac{1}{2}]}, 1_{[\frac{1}{2},\frac{3}{4}]}). \tag{19}$$

By (17), (18) and (19),

$$Q(1) = 4Q(1_{[0,\frac{1}{4}]}) + 36A_Q(1_{[0,\frac{1}{4}]}, 1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{4},\frac{1}{2}]}) + 24A_Q(1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{4},\frac{1}{2}]}, 1_{[\frac{1}{2},\frac{3}{4}]}). \tag{20}$$

Thus,

$$(21)\quad \big(Q(1), Q(1_{[0,\frac{1}{2}]}), Q(1_{[0,\frac{1}{4}]})\big)^T = K\big(Q(1_{[0,\frac{1}{4}]}), A_Q(1_{[0,\frac{1}{4}]}, 1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{4},\frac{1}{2}]}), A_Q(1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{4},\frac{1}{2}]}, 1_{[\frac{1}{2},\frac{3}{4}]})\big)^T,$$

where

$$K = \begin{pmatrix} 4 & 36 & 24 \\ 2 & 6 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

By (15) and (21),

$$\big(\alpha, \beta, \gamma\big)^T = M^{-1}K\big(Q(1_{[0,\frac{1}{4}]}), A_Q(1_{[0,\frac{1}{4}]}, 1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{4},\frac{1}{2}]}), A_Q(1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{4},\frac{1}{2}]}, 1_{[\frac{1}{2},\frac{3}{4}]})\big)^T.$$

We obtain

$$\alpha = 64A_Q(1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{4},\frac{1}{2}]}, 1_{[\frac{1}{2},\frac{3}{4}]}),$$

$$\beta = 48A_Q(1_{[0,\frac{1}{4}]}, 1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{4},\frac{1}{2}]}) - 48A_Q(1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{4},\frac{1}{2}]}, 1_{[\frac{1}{2},\frac{3}{4}]}),$$

$$\gamma = 4Q(1_{[0,\frac{1}{4}]}) - 12A_Q(1_{[0,\frac{1}{4}]}, 1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{4},\frac{1}{2}]}) + 8A_Q(1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{4},\frac{1}{2}]}, 1_{[\frac{1}{2},\frac{3}{4}]}).$$

□

4. Symmetric 3-homogeneous polynomials on $(L_\infty)^2$

In this section we shall construct a Hamel basis of the space of continuous Ξ-symmetric 3-homogeneous polynomials on $(L_\infty)^2$.

Let $P : (L_\infty)^2 \to \mathbb{C}$ be a continuous Ξ-symmetric 3-homogeneous polynomial and let A_P be the symmetric 3-linear form associated with P. By the Polarization Formula and by the Ξ-symmetry of P,

$$(22)\quad A_P((x_1\circ\sigma, y_1\circ\sigma), (x_2\circ\sigma, y_2\circ\sigma), (x_3\circ\sigma, y_3\circ\sigma)) = A_P((x_1, y_1), (x_2, y_2), (x_3, y_3))$$

for every $x_1, x_2, x_3, y_1, y_2, y_3 \in L_\infty$ and $\sigma \in \Xi$.

Let us define mappings $p_1, p_2 : L_\infty \to (L_\infty)^2$ by

$$p_1(x) = (x, 0), \quad p_2(x) = (0, x).$$

For a given $J = (j_1, j_2, j_3)$, $j_1, j_2, j_3 \in \{1, 2\}$, we define a mapping $A_J : (L_\infty)^3 \to \mathbb{C}$ by

$$A_J(x_1, x_2, x_3) = A_P(p_{j_1}(x_1), p_{j_2}(x_2), p_{j_3}(x_3)).$$

Note that both p_1 and p_2 are continuous linear operators and so A_J is a continuous 3-linear form. By (22),

$$(23)\quad A_J(x_1\circ\sigma, x_2\circ\sigma, x_3\circ\sigma) = A_J(x_1, x_2, x_3)$$

for every $x_1, x_2, x_3 \in L_\infty$ and $\sigma \in \Xi$.

Let us denote

$$a_J^{(111)} = A_J(1_{[0,\frac{1}{4}]}, 1_{[0,\frac{1}{4}]}, 1_{[0,\frac{1}{4}]}),$$

$$a_J^{(112)} = A_J(1_{[0,\frac{1}{4}]}, 1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{4},\frac{1}{2}]}),$$

$$a_J^{(121)} = A_J(1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{4},\frac{1}{2}]}, 1_{[0,\frac{1}{4}]}),$$

$$a_J^{(211)} = A_J(1_{[\frac{1}{4},\frac{1}{2}]}, 1_{[0,\frac{1}{4}]}, 1_{[0,\frac{1}{4}]}),$$

$$a_J^{(123)} = A_J(1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{4},\frac{1}{2}]}, 1_{[\frac{1}{2},\frac{3}{4}]}).$$

Lemma 4.1. *Let* $C, D, E \subset [0,1]$ *be disjoint measurable sets such that* $\mu(C), \mu(D), \mu(E) \le \frac{1}{4}$. *Then*

$$A_J(1_C, 1_C, 1_E) = 8\lambda_J \mu(C)^2 \mu(E) + 4\nu_J^{(3)} \mu(C)\mu(E), \tag{24}$$

$$A_J(1_C, 1_E, 1_C) = 8\lambda_J \mu(C)^2 \mu(E) + 4\nu_J^{(2)} \mu(C)\mu(E), \tag{25}$$

$$A_J(1_E, 1_C, 1_C) = 8\lambda_J \mu(C)^2 \mu(E) + 4\nu_J^{(1)} \mu(C)\mu(E), \tag{26}$$

$$A_J(1_C, 1_D, 1_E) = 8\lambda_J \mu(C)\mu(D)\mu(E), \tag{27}$$

where

$$\lambda_J = 8a_J^{(123)},$$

$$\nu_J^{(1)} = 4a_J^{(211)} - 4a_J^{(123)}, \quad \nu_J^{(2)} = 4a_J^{(121)} - 4a_J^{(123)}, \quad \nu_J^{(3)} = 4a_J^{(112)} - 4a_J^{(123)}.$$

Proof. For a fixed function $z \in L_\infty$ let us define $B_z : (L_\infty)^2 \to \mathbb{C}$,

$$B_z(x, y) = A_J(\varkappa_1(x), \varkappa_1(y), \varkappa_2(z)).$$

Note that B_z is a continuous bilinear form by the linearity and the continuity of $\varkappa_1$ and by the 3-linearity and the continuity of A_J. For $\sigma \in \Xi$ and $x \in L_\infty$, by Lemma 3.1 we have $\varkappa_1(x \circ \sigma) = \varkappa_1(x) \circ v_1(\sigma)$ and $\varkappa_2(x) = \varkappa_2(x) \circ v_1(\sigma)$. Therefore,

$$\begin{aligned} B_z(x \circ \sigma, y \circ \sigma) &= A_J(\varkappa_1(x \circ \sigma), \varkappa_1(y \circ \sigma), \varkappa_2(z)) \\ &= A_J(\varkappa_1(x) \circ v_1(\sigma), \varkappa_1(y) \circ v_1(\sigma), \varkappa_2(z) \circ v_1(\sigma)). \end{aligned}$$

By (23),

$$A_J(\varkappa_1(x) \circ v_1(\sigma), \varkappa_1(y) \circ v_1(\sigma), \varkappa_2(z) \circ v_1(\sigma)) = A_J(\varkappa_1(x), \varkappa_1(y), \varkappa_2(z)).$$

Therefore,

$$B_z(x \circ \sigma, y \circ \sigma) = B_z(x, y).$$

By Proposition 3.2, the form B_z is symmetric, i.e. $B_z(y, x) = B_z(x, y)$. By Proposition 3.1,

$$B_z(x, y) = \alpha R_1(x) R_1(y) + \beta \int_0^1 x(t) y(t)\, dt,$$

where $\alpha = B_z(1, 1) - B_z(r, r)$, $\beta = B_z(r, r)$ (we remind that $r = 1_{[0, \frac{1}{2}]} - 1_{[\frac{1}{2}, 1]}$).

Thus,

$$A_J(\varkappa_1(x), \varkappa_1(y), \varkappa_2(z)) = \alpha R_1(x) R_1(y) + \beta \int_0^1 x(t) y(t)\, dt, \tag{28}$$

where

$$\alpha = A_J(\varkappa_1(1), \varkappa_1(1), \varkappa_2(z)) - A_J(\varkappa_1(r), \varkappa_1(r), \varkappa_2(z)),$$
$$\beta = A_J(\varkappa_1(r), \varkappa_1(r), \varkappa_2(z)).$$

For fixed x and y the mapping $f : z \mapsto A_J(\varkappa_1(x), \varkappa_1(y), \varkappa_2(z))$ is a continuous linear Ξ-symmetric functional, therefore, by (5)

$$A_J(\varkappa_1(x), \varkappa_1(y), \varkappa_2(z)) = A_J(\varkappa_1(x), \varkappa_1(y), \varkappa_2(1)) R_1(z).$$

Thus,

$$\alpha = A_J(\varkappa_1(1), \varkappa_1(1), \varkappa_2(1)) R_1(z) - A_J(\varkappa_1(r), \varkappa_1(r), \varkappa_2(1)) R_1(z),$$
$$\beta = A_J(\varkappa_1(r), \varkappa_1(r), \varkappa_2(1)) R_1(z).$$

Let us denote

$$\lambda_J = A_J(\varkappa_1(1), \varkappa_1(1), \varkappa_2(1)) - A_J(\varkappa_1(r), \varkappa_1(r), \varkappa_2(1)), \tag{29}$$

$$\nu_J^{(3)} = A_J(\varkappa_1(r), \varkappa_1(r), \varkappa_2(1)). \tag{30}$$

Now, by (28),

$$A_J(\varkappa_1(x), \varkappa_1(y), \varkappa_2(z)) = \lambda_J R_1(x)R_1(y)R_1(z) + \nu_J^{(3)} \int_0^1 x(t)y(t)\,dt\, R_1(z). \tag{31}$$

Note that $\varkappa_1(1) = 1_{[0,\frac{1}{2}]}, \varkappa_1(r) = 1_{[0,\frac{1}{4}]} - 1_{[\frac{1}{4},\frac{1}{2}]}$ and $\varkappa_2(1) = 1_{[\frac{1}{2},1]}$. Substituting this in (29) and (30), we obtain

$$\lambda_J = 8A_J(1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{4},\frac{1}{2}]}, 1_{[\frac{1}{2},\frac{3}{4}]}) = 8a_J^{(123)},$$

$$\nu_J^{(3)} = 4A_J(1_{[0,\frac{1}{4}]}, 1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{2},\frac{3}{4}]}) - 4A_J(1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{4},\frac{1}{2}]}, 1_{[\frac{1}{2},\frac{3}{4}]}) = 4a_J^{(112)} - 4a_J^{(123)}.$$

Let us prove the equality (27). Let $c = \mu(C), d = \mu(D)$ and $e = \mu(E)$. We substitute $x = 1_{[0,2c]}, y = 1_{[\frac{1}{2},\frac{1}{2}+2d]}$ and $z = 1_{[0,2e]}$ into (31). In this case

$$\varkappa_1(x) = 1_{[0,c]}, \quad \varkappa_1(y) = 1_{[\frac{1}{4},\frac{1}{4}+d]}, \quad \varkappa_2(z) = 1_{[\frac{1}{2},\frac{1}{2}+e]},$$
$$R_1(x) = 2c, \quad R_1(y) = 2d, \quad R_1(z) = 2e.$$

Note that $\mu\big([0,2c] \cap [\frac{1}{2}, \frac{1}{2} + 2d]\big) = 0$, since $c \le \frac{1}{4}$. So,

$$\int_0^1 x(t)y(t)\,dt = 0.$$

Thus, by (31),

$$A_J(1_{[0,c]}, 1_{[\frac{1}{4},\frac{1}{4}+d]}, 1_{[\frac{1}{2},\frac{1}{2}+e]}) = 8\lambda_J cde. \tag{32}$$

By [7, Proposition 2.2], there exists a mapping $\sigma_{C,D,E} \in \Xi$ such that

$$1_C = 1_{[0,c]} \circ \sigma_{C,D,E}, \quad 1_D = 1_{[c,c+d]} \circ \sigma_{C,D,E}, \quad 1_E = 1_{[c+d,c+d+e]} \circ \sigma_{C,D,E}.$$

We define the mapping $\widetilde{\sigma} : [0,1] \to [0,1]$ by

$$\widetilde{\sigma}(t) = \begin{cases} t, & \text{if } t \in [0,c) \\ t+d+e, & \text{if } t \in [c,\frac{1}{4}) \\ t - \frac{1}{4} + c, & \text{if } t \in [\frac{1}{4}, \frac{1}{4}+d) \\ t+e, & \text{if } t \in [\frac{1}{4}+d, \frac{1}{2}) \\ t - \frac{1}{2} + c + d, & \text{if } t \in [\frac{1}{2}, \frac{1}{2}+e) \\ t, & \text{if } t \in [\frac{1}{2}+e, 1]. \end{cases}$$

It can be checked that $\widetilde{\sigma} \in \Xi$ and

$$1_{[0,c]} = 1_{[0,c]} \circ \widetilde{\sigma}, \quad 1_{[\frac{1}{4},\frac{1}{4}+d]} = 1_{[c,c+d]} \circ \widetilde{\sigma}, \quad 1_{[\frac{1}{2},\frac{1}{2}+e]} = 1_{[c+d,c+d+e]} \circ \widetilde{\sigma}.$$

Hence,

$$1_{[0,c]} \circ \widetilde{\sigma}^{-1} = 1_{[0,c]}, \quad 1_{[\frac{1}{4},\frac{1}{4}+d]} \circ \widetilde{\sigma}^{-1} = 1_{[c,c+d]}, \quad 1_{[\frac{1}{2},\frac{1}{2}+e]} \circ \widetilde{\sigma}^{-1} = 1_{[c+d,c+d+e]}.$$

Thus,

$$1_C = 1_{[0,c]} \circ \widetilde{\sigma}^{-1} \circ \sigma_{C,D,E}, \quad 1_D = 1_{[\frac{1}{4},\frac{1}{4}+d]} \circ \widetilde{\sigma}^{-1} \circ \sigma_{C,D,E}, \quad 1_E = 1_{[\frac{1}{2},\frac{1}{2}+e]} \circ \widetilde{\sigma}^{-1} \circ \sigma_{C,D,E}. \tag{33}$$

Therefore, by (23),

$$A_J(1_C, 1_D, 1_E) = A_J(1_{[0,c]}, 1_{[\frac{1}{4},\frac{1}{4}+d]}, 1_{[\frac{1}{2},\frac{1}{2}+e]}).$$

By (32),

$$A_J(1_C, 1_D, 1_E) = 8\lambda_J cde.$$

Thus, the equality (27) is proved.

Let us prove the equality (24). We substitute $x = y = 1_{[0,2c]}$ and $z = 1_{[0,2e]}$ into (31). Note that in this case

$$\int_0^1 x(t)y(t)\,dt = 2c.$$

Therefore,

$$A_J(1_{[0,c]}, 1_{[0,c]}, 1_{[\frac{1}{2}, \frac{1}{2}+e]}) = 8\lambda_J c^2 e + 4\nu_J^{(3)} ce. \tag{34}$$

By (33), $1_C = 1_{[0,c]} \circ \widetilde{\sigma}^{-1} \circ \sigma_{C,D,E}$ and $1_E = 1_{[\frac{1}{2}, \frac{1}{2}+e]} \circ \widetilde{\sigma}^{-1} \circ \sigma_{C,D,E}$, and so by (23),

$$A_J(1_C, 1_C, 1_E) = A_J(1_{[0,c]}, 1_{[0,c]}, 1_{[\frac{1}{2}, \frac{1}{2}+e]}). \tag{35}$$

Equalities (34) and (35) imply (24). Equalities (25) and (26) can be proved analogically to (24). □

Lemma 4.2. *Let $C \subset [0,1]$ be a measurable set. Then*

$$A_J(1_C, 1_C, 1_C) = \alpha_J \mu(C)^3 + \beta_J \mu(C)^2 + \gamma_J \mu(C),$$

where

$$\alpha_J = 64 a_J^{(123)}, \tag{36}$$

$$\beta_J = 16\Big(a_J^{(112)} + a_J^{(121)} + a_J^{(211)}\Big) - 48 a_J^{(123)}, \tag{37}$$

$$\gamma_J = 4 a_J^{(111)} - 4\Big(a_J^{(112)} + a_J^{(121)} + a_J^{(211)}\Big) + 8 a_J^{(123)}. \tag{38}$$

Proof. The restriction of A_J to the diagonal, which we denote by $\widehat{A_J}$, is a continuous 3-homogeneous polynomial. By (23), the polynomial $\widehat{A_J}$ is Ξ-symmetric. Note that the form A_J^s, which is the symmetrization of A_J, is associated with the polynomial $\widehat{A_J}$. By Proposition 3.3,

$$\widehat{A_J} = \alpha_J R_1^3 + \beta_J R_1 R_2 + \gamma_J R_3,$$

where

$$\alpha_J = 64 A_J^s(1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{4},\frac{1}{2}]}, 1_{[\frac{1}{2},\frac{3}{4}]}),$$

$$\beta_J = 48 A_J^s(1_{[0,\frac{1}{4}]}, 1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{4},\frac{1}{2}]}) - 48 A_J^s(1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{4},\frac{1}{2}]}, 1_{[\frac{1}{2},\frac{3}{4}]}),$$

$$\gamma_J = 4 A_J^s(1_{[0,\frac{1}{4}]}, 1_{[0,\frac{1}{4}]}, 1_{[0,\frac{1}{4}]}) - 12 A_J^s(1_{[0,\frac{1}{4}]}, 1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{4},\frac{1}{2}]}) + 8 A_J^s(1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{4},\frac{1}{2}]}, 1_{[\frac{1}{2},\frac{3}{4}]}).$$

By the definition of the symmetrization operator,

$$A_J^s(1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{4},\frac{1}{2}]}, 1_{[\frac{1}{2},\frac{3}{4}]}) = \frac{1}{3!} \sum_{\tau \in S_3} A_J\Big(1_{[\frac{\tau(1)-1}{4}, \frac{\tau(1)}{4}]}, 1_{[\frac{\tau(2)-1}{4}, \frac{\tau(2)}{4}]}, 1_{[\frac{\tau(3)-1}{4}, \frac{\tau(3)}{4}]}\Big),$$

where S_3 is the group of all permutations on the set $\{1,2,3\}$. For every $\tau \in S_3$ let us define $\sigma_\tau \in \Xi$ such that

$$1_{[\frac{\tau(j)-1}{4}, \frac{\tau(j)}{4}]} \circ \sigma_\tau = 1_{[\frac{j-1}{4}, \frac{j}{4}]}$$

for every $j \in \{1,2,3\}$, i.e., if $t \in [\frac{j-1}{4}, \frac{j}{4}]$, where $j \in \{1,2,3\}$, we set

$$\sigma_\tau(t) = t - \frac{j-1}{4} + \frac{\tau(j)-1}{4};$$

if $t \in [\frac{3}{4}, 1]$, we set $\sigma_\tau(t) = t$. By (23), where we substitute $\sigma = \sigma_\tau$,

$$A_J\Big(1_{[\frac{\tau(1)-1}{4}, \frac{\tau(1)}{4}]}, 1_{[\frac{\tau(2)-1}{4}, \frac{\tau(2)}{4}]}, 1_{[\frac{\tau(3)-1}{4}, \frac{\tau(3)}{4}]}\Big) = A_J(1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{4},\frac{1}{2}]}, 1_{[\frac{1}{2},\frac{3}{4}]}).$$

Consequently,

$$A_J^s(1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{4},\frac{1}{2}]}, 1_{[\frac{1}{2},\frac{3}{4}]}) = A_J(1_{[0,\frac{1}{4}]}, 1_{[\frac{1}{4},\frac{1}{2}]}, 1_{[\frac{1}{2},\frac{3}{4}]}).$$

Therefore, $\alpha_J = 64 a_J^{(123)}$.

Note that

$$A_J^s(1_{[0,\frac14]},1_{[0,\frac14]},1_{[\frac14,\frac12]}) = \frac13\Big(A_J(1_{[0,\frac14]},1_{[0,\frac14]},1_{[\frac14,\frac12]}) + A_J(1_{[0,\frac14]},1_{[\frac14,\frac12]},1_{[0,\frac14]}) + A_J(1_{[\frac14,\frac12]},1_{[0,\frac14]},1_{[0,\frac14]})\Big) = \frac13\Big(a_J^{(112)}+a_J^{(121)}+a_J^{(211)}\Big).$$

Hence,

$$\beta_J = 16\Big(a_J^{(112)}+a_J^{(121)}+a_J^{(211)}\Big) - 48a_J^{(123)},$$

$$\gamma_J = 4a_J^{(111)} - 4\Big(a_J^{(112)}+a_J^{(121)}+a_J^{(211)}\Big) + 8a_J^{(123)}.$$

Since $R_1(1_C)=R_2(1_C)=R_3(1_C)=\mu(C)$, it follows that

$$A_J(1_C,1_C,1_C) = \widehat{A_J}(1_C) = \alpha_J\mu(C)^3+\beta_J\mu(C)^2+\gamma_J\mu(C).$$

□

Lemma 4.3. *For $x,y,z\in L_\infty$*

$$A_J(x,y,z) = \alpha_J\int_0^1 x(t)\,dt\int_0^1 y(t)\,dt\int_0^1 z(t)\,dt + 4\nu_J^{(1)}\int_0^1 x(t)\,dt\int_0^1 y(t)z(t)\,dt + 4\nu_J^{(2)}\int_0^1 y(t)\,dt\int_0^1 x(t)z(t)\,dt + 4\nu_J^{(3)}\int_0^1 z(t)\,dt\int_0^1 x(t)y(t)\,dt + \gamma_J\int_0^1 x(t)y(t)z(t)\,dt, \tag{39}$$

where

$$\alpha_J = 64a_J^{(123)}, \tag{40}$$

$$\nu_J^{(1)} = 4a_J^{(211)} - 4a_J^{(123)}, \tag{41}$$

$$\nu_J^{(2)} = 4a_J^{(121)} - 4a_J^{(123)}, \tag{42}$$

$$\nu_J^{(3)} = 4a_J^{(112)} - 4a_J^{(123)}, \tag{43}$$

$$\gamma_J = 4a_J^{(111)} - 4\Big(a_J^{(112)}+a_J^{(121)}+a_J^{(211)}\Big) + 8a_J^{(123)}. \tag{44}$$

Proof. Let x,y,z be simple measurable functions. There are a finite sequence of disjoint measurable sets $(E_1,\ldots,E_N)$, such that $\mu(E_j)\le\frac14$ for every $j=1,\ldots,N$, and finite sequences of complex numbers $(x_1,\ldots,x_N)$, $(y_1,\ldots,y_N)$ and $(z_1,\ldots,z_N)$ such that $x=\sum_{k=1}^N x_k1_{E_k}$, $y=\sum_{k=1}^N y_k1_{E_k}$ and $z=\sum_{k=1}^N z_k1_{E_k}$. By the 3-linearity of A_J,

$$A_J(x,y,z) = \sum_{k_1=1}^N\sum_{k_2=1}^N\sum_{k_3=1}^N x_{k_1}y_{k_2}z_{k_3}A_J(1_{E_{k_1}},1_{E_{k_2}},1_{E_{k_3}}).$$

Let us represent this expression as

$$A_J(x,y,z) = \sum_{k=1}^N x_k y_k z_k A_J(1_{E_k}, 1_{E_k}, 1_{E_k}) + \sum_{k=1}^N \sum_{k_3 \neq k} x_k y_k z_{k_3} A_J(1_{E_k}, 1_{E_k}, 1_{E_{k_3}})$$
$$+ \sum_{k=1}^N \sum_{k_2 \neq k} x_k y_{k_2} z_k A_J(1_{E_k}, 1_{E_{k_2}}, 1_{E_k}) + \sum_{k=1}^N \sum_{k_1 \neq k} x_{k_1} y_k z_k A_J(1_{E_{k_1}}, 1_{E_k}, 1_{E_k})$$
$$+ \sum_{k_1 \neq k_2 \neq k_3} x_{k_1} y_{k_2} z_{k_3} A_J(1_{E_{k_1}}, 1_{E_{k_2}}, 1_{E_{k_3}}).$$

By Lemma 4.2,

$$A_J(1_{E_k}, 1_{E_k}, 1_{E_k}) = \alpha_J \mu(E_k)^3 + \beta_J \mu(E_k)^2 + \gamma_J \mu(E_k),$$

where

$$\alpha_J = 64 a_J^{(123)},$$
$$\beta_J = 16\Big(a_J^{(112)} + a_J^{(121)} + a_J^{(211)}\Big) - 48 a_J^{(123)},$$
$$\gamma_J = 4a_J^{(111)} - 4\Big(a_J^{(112)} + a_J^{(121)} + a_J^{(211)}\Big) + 8a_J^{(123)}.$$

By Lemma 4.1,

$$A_J(1_{E_k}, 1_{E_k}, 1_{E_{k_3}}) = 8\lambda_J \mu(E_k)^2 \mu(E_{k_3}) + 4\nu_J^{(3)} \mu(E_k)\mu(E_{k_3}),$$
$$A_J(1_{E_k}, 1_{E_{k_2}}, 1_{E_k}) = 8\lambda_J \mu(E_k)^2 \mu(E_{k_2}) + 4\nu_J^{(2)} \mu(E_k)\mu(E_{k_2}),$$
$$A_J(1_{E_{k_1}}, 1_{E_k}, 1_{E_k}) = 8\lambda_J \mu(E_k)^2 \mu(E_{k_1}) + 4\nu_J^{(1)} \mu(E_k)\mu(E_{k_1}),$$
$$A_J(1_{E_{k_1}}, 1_{E_{k_2}}, 1_{E_{k_3}}) = 8\lambda_J \mu(E_{k_1})\mu(E_{k_2})\mu(E_{k_3}),$$

where

$$\lambda_J = 8a_J^{(123)},$$
$$\nu_J^{(1)} = 4a_J^{(211)} - 4a_J^{(123)}, \quad \nu_J^{(2)} = 4a_J^{(121)} - 4a_J^{(123)}, \quad \nu_J^{(3)} = 4a_J^{(112)} - 4a_J^{(123)}.$$

Note that $\alpha_J = 8\lambda_J$ and $\beta_J = 4\nu_J^{(1)} + 4\nu_J^{(2)} + 4\nu_J^{(3)}$. Thus,

$$A_J(x,y,z) = \sum_{k=1}^N x_k y_k z_k \Big(\alpha_J \mu(E_k)^3 + \big(4\nu_J^{(1)} + 4\nu_J^{(2)} + 4\nu_J^{(3)}\big)\mu(E_k)^2 + \gamma_J \mu(E_k)\Big)$$
$$+ \sum_{k=1}^N \sum_{k_3 \neq k} x_k y_k z_{k_3} \Big(\alpha_J \mu(E_k)^2 \mu(E_{k_3}) + 4\nu_J^{(3)} \mu(E_k)\mu(E_{k_3})\Big)$$
$$+ \sum_{k=1}^N \sum_{k_2 \neq k} x_k y_{k_2} z_k \Big(\alpha_J \mu(E_k)^2 \mu(E_{k_2}) + 4\nu_J^{(2)} \mu(E_k)\mu(E_{k_2})\Big)$$
$$+ \sum_{k=1}^N \sum_{k_1 \neq k} x_{k_1} y_k z_k \Big(\alpha_J \mu(E_k)^2 \mu(E_{k_1}) + 4\nu_J^{(1)} \mu(E_k)\mu(E_{k_1})\Big)$$
$$+ \sum_{k_1 \neq k_2 \neq k_3} x_{k_1} y_{k_2} z_{k_3} \alpha_J \mu(E_{k_1})\mu(E_{k_2})\mu(E_{k_3}).$$

After reordering of terms we have

$$
\begin{aligned}
A_J(x,y,z) = \alpha_J\Bigg(&\sum_{k=1}^{N} x_k\mu(E_k)\cdot y_k\mu(E_k)\cdot z_k\mu(E_k) \\
&+ \sum_{k=1}^{N}\sum_{k_3\neq k} x_k\mu(E_k)\cdot y_k\mu(E_k)\cdot z_{k_3}\mu(E_{k_3}) + \sum_{k=1}^{N}\sum_{k_2\neq k} x_k\mu(E_k)\cdot y_{k_2}\mu(E_{k_2})\cdot z_k\mu(E_k) \\
&+ \sum_{k=1}^{N}\sum_{k_1\neq k} x_{k_1}\mu(E_{k_1})\cdot y_k\mu(E_k)\cdot z_k\mu(E_k) + \sum_{k_1\neq k_2\neq k_3} x_{k_1}\mu(E_{k_1})\cdot y_{k_2}\mu(E_{k_2})\cdot z_{k_3}\mu(E_{k_3})\Bigg) \\
&+ 4\nu_J^{(1)}\Bigg(\sum_{k=1}^{N} x_k\mu(E_k)\cdot y_kz_k\mu(E_k) + \sum_{k=1}^{N}\sum_{k_1\neq k} x_{k_1}\mu(E_{k_1})\cdot y_kz_k\mu(E_k)\Bigg) \\
&+ 4\nu_J^{(2)}\Bigg(\sum_{k=1}^{N} y_k\mu(E_k)\cdot x_kz_k\mu(E_k) + \sum_{k=1}^{N}\sum_{k_2\neq k} y_{k_2}\mu(E_{k_2})\cdot x_kz_k\mu(E_k)\Bigg) \\
&+ 4\nu_J^{(3)}\Bigg(\sum_{k=1}^{N} z_k\mu(E_k)\cdot x_ky_k\mu(E_k) + \sum_{k=1}^{N}\sum_{k_3\neq k} z_{k_3}\mu(E_{k_3})\cdot x_ky_k\mu(E_k)\Bigg) \\
&+ \gamma_J\sum_{k=1}^{N} x_ky_kz_k\mu(E_k).
\end{aligned}
$$

Note that the expression in the first brackets is equal to

$$\Bigg(\sum_{k=1}^{N} x_k\mu(E_k)\Bigg)\Bigg(\sum_{k=1}^{N} y_k\mu(E_k)\Bigg)\Bigg(\sum_{k=1}^{N} z_k\mu(E_k)\Bigg),$$

which in turn is equal to

$$\int_0^1 x(t)\,dt\int_0^1 y(t)\,dt\int_0^1 z(t)\,dt.$$

Also note that

$$
\begin{aligned}
&\sum_{k=1}^{N} x_k\mu(E_k)\cdot y_kz_k\mu(E_k) + \sum_{k=1}^{N}\sum_{k_1\neq k} x_{k_1}\mu(E_{k_1})\cdot y_kz_k\mu(E_k) \\
&\qquad = \Bigg(\sum_{k=1}^{N} x_k\mu(E_k)\Bigg)\Bigg(\sum_{k=1}^{N} y_kz_k\mu(E_k)\Bigg) = \int_0^1 x(t)\,dt\int_0^1 y(t)z(t)\,dt.
\end{aligned}
$$

Analogically,

$$\sum_{k=1}^{N} y_k\mu(E_k)\cdot x_kz_k\mu(E_k) + \sum_{k=1}^{N}\sum_{k_2\neq k} y_{k_2}\mu(E_{k_2})\cdot x_kz_k\mu(E_k) = \int_0^1 y(t)\,dt\int_0^1 x(t)z(t)\,dt$$

and

$$\sum_{k=1}^{N} z_k\mu(E_k)\cdot x_ky_k\mu(E_k) + \sum_{k=1}^{N}\sum_{k_3\neq k} z_{k_3}\mu(E_{k_3})\cdot x_ky_k\mu(E_k) = \int_0^1 z(t)\,dt\int_0^1 x(t)y(t)\,dt.$$

Also note that

$$\sum_{k=1}^{N} x_ky_kz_k\mu(E_k) = \int_0^1 x(t)y(t)z(t)\,dt.$$

So, we have proved that the equality (39) holds for every simple functions x, y and z. Since the set of such functions is dense in L_∞ and the form A_J is continuous, the equality (39) holds for every $x, y, z \in L_\infty$. □

Let us define mappings $R_{mn} : (L_\infty)^2 \to \mathbb{C}$ by

$$R_{mn}((x,y)) = \int_0^1 x^m(t) y^n(t)\, dt, \tag{45}$$

where $m, n \in \mathbb{N} \cup \{0\}$. Note that every mapping R_{mn} is a continuous $(m+n)$-homogeneous polynomial as the restriction to the diagonal of the continuous $(m+n)$-linear form

$$((x_1, y_1), \ldots, (x_{m+n}, y_{m+n})) \mapsto \int_0^1 x_1(t) \ldots x_m(t) y_{m+1}(t) \ldots y_{m+n}(t)\, dt.$$

It is clear that every polynomial R_{mn} is Ξ-symmetric. Let us call R_{mn} by *elementary Ξ-symmetric polynomials* on $(L_\infty)^2$. Following theorem shows that every continuous 3-homogeneous Ξ-symmetric polynomial on $(L_\infty)^2$ can be uniquely represented as an algebraic combination of elementary Ξ-symmetric polynomials on $(L_\infty)^2$.

Theorem 4.1. *Let P be a continuous 3-homogeneous Ξ-symmetric polynomial on $(L_\infty)^2$. Then*

$$\begin{aligned} P = {} & \alpha_{J_0} R_{10}^3 + 4\big(\nu_{J_0}^{(1)} + \nu_{J_0}^{(2)} + \nu_{J_0}^{(3)}\big) R_{10} R_{20} + \gamma_{J_0} R_{30} \\ & + 3\alpha_{J_1} R_{10}^2 R_{01} + 12\big(\nu_{J_1}^{(1)} + \nu_{J_1}^{(2)}\big) R_{10} R_{11} + 12\nu_{J_1}^{(3)} R_{20} R_{01} + 3\gamma_{J_1} R_{21} \\ & + 3\alpha_{J_2} R_{10} R_{01}^2 + 12\nu_{J_2}^{(1)} R_{10} R_{02} + 12\big(\nu_{J_2}^{(2)} + \nu_{J_2}^{(3)}\big) R_{11} R_{01} + 3\gamma_{J_2} R_{12} \\ & + \alpha_{J_3} R_{01}^3 + 4\big(\nu_{J_3}^{(1)} + \nu_{J_3}^{(2)} + \nu_{J_3}^{(3)}\big) R_{01} R_{02} + \gamma_{J_3} R_{03}, \end{aligned} \tag{46}$$

where $J_0 = (1,1,1), J_1 = (1,1,2), J_2 = (1,2,2), J_3 = (2,2,2)$, and coefficients α_J, $\nu_J^{(1)}$, $\nu_J^{(2)}$, $\nu_J^{(3)}$ and γ_J are defined by formulas (40)–(44).

Proof. For $(x,y) \in (L_\infty)^2$, by the Binomial Formula,

$$\begin{aligned} P((x,y)) = P((x,0)+(0,y)) = {} & A_P((x,0),(x,0),(x,0)) + 3A_P((x,0),(x,0),(0,y)) \\ & + 3A_P((x,0),(0,y),(0,y)) + A_P((0,y),(0,y),(0,y)) \\ = {} & A_{J_0}(x,x,x) + 3A_{J_1}(x,x,y) + 3A_{J_2}(x,y,y) + A_{J_3}(y,y,y). \end{aligned} \tag{47}$$

By Lemma 4.3,

$$A_{J_0}(x,x,x) = \alpha_{J_0} R_1^3(x) + 4\big(\nu_{J_0}^{(1)} + \nu_{J_0}^{(2)} + \nu_{J_0}^{(3)}\big) R_1(x) R_2(x) + \gamma_{J_0} R_3(x), \tag{48}$$

$$\begin{aligned} A_{J_1}(x,x,y) = {} & \alpha_{J_1} R_1^2(x) R_1(y) + 4\big(\nu_{J_1}^{(1)} + \nu_{J_1}^{(2)}\big) R_1(x) \int_0^1 x(t) y(t)\, dt \\ & + 4\nu_{J_1}^{(3)} R_2(x) R_1(y) + \gamma_{J_1} \int_0^1 x^2(t) y(t)\, dt, \end{aligned} \tag{49}$$

$$\begin{aligned} A_{J_2}(x,y,y) = {} & \alpha_{J_2} R_1(x) R_1^2(y) + 4\nu_{J_2}^{(1)} R_1(x) R_2(y) \\ & + 4\big(\nu_{J_2}^{(2)} + \nu_{J_2}^{(3)}\big) R_1(y) \int_0^1 x(t) y(t)\, dt + \gamma_{J_2} \int_0^1 x(t) y^2(t)\, dt, \end{aligned} \tag{50}$$

$$A_{J_3}(y,y,y) = \alpha_{J_3} R_1^3(y) + 4\big(\nu_{J_3}^{(1)} + \nu_{J_3}^{(2)} + \nu_{J_3}^{(3)}\big) R_1(y) R_2(y) + \gamma_{J_3} R_3(y). \tag{51}$$

Taking into account the equality (45) and on substituting (48)–(51) into (47), we get (46). □

Corollary 4.1. *The set of polynomials*

$$\{R_{10}^3, R_{10}R_{20}, R_{30}, R_{10}^2R_{01}, R_{10}R_{11}, R_{20}R_{01}, R_{21}, R_{10}R_{01}^2, R_{10}R_{02}, R_{11}R_{01}, R_{12}, R_{01}^3, R_{01}R_{02}, R_{03}\}$$

is a Hamel basis of the space of all continuous Ξ*-symmetric 3-homogeneous polynomials on* $(L_\infty)^2$.

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