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MEASURE OF NONCOMPACTNESS, ESSENTIAL APPROXIMATION AND DEFECT PSEUDOSPECTRUM

AYMEN AMMAR, AREF JERIBI, AND KAMEL MAHFOUDHI

Abstract. The scope of the present research is to establish some findings concerning the essential approximation pseudospectra and the essential defect pseudospectra of closed, densely defined linear operators in a Banach space, building upon the notion of the measure of noncompactness. We start by giving a refinement of the definition of the essential approximation pseudospectra and that of the essential defect pseudospectra by means of the measure of noncompactness. From these characterizations we shall deduce several results and we shall give sufficient conditions on the perturbed operator to have its invariance.

1. Introduction

Let X be an infinite-dimensional Banach space. We denote by $\mathcal{L}(X)$ (resp. $\mathcal{C}(X)$) the set of all bounded (resp. closed, densely defined) linear operators from X into X. The set of all compact operators of $\mathcal{L}(X)$ is denoted by $\mathcal{K}(X)$. For $T \in \mathcal{C}(X)$, we denote by $\sigma(T)$, $\rho(T)$, $\sigma_{ap}(T)$, $\sigma_{\delta}(T)$, $\mathcal{N}(T)$ and $\mathcal{R}(T)$ (resp. the spectrum, the resolvent, the approximation spectrum, the defect spectrum, the null space and the range of T). The nullity of T, $\alpha(T)$, is defined as the dimension of $\mathcal{N}(T)$ and the deficiency of T, $\beta(T)$ is defined as the codimension of $\mathcal{R}(T)$ in X.

In what follows, we need to introduce some important classes of operators. The set of upper semi-Fredholm operators from X into X is defined by

$$\Phi_+(X) := \{T \in \mathcal{C}(X) : \ \alpha(T) < \infty, \ \mathcal{R}(T) \text{ is closed in } X\},$$

the set of all lower semi-Fredholm operators is defined by

$$\Phi_-(X) := \{T \in \mathcal{C}(X) : \ \beta(T) < \infty, \ \mathcal{R}(T) \text{ is closed in } X\}.$$

The set of all semi-Fredholm operators is defined by

$$\Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X)$$

and the class $\Phi(X)$ of all Fredholm operators is defined by

$$\Phi(X) := \Phi_+(X) \cap \Phi_-(X).$$

The index of a semi-Fredholm operator T is defined by $i(T) = \alpha(T) - \beta(T)$. The set of bounded Fredholm operators from X into X is defined by

$$\Phi^b(X) := \Phi(X) \cap \mathcal{L}(X).$$

The set of bounded upper (resp. lower) semi-Fredholm operators from X into X is defined by

$$\Phi^b_+(X) := \Phi_+(X) \cap \mathcal{L}(X) \text{ (resp. } \Phi^b_-(X) := \Phi_-(X) \cap \mathcal{L}(X)).$$

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Now, we define the minimum modulus

$$m(T) \quad := \quad \inf\Big\{\|Tx\|:\ x\in\mathcal{D}(X) \text{ and } \|x\|=1\Big\},$$

and the surjectivity modulus

$$q(T) \quad := \quad \sup\Big\{r>0 : rB_X\subset TB_X\Big\},$$

where B_X is the closed unit ball of X. Note that $m(T)>0$ if, and only if, T is bounded below, i.e. T is injective and has closed range, and $q(T)>0$ if, and only if, T is surjective. Recall also that $m(T^*)=q(T)$ and $q(T^*)=m(T)$ where, $T^*\in\mathcal{L}(X^*)$ is the adjoint of T acting on X^* (dual space), for more information see [13]. It is clear that

$$\sigma_{ap}(T):=\Big\{\lambda\in\mathbb{C}:\ m(\lambda-T)=0\Big\}$$

and

$$\sigma_{\delta}(T):=\Big\{\lambda\in\mathbb{C}:\ q(\lambda-T)=0\Big\}.$$

Let T be a closed linear operator on a Banach space X. For $x\in\mathcal{D}(T)$ the graph norm of x is defined by

$$\|x\|_T:=\|x\|+\|Tx\|.$$

It follows from the closedness of T that $\mathcal{D}(T)$ endowed with the norm $\|.\|_T$ is a Banach space. Let X_T denote $(\mathcal{D}(T),\|.\|_T)$. In this new space the operator T satisfies $\|Tx\|\leq\|x\|_T$ and consequently, T is a bounded operator from X_T into X. If $\hat{T}$ denotes the restriction of T to $\mathcal{D}(T)$, we observe that

$$\left\{\begin{array}{ll}\alpha(\hat{T})=\alpha(T), & \mathcal{N}(\hat{T})=\mathcal{N}(T),\\ \beta(\hat{T})=\beta(T) & \text{and}\quad \mathcal{R}(\hat{T})=\mathcal{R}(T).\end{array}\right. \tag{1.1}$$

In this paper we are concerned with the following essential pseudospectra:

$$\sigma_{eap,\varepsilon}(T) \quad = \quad \bigcap_{K\in\mathcal{K}(X)}\sigma_{ap,\varepsilon}(T+K),$$

$$\sigma_{e\delta,\varepsilon}(T) \quad = \quad \bigcap_{K\in\mathcal{K}(X)}\sigma_{\delta,\varepsilon}(T+K),$$

where

$$\sigma_{ap,\varepsilon}(T):=\sigma_{ap}(T)\cup\Big\{\lambda\in\mathbb{C}:\ m(\lambda-T)<\varepsilon\Big\}$$

and

$$\sigma_{\delta,\varepsilon}(T):=\sigma_{\delta}(T)\cup\Big\{\lambda\in\mathbb{C}:\ q(\lambda-T)<\varepsilon\Big\}.$$

The subsets $\sigma_{eap,\varepsilon}(T),\sigma_{e\delta,\varepsilon}(T),\sigma_{ap,\varepsilon}(T)$ and $\sigma_{\delta,\varepsilon}(T)$ are the essential approximation pseudospectrum, the essential defect pseudospectra, the approximation pseudospectrum [19] and the defect pseudospectra [11].

This paper is a continuation of the research which was undertaken by A. Ammar and A. Jeribi in works [1, 2, 3, 4], and was devoted to special subsets of the essential pseudospectrum of closed, densely defined linear operators

$$\sigma_{e,\varepsilon}(T) \quad := \quad \bigcap_{K\in\mathcal{K}(X)}\sigma_{\varepsilon}(T+K),$$

where

$$\sigma_{\varepsilon}(T):=\sigma(T)\cup\left\{\lambda\in\mathbb{C}:\|(\lambda-T)^{-1}\|>\frac{1}{\varepsilon}\right\}.$$

For further details about pseudospectrum, we can refer to [8, 15, 16, 17].

The first purpose of this work is inspired by [2] when applying the notion of measure of noncompactness to investigate the characterization of the essential pseudospectra on a Banach space. The aim of this paper is to pursue the analysis started in [1, 2] and to extend it to the essential approximation pseudospectra (resp. the essential defect pseudospectra) of closed, densely defined linear operators in a Banach space (Theorem 3.1). We also develop a refinement of the definition of the essential approximation pseudospectra (resp. the essential defect pseudospectra) by means of $T+D$-bounded perturbations (Theorem 3.2). We use this fruitful approach to study the invariance of the essential approximation pseudospectra (resp. the essential defect pseudospectra) of these operators, subject to various kinds of perturbation (Theorems 4.1 and 4.2).

The paper is structured in this way. Section 2 contains preliminary and auxiliary properties that we will need in order to prove the main results of the other sections. The main aim of Section 3 is to characterize the essential approximation pseudospectrum (resp. the essential defect pseudospectrum) of closed, densely defined linear operators on a Banach space by means of the measure of noncompactness. Finally, we will prove the invariance of the essential approximation pseudospectrum (resp. the essential defect pseudospectrum) and establish some results of perturbation on the context of closed, densely defined linear operators on a Banach space.

2. Preliminaries

In order to recall the measure of noncompactness, we denote by $\mathcal{M}_X$ the family of all nonempty and bounded subsets of X, while $\mathcal{N}_X$ denotes its subfamily consisting of all relatively compact sets. Moreover, let us denote the convex hull of a set $A \subset X$ by $conv(A)$. Let us recall the following definition.

Definition 2.1 ([7])**.** A mapping $\gamma : \mathcal{M}_X \longrightarrow [0,\infty[$ is said to be a measure of noncompactness in the Banach space X if it satisfies the following conditions:

(i) The family $\ker(\gamma) = \Big\{D \in \mathcal{M}_X : \ \gamma(D) = 0\Big\}$ is nonempty and $\ker(\gamma) \subset \mathcal{N}_X$. The family $\ker(\gamma)$ is called the kernel of the measure of noncompactness γ.

For $A, B \subset \mathcal{M}_X$:

(ii) If $A \subset B$, then $\gamma(A) \leq \gamma(B)$.

(iii) $\gamma(\overline{A}) = \gamma(A)$.

(iv) $\gamma(\overline{conv(A)}) = \gamma(A)$.

(v) $\gamma(\lambda A + (1-\lambda)B) \leq \lambda\gamma(A) + (1-\lambda)\gamma(B)$, for all $\lambda \in [0,1]$.

(vi) If $(A_n)_{n\in\mathbb{N}}$ is a sequence of sets from $\mathcal{M}_X$ such that $A_{n+1} \subset A_n$, $\overline{A_n} \subset A_n$, for $(n = 1, 2, \ldots)$ and $\lim \gamma(A_n) = 0$, then $A_\infty = \bigcap_{n=1}^{\infty} A_n$ is nonempty and $A_\infty \in \ker(\gamma)$.

A measure of noncompactness γ is said to be sublinear if, for all $A, B \subset \mathcal{M}_X$, it satisfies the following two conditions:

(i) $\gamma(\lambda A) = |\lambda|\gamma(A)$ for all $\lambda \in \mathbb{R}$ (γ is said to be homogeneous).

(ii) $\gamma(A + B) \leq \gamma(A) + \gamma(B)$ (γ is said to be subadditive).

A measure of noncompactness γ is said to be with maximum property, if

$$max(\gamma(A), \gamma(B)) = \gamma(A \cup B).$$

A measure of noncompactness γ is said to be regular if it is sublinear, has the maximum property and $\ker(\gamma) = \mathcal{N}_X$.

To take an instance of the regular measure of noncompactness in a Banach space X, we may refer to the measure of Kuratowski defined, for all $A \in \mathcal{M}_X$ by

$$\gamma(A) := \inf \Big\{ \varepsilon > 0 : A \text{ may be covered by a finite number of sets of } diam \leq \varepsilon \Big\}.$$

Let $T \in \mathcal{L}(X)$. We say that T is k-set-contraction if for every set $D \in \mathcal{M}_X$, we have

$$\gamma(T(D)) \leq k\gamma(D).$$

We define $\gamma(T)$ by

$$\gamma(T) = \inf \big\{ k : T \text{ is } k\text{-set-contraction} \big\}.$$

In the following theorem, we give some results and properties of $\gamma(A)$.

Theorem 2.1. (i) [5, Theorem 3.1] *Let $T \in \mathcal{L}(X)$ and P, Q be two complex polynomials satisfying Q divides $P - 1$.*

(i_1) *If $\gamma(P(T)) < 1$, then $Q(T) \in \Phi_+(X)$.*

(i_2) *If $\gamma(P(T)) < \frac{1}{2}$, then $Q(T) \in \Phi(X)$.*

(ii) [6] *Let X be a Banach space and $T \in \mathcal{L}(X)$.*

(ii_1) *$\gamma(T) = 0$ if, and only if, T is compact.*

(ii_2) *If $T, B \in \mathcal{L}(X)$, then $\gamma(TB) \leq \gamma(T)\gamma(B)$.*

(ii_3) *If $K \in \mathcal{K}(X)$, then $\gamma(T + K) = \gamma(T)$.*

(iii) [5, Corollary 2.3] *Let $T \in \mathcal{L}(X)$. If $\gamma(T^n) < 1$ for some $n > 0$, then $I - T \in \Phi(X)$ and $i(I - T) = 0$.*

Definition 2.2. Let X a Banach spaces and let $K \in \mathcal{L}(X)$.
(i) The operator K is called Fredholm perturbation if, $T + K \in \Phi^b(X)$ whenever, $T \in \Phi^b(X)$. The set of Fredholm perturbations is denoted by $\mathcal{F}^b(X)$.

(ii) The operator K is called an upper semi-Fredholm perturbation if $T + K \in \Phi^b_+(X)$ whenever, $T \in \Phi^b_+(X)$. The set of upper semi-Fredholm perturbations is denoted by $\mathcal{F}^b_+(X)$.

(iii) The operator K is called a lower semi-Fredholm perturbation if $T + K \in \Phi^b_-(X)$ whenever, $T \in \Phi^b_-(X)$. The set of lower semi-Fredholm perturbations is denoted by $\mathcal{F}^b_-(X)$.

Definition 2.3. Let X a Banach spaces.
(i) An operator $T \in \mathcal{C}(X)$ is said to have a left Fredholm inverse if there are maps $R_l \in \mathcal{L}(X)$ and $K \in \mathcal{K}(X)$ such that $I + K$ extends $R_l T$. The operator R_l is called left Fredholm inverse of T.

(ii) An operator $T \in \mathcal{C}(X)$ is said to have a right Fredholm inverse if there is a map $R_r \in \mathcal{L}(X)$ such that $R_r(X) \in \mathcal{D}(T)$ and $TR_r - I \in \mathcal{K}(X)$. The operator R_r is called right Fredholm inverse of T.

Definition 2.4. Let $T, B \in \mathcal{C}(X)$.
(i) An operator B is called T-bounded if $\mathcal{D}(T) \subset \mathcal{D}(B)$ and there exists nonnegative constant c such that

$$\|Bx\| \leq c(\|x\| + \|Tx\|).$$

(*ii*) An operator B is called T-compact if $\mathcal{D}(T) \subset \mathcal{D}(B)$ and whenever a sequence (x_k) of elements of $\mathcal{D}(T)$ satisfies

$$\|x_k\| + \|Tx_k\| \leq c, \quad k = 1, 2, \dots,$$

then (Bx_k) has a subsequence convergent in X.

(*iii*) An operator B is called T-pseudocompact if $\mathcal{D}(T) \subset \mathcal{D}(B)$ and whenever a sequence (x_k) of elements of $\mathcal{D}(T)$ satisfies

$$\|x_k\| + \|Tx_k\| + \|Bx_k\| \leq c, \quad k = 1, 2, \dots,$$

then (Bx_k) has a subsequence.

Our first result is the following.

Theorem 2.2. *Let $T \in \mathcal{C}(X)$ and $\varepsilon > 0$. Then*

(*i*) *$\lambda \in \sigma_{ap,\varepsilon}(T)$ if, and only if, there exists a bounded operator $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ and $\lambda \in \sigma_{ap}(T + D)$.*

(*ii*) *$\lambda \in \sigma_{\delta,\varepsilon}(T)$ if, and only if, there exists a bounded operator $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ and $\lambda \in \sigma_{\delta}(T + D)$.*

Proof. The proof of (*i*) and (*ii*) may be achieved in the same way as the proof of [1, Theorem 3.3]. □

Remark 2.1. Let $T \in \mathcal{C}(X)$ and $\varepsilon > 0$. Then, from Theorem 2.2 we can derive,

$$\sigma_{ap,\varepsilon}(T) = \bigcup_{\|D\|<\varepsilon} \sigma_{ap}(T + D) \quad \text{and} \quad \sigma_{\delta,\varepsilon}(T) = \bigcup_{\|D\|<\varepsilon} \sigma_{\delta}(T + D).$$

We will give a characterization of the essential approximation pseudospectrum and the essential defect pseudospectrum by means of semi-Fredholm operators.

Theorem 2.3. *Let $T \in \mathcal{C}(X)$ and $\varepsilon > 0$. Then*

(*i*) *$\lambda \notin \sigma_{eap,\varepsilon}(T)$ if, and only if, for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$, we have*

$$\lambda - T - D \in \Phi_{+}(X) \text{ and } i(\lambda - T - D) \leq 0.$$

(*ii*) *$\lambda \notin \sigma_{e\delta,\varepsilon}(T)$ if, and only if, for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$, we have*

$$\lambda - T - D \in \Phi_{-}(X) \text{ and } i(\lambda - T - D) \geq 0.$$

Proof. A similar reasoning as before [1, Theorem 4.1]. □

3. Characterization of $\sigma_{eap,\varepsilon}(T)$ and $\sigma_{e\delta,\varepsilon}(T)$

We will give a fine description of the essential approximation pseudospectrum and the essential defect pseudospectrum of a closed, densely defined linear operator by means of the measure of noncompactness. If $T \in \mathcal{C}(X)$, we define the sets

$$\mathcal{M}_n^{\varepsilon}(X) = \Big\{ M \in \mathcal{L}(X) : \ \gamma\Big([\lambda - T - M - D)^{-1} M]^n \Big) < 1 \text{ for all } D \in \mathcal{L}(X) \\ : \ \|D\| < \varepsilon \text{ and } \lambda \in \rho(T + M + D) \Big\}.$$

and

$$\mathcal{T}_n^{\varepsilon}(X) = \Big\{ M \in \mathcal{L}(X) : \ \gamma\Big([M^*(\lambda - T - M - D)^{*-1}]^n \Big) < 1 \text{ for all } D \in \mathcal{L}(X) \\ : \ \|D\| < \varepsilon \text{ and } \lambda \in \rho(T^* + M^* + D^*) \Big\}.$$

Theorem 3.1. *Let $T \in \mathcal{C}(X)$ and $\varepsilon > 0$. Then*

(i) $\sigma_{eap,\varepsilon}(T) = \bigcap\limits_{M \in \mathcal{M}_n^\varepsilon(X)} \sigma_{ap,\varepsilon}(T+M)$.

(ii) $\sigma_{e\delta,\varepsilon}(T) = \bigcap\limits_{M \in \mathcal{T}_n^\varepsilon(X)} \sigma_{\delta,\varepsilon}(T+M)$.

Proof. (i) We will prove this theorem by the similar ways in [2, Theorem 2.1]. Let $M \in \mathcal{K}(X)$, then $\gamma(M) = 0$. Hence, $\gamma\Big([\lambda - T - M - D)^{-1}M]^n\Big) < 1$ for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ and $\lambda \in \rho(T+M+D)$. Therefore, $\mathcal{K}(X) \subset \mathcal{M}_n^\varepsilon(X)$. Consequently,

$$\bigcap_{M \in \mathcal{M}_n^\varepsilon(X)} \sigma_{ap,\varepsilon}(T+M) \subset \bigcap_{M \in \mathcal{K}(X)} \sigma_{ap,\varepsilon}(T+M) = \sigma_{eap,\varepsilon}(T).$$

Conversely, we suppose that $\lambda \notin \bigcap\limits_{M \in \mathcal{M}_n^\varepsilon(X)} \sigma_{ap,\varepsilon}(T+M)$. Then, there exists $M \in \mathcal{M}_n^\varepsilon(X)$ such that for every $\|D\| < \varepsilon$ and $\lambda \in \rho(T+M+D)$, we have

$$\gamma\Big([(\lambda - T - M - D)^{-1}M]^n\Big) \quad < \quad 1 \quad \text{and} \quad \lambda \notin \sigma_{ap,\varepsilon}(T+M).$$

Then, we apply Theorem 2.1-(i) for $P(z) = z^n$ and $Q(z) = 1 - z$, we infer that

$$Q(T) \quad = \quad I - (\lambda - T - M - D)^{-1}M \in \Phi_+(X).$$

Thereby, we can write

$$(\lambda - T - D) = (\lambda - T - M - D)((I + \lambda - T - M - D)^{-1}M).$$

Now, let $t \in [0,1]$, then

$$\gamma\Big((t(\lambda - T - M - D)^{-1}M)^n\Big) \quad < \quad 1$$

and also $I - t(\lambda - T - M - D)^{-1}M \in \Phi_+(X)$. It follows from [14, Theorem 7.25] that

$$i\big(I - (\lambda - T - M - D)^{-1}M\big) = 0.$$

According to [1, Theorem 3.3], we have for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ and

$$\lambda \notin \sigma_{ap}(T + D + M).$$

We conclude that for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$

$$\lambda - T - D \in \Phi_+(X) \text{ and } i(\lambda - T - D) = i(\lambda - T - D - M) \leq 0.$$

By using Theorem 2.3, we obtain $\lambda \notin \sigma_{eap,\varepsilon}(T)$.

(ii) It follows directly from (i), if we replace T by the adjoint operator T^*. □

Remark 3.1. Let $T \in \mathcal{C}(X)$ and $\varepsilon > 0$.

(i) Let $\mathfrak{I}(X)$ $\Big($resp. $\mathfrak{V}(X)\Big)$ be a subset of $\mathcal{L}(X)$. If $\mathcal{K}(X) \subset \mathfrak{I}(X) \subset \mathcal{M}_n^\varepsilon(X)$ $\Big($resp. $\mathcal{K}(X) \subset \mathfrak{V}(X) \subset \mathcal{T}_n^\varepsilon(X)\Big)$, then

$$\sigma_{eap,\varepsilon}(T) = \bigcap_{M \in \mathfrak{I}(X)} \sigma_{ap,\varepsilon}(T+M) \ \Big(\text{resp. } \sigma_{e\delta,\varepsilon}(T) = \bigcap_{M \in \mathfrak{V}(X)} \sigma_{\delta,\varepsilon}(T+M)\Big).$$

(ii) If for all $J, J_2 \in \mathfrak{I}(X)$ $\Big($resp. $\mathfrak{V}(X)\Big)$, we have $J \pm J_2 \in \mathfrak{I}(X)$ $\Big($ resp. $\mathfrak{V}(X)\Big)$, then for all $J \in \mathfrak{I}(X)$, we have

$$\sigma_{eap,\varepsilon}(T+J)=\sigma_{eap,\varepsilon}(T)\ \Big(\text{resp. } \sigma_{e\delta,\varepsilon}(T+J)=\sigma_{e\delta,\varepsilon}(T)\Big).$$

In the next theorem, we will give a fine characterization of $\sigma_{eap,\varepsilon}(.)$ and $\sigma_{e\delta,\varepsilon}(.)$ by means of $T+D$-bounded perturbations. For this end we define the sets

$$\mathcal{H}_\varepsilon(X)=\Big\{K\in\mathcal{C}(X): K \text{ is } T+D\text{-bounded and } K(\mu-T-D)^{-1}\in\mathcal{F}_+^b(X)$$
$$\text{for all } D\in\mathcal{L}(X):\ \|D\|<\varepsilon \text{ for some } \mu\in\rho(T+D)\Big\}$$

and

$$\mathcal{Q}_\varepsilon(X)=\Big\{K\in\mathcal{C}(X):\ K \text{ is } T+D\text{-bounded and } ((\mu-T-D)^{-1}\hat{K})^*\in\mathcal{F}_+^b(X_T^*)$$
$$\text{for all } D\in\mathcal{L}(X):\ \|D\|<\varepsilon,\ \text{for some } \mu\in\rho(T+D)\Big\}.$$

Theorem 3.2. *Let $T\in\mathcal{C}(X)$ and $\varepsilon>0$. Then*

(i) $\sigma_{eap,\varepsilon}(T)=\bigcap\limits_{K\in\mathcal{H}^\varepsilon(X)}\sigma_{ap,\varepsilon}(T+K)$.

(ii) $\sigma_{e\delta,\varepsilon}(T)=\bigcap\limits_{K\in\mathcal{Q}^\varepsilon(X)}\sigma_{\delta,\varepsilon}(T+K)$.

Proof. (i) Since $\mathcal{K}(X)\subset\mathcal{H}^\varepsilon(X)$, then $\bigcap\limits_{K\in\mathcal{H}^\varepsilon(X)}\sigma_{ap,\varepsilon}(T+K)\subset\sigma_{eap,\varepsilon}(T)$. To prove the inverse inclusions of (i), let $\lambda\notin\bigcap\limits_{K\in\mathcal{H}^\varepsilon(X)}\sigma_{ap,\varepsilon}(T+K)$, then there exists $K\in\mathcal{H}^\varepsilon(X)$ such that $\lambda\notin\sigma_{ap,\varepsilon}(T+K)$. Therefore

$$\lambda-T-D-K\in\Phi_+(X).$$

Since $Y:=\mathcal{R}(\lambda-T-D-K)$ is a closed subspace of X, then Y is a Banach space with the same norm, hence $(\lambda-\hat{T}-\hat{D}-\hat{K})^{-1}\in\mathcal{L}(Y,X_T)$. Let $\mu\in\rho(T+D)$ such that $K(\mu-T-D)^{-1}\in\mathcal{F}_+^b(X)$. Then, we can write

$$\hat{K}(\lambda-\hat{T}-\hat{D}-\hat{K})^{-1}=K(\mu-\hat{T}-\hat{D})^{-1}(\mathcal{J}+(\mu-\lambda+\hat{K})(\lambda-\hat{T}-\hat{D}-\hat{K})^{-1}),$$

where $\mathcal{J}$ denotes the embedding operator which maps every $x\in Y$ onto the same element in X. Since $\mu-\lambda+\hat{K}\in\mathcal{L}(X_T,X)$ and $K(\mu-\hat{T}-\hat{D})^{-1}\in\mathcal{F}_+^b(X)$, it follows from [12, p. 70] that

$$\hat{K}(\lambda-\hat{T}-\hat{D}-\hat{K})^{-1}\in\mathcal{F}_+^b(Y,X).\tag{3.1}$$

Now, we can write $\lambda-\hat{T}-\hat{D}$ in the form

$$\lambda-\hat{T}-\hat{D}=(\mathcal{J}+\hat{K}(\lambda-\hat{T}-\hat{D}-\hat{K})^{-1})(\lambda-\hat{T}-\hat{D}-\hat{K}).$$

We see that $\mathcal{J}$ is injective (i.e., $\mathcal{N}(\mathcal{J})=0$) and $\mathcal{R}(\mathcal{J})=Y$. Hence

$$\mathcal{J}\in\Phi_+^b(Y,X)\quad\text{and}\quad i(\mathcal{J})\le 0.$$

By using Eq. (3.1) and [9, Lemma 2.1], we obtain

$$\mathcal{J}+\hat{K}(\lambda-\hat{T}-\hat{D}-\hat{K})^{-1}\in\Phi_+^b(Y,X)\quad\text{and}\quad i(\mathcal{J}+\hat{K}(\lambda-\hat{T}-\hat{D}-\hat{K})^{-1})\le 0.$$

It follows from [13, Theorems 5 and 12] that

$$\lambda-\hat{T}-\hat{D}\in\Phi_+^b(X_T,X)\quad\text{and}\quad i(\lambda-\hat{T}-\hat{D})\le 0.$$

From Eq.(1.1) we deduce that

$$\lambda-T-D\in\Phi_+^b(X)\quad\text{and}\quad i(\lambda-T-D)\le 0.$$

Consequently, $\lambda \notin \sigma_{eap,\varepsilon}(T)$.

(ii) Let $K \in \mathcal{K}(X)$, then $\hat{K} \in \mathcal{K}(X_T, X)$. Since, $(\mu - T - D)^{-1} \in \mathcal{L}(X, X_T)$ and $\hat{K} \in \mathcal{K}(X_T, X)$, we have

$$((\mu - T - D)^{-1}\hat{K})^* \in \mathcal{K}(X_T^*).$$

Because $\mathcal{K}(X_T^*) \subset \mathcal{F}_+^b(X_T^*)$, then $\mathcal{K}(X) \subset \mathcal{Q}^\varepsilon(X)$. Therefore,

$$\bigcap_{K \in \mathcal{Q}^\varepsilon(X)} \sigma_{\delta,\varepsilon}(T + K) \subset \sigma_{e\delta,\varepsilon}(T). \tag{3.2}$$

Now, it remains to prove the inverse inclusion of (3.2). Let $\lambda \notin \bigcap_{K \in \mathcal{Q}^\varepsilon(X)} \sigma_{\delta,\varepsilon}(T + K)$, then there exists $K \in \mathcal{Q}^\varepsilon(X)$ such that $\lambda \notin \sigma_{\delta,\varepsilon}(T+K)$ which means that $\lambda - T - D - K$ is surjective. Thus,

$$\lambda - T - D - K \in \Phi_-(X) \quad \text{and} \quad \beta(\lambda - T - D - K) = 0.$$

Hence, $\lambda - \hat{T} - \hat{D} - \hat{K} \in \Phi_-^b(X_T, X)$, we deduce that

$$\lambda - \hat{T}^* - \hat{D}^* - \hat{K}^* \in \Phi_+^b(X^*, X_T^*) \quad \text{and} \quad \alpha(\lambda - \hat{T}^* - \hat{D}^* - \hat{K}^*) = 0.$$

Now, reasoning in the same way as in the proof of (i), we obtain that

$$\lambda - \hat{T} - \hat{D} \in \Phi_-^b(X_T, X) \quad \text{and} \quad i(\lambda - \hat{T} - \hat{D}) \geq 0.$$

Then, from Eq. (1.1), we infer that

$$\lambda - T - D \in \Phi_+^b(X) \quad \text{and} \quad i(\lambda - T - D) \leq 0.$$

Hence, $\lambda \notin \sigma_{e\delta,\varepsilon}(T)$. □

4. Stability of $\sigma_{eap,\varepsilon}(T)$ and $\sigma_{e\delta,\varepsilon}(T)$

In this section, we will establish some findings of stability of the essential approximation pseudospectrum (resp. essential defect pseudospectrum). Before beginning, we denote by $\mathcal{P}_{\gamma,n}(.)$ the set defined by

$$\mathcal{P}_{\gamma,n}(X) = \Big\{T \in \mathcal{L}(X) : \ \gamma(T^n) < 1 \text{ for some } n > 0\Big\}.$$

Theorem 4.1. *Let $T, B \in \mathcal{C}(X)$ and $\varepsilon > 0$. Then the following statements hold:*

(i) Assume that $\lambda - T - D \in \Phi_+(X)$. There exists a left Fredholm inverse $T_{\lambda,\varepsilon}$ of $\lambda - T - D$ such that $BT_{\lambda,\varepsilon} \in \mathcal{P}_{\gamma,n}(X)$, then $\sigma_{eap,\varepsilon}(T + B) \subseteq \sigma_{eap,\varepsilon}(T)$.

(ii) Assume that $\lambda - T - D \in \Phi_-(X)$. There exists a left Fredholm inverse $T_{\lambda,\varepsilon}$ of $\lambda - T - D$ such that $BT_{\lambda,\varepsilon} \in \mathcal{P}_{\gamma,n}(X)$, then $\sigma_{e\delta,\varepsilon}(T + B) \subseteq \sigma_{e\delta,\varepsilon}(T)$.

Proof. (i) Let $\lambda \notin \sigma_{eap,\varepsilon}(T)$, then for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ we have

$$\lambda - T - D \in \Phi_+(X) \quad \text{and} \quad i(\lambda - T - D) \leq 0.$$

Let $T_{\lambda,\varepsilon}$ be the left Fredholm inverse of $\lambda - T - D$, then there exists $K \in \mathcal{K}(X)$ such that

$$T_{\lambda,\varepsilon}(\lambda - T - D) = I - K \quad \text{on} \quad X. \tag{4.1}$$

We infer from Eq. (4.1) that the operator $\lambda - T - B - D$ can be written in the form

$$\begin{aligned} \lambda - T - B - D &= \lambda - T - D - (BT_{\lambda,\varepsilon}(\lambda - T - D) + BK) \\ &= (I - BT_{\lambda,\varepsilon})(\lambda - T - D) - BK. \end{aligned} \tag{4.2}$$

Since $BT_{\lambda,\varepsilon} \in \mathcal{P}_{\gamma,n}(X)$ and applying Theorem 2.1-(iii) we obtain that

$$I - BT_{\lambda,\varepsilon} \in \Phi(X) \quad \text{and} \quad i(I - BT_{\lambda,\varepsilon}) = 0.$$

Consequently, $I - BT_{\lambda,\varepsilon} \in \Phi_+(X)$. By using [13, Theorem 5], we have

$$\begin{aligned} (I - BT_{\lambda,\varepsilon})(\lambda - T - D) &\in \Phi_+(X), \quad \text{and} \\ i((I - BT_{\lambda,\varepsilon})(\lambda - T - D)) &= i(I - BT_{\lambda,\varepsilon}) + i(\lambda - T - D) \\ &= i(\lambda - T - D) \leq 0. \end{aligned}$$

It follows from Eq. (4.2) and [9, Lemma 2.1] that

$$\lambda - T - B - D \in \Phi_+(X) \quad \text{and} \quad i(\lambda - T - B - D) \leq 0.$$

Then $\lambda \notin \sigma_{eap,\varepsilon}(T + B)$. Thus, $\sigma_{eap,\varepsilon}(T + B) \subseteq \sigma_{eap,\varepsilon}(T)$.

(ii) Let $\lambda \notin \sigma_{e\delta,\varepsilon}(T)$. Then, for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ we have

$$\lambda - T - D \in \Phi_-(X) \quad \text{and} \quad i(\lambda - T - D) \geq 0.$$

Since $T_{\lambda,\varepsilon}$ is the right Fredholm inverse of $\lambda - T - D$, then there exists $K \in \mathcal{K}(X)$ such that

$$(\lambda - T - D)T_{\lambda,\varepsilon} = I - K \text{ on } X. \tag{4.3}$$

It follows from Eq. (4.3) that the operator $\lambda - T - B - D$ can be written in the form

$$\begin{aligned} \lambda - T - B - D &= \lambda - T - D - ((\lambda - T - D)T_{\lambda,\varepsilon}B + KB) \\ &= (\lambda - T - D)(I - T_{\lambda,\varepsilon}B) - KB. \end{aligned} \tag{4.4}$$

(ii) As a similar proof to (i), it suffices to replace $\Phi_+(.)$, $\sigma_{eap,\varepsilon}(.)$, Eq. (4.2) and [9, Lemma 2.1(ii)] by $\Phi_-(.)$, $\sigma_{e\delta,\varepsilon}(.)$, Eq. (4.4) and [9, Lemma 2.1-(iii)] respectively. Hence, we deduce that

$$\sigma_{e\delta,\varepsilon}(T + B) \subseteq \sigma_{e\delta,\varepsilon}(T). \qquad \square$$

Remark 4.1. *Let $T \in \mathcal{C}(X)$ and $\varepsilon > 0$. Then,*

$$\bigcap_{B\in\mathcal{P}_{\gamma,n}(X)} \sigma_{ap,\varepsilon}(T + M) \subseteq \sigma_{eap,\varepsilon}(T) \quad \text{and} \quad \bigcap_{B\in\mathcal{P}_{\gamma,n}(X)} \sigma_{\delta,\varepsilon}(T + M) \subseteq \sigma_{e\delta,\varepsilon}(T).$$

Finally, we close this section by the stability of the essential approximation pseudospectrum (resp. essential defect pseudospectrum) by means of a pseudocompact perturbation.

Theorem 4.2. *Let $T, B \in \mathcal{C}(X)$, $\varepsilon > 0$ and $\lambda \in \rho(T + D) \cap \rho(T + B + D)$. If for all bounded operators D such that $\|D\| < \varepsilon$ and B is $(T + D)$-pseudo-compact, then*

(i) $\sigma_{eap,\varepsilon}(T + B) = \sigma_{eap,\varepsilon}(T)$.

(ii) $\sigma_{e\delta,\varepsilon}(T + B) = \sigma_{e\delta,\varepsilon}(T)$.

Proof. (i) For all $\lambda \in \mathbb{C}$, we can write

$$(\mu-T-B-D)-(\mu-T-D)(\lambda-T-D)^{-1}(\lambda-T-B-D) = (\mu-\lambda)(\lambda-T-D)^{-1})B. \tag{4.5}$$

Now, we have $\rho(T + D)$ and $\rho(T + B + D)$ are not empty, then $T + D$ and $T + B + D$ are closed. Therefore, $T + B + D$ is $T + D$-bounded and $(\lambda - T - D)^{-1})B$ is $T + D$-compact. It follows from that $(T + B + D)$-compact. First, let $\mu \notin \sigma_{eap,\varepsilon}(T + B)$, then

$$\mu - T - B - D \in \Phi_+(X) \quad \text{and} \quad i(\mu - T - B - D) \leq 0.$$

By using Eq. (4.5), we obtain that

$$(\mu - T - D)(\lambda - T - D)^{-1}(\lambda - T - B - D) \in \Phi_+(X) \quad \text{and}$$

$$i((\mu - T - D)(\lambda - T - D)^{-1}(\lambda - T - B - D)) \leq 0.$$

Since $\lambda \in \rho(T + B + D)$ and by using Theorem 2.3, we deduce that

$$(\lambda - T - B - D) \in \Phi_+(X) \quad \text{and} \quad i(\lambda - T - B - D) \leq 0.$$

Applying [14, Theorem 5.32], we conclude that

$$(\mu - T - D)(\lambda - T - D)^{-1} \in \Phi_+(X) \quad \text{and} \quad i((\mu - T - D)(\lambda - T - D)^{-1}) \leq 0.$$

From this and the identity $(\mu - T - D) = (\mu - T - D)(\lambda - T - D)(\lambda - T - D)^{-1}$ we obtain

$$\mu - T - D \in \Phi_+(X) \quad \text{and} \quad i(\mu - T - D) \leq 0.$$

This implies that $\mu \notin \sigma_{eap,\varepsilon}(T)$. Therefore $\sigma_{eap,\varepsilon}(T) \subseteq \sigma_{eap,\varepsilon}(T + B)$.

Next, if $\mu \notin \sigma_{eap,\varepsilon}(T)$, then

$$\mu - T - D \in \Phi_+(X) \quad \text{and} \quad i(\mu - T - D) \leq 0. \qquad .$$

Since $\lambda \in \rho(T + B + D)$ and the use of Theorem 2.3 we infer that

$$(\lambda - T - B - D) \in \Phi_+(X) \quad \text{and} \quad i(\lambda - T - B - D) \leq 0.$$

We deduce that

$$(\mu - T - D)(\lambda - T - D)^{-1}(\lambda - T - B - D) \in \Phi_+(X) \quad \text{and}$$

$$i((\mu - T - D)(\lambda - T - D)^{-1}(\lambda - T - B - D)) \leq 0.$$

By referring to Eq. (4.5), we have

$$(\lambda - T - B - D) \in \Phi_+(X) \quad \text{and} \quad i(\lambda - T - B - D) \leq 0.$$

Then, $\mu \notin \sigma_{eap,\varepsilon}(T + B)$.

(ii) A similar reasoning allows us to deduce that $\sigma_{e\delta,\varepsilon}(T + B) = \sigma_{e\delta,\varepsilon}(T)$. □

References

1. A. Ammar, A. Jeribi, and K. Mahfoudhi, *A characterization of the essential approximation pseudospectrum on a Banach space*, Filomat **31** (2017), no. 11, 3599–3610.
2. A. Ammar, A. Jeribi, *Measures of noncompactness and essential pseudospectra on Banach space*, Math. Methods Appl. Sci. **37** (2014), no. 3, 447–452.
3. A. Ammar and A. Jeribi, *A characterization of the essential pseudospectra on a Banach space*, Arab. J. Math. **2** (2013), 139–145.
4. A. Ammar and A. Jeribi, *A characterization of the essential pseudospectra and application to a transport equation*, Extra. Math. **28** (2013), 95–112.
5. B. Abdelmoumen, A. Dehici, A. Jeribi, and M. Mnif, *Some new properties of Fredholm theory, Schechter essential spectrum, and application to transport theory*, J. Inequal. Appl., Art. ID 852676 (2008), 1–14.
6. K. Astala, *On measures of noncompactness and ideal variations in Banach spaces*, Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes **29** (1980), 1–42.
7. J. Banás, K. Goebel, *Measures of Noncompactness in Banach Spaces*, Lect. Notes Pure Appl. Math., vol. 60, Marcel Dekker, New York, 1980.
8. E. B. Davies, *Linear Operators and their Spectra*, Cambridge Studies in Advanced Mathematics, vol. 106, Cambridge University Press, Cambridge, 2007.
9. A. Jeribi, *Spectral Theory and Applications of Linear Operators and Block Operator Matrices*, Springer-Verlag, New York, 2015.
10. A. Jeribi and N. Moalla, *A characterization of some subsets of Schechter's essential spectrum and application to singular transport equation*, J. Math. Anal. Appl. **358** (2009), 434–444.
11. M. Elhodaibi and A. Jaatit, *On linear maps approximately preserving the approximate point spectrum or the surjectivity spectrum*, Mat. Vesnik **366** (2014), 333–342.
12. I. C. Gohberg, A. S. Markus, and I. A. Fel'dman, *Normally solvable operators and ideals associated with them*, Amer. Math. Soc. Transl. Ser. 2 **61** (1967), 63–84.
13. V. Müller, *Spectral theory of linear operators and spectral system in Banach algebras*, Oper. Theory Adv. Appl., vol. 139, Birkhäuser, Basel, 2003.

14. M. Schechter, *Principles of Functional Analysis*, 2nd ed., Graduate Studies in Mathematics, vol. 36, American Mathematical Society, Providence, RI, 2002.
15. E. Shargorodsky, *On the definition of pseudospectra*, Bull. London Math. Soc. **41** (2009), 524–534.
16. L. N. Trefethen, *Approximation theory and numerical linear algebra*, in: Algorithms for Approximation II, eds. J. C. Mason and M. G. Cox, Chapman and Hall, London, 1990, pp. 336–360.
17. L. N. Trefethen, *Pseudospectra of matrices*, in: Numerical Analysis, 1991 (Dundee, 1991), vol. 260 of Pitman Research Notes Math. Ser., Longman Science and Technology, Harlow, 1992, pp. 234–266.
18. L. N. Trefethen, *Pseudospectra of linear operators*, SAIM Rev. **39** (1997), no. 3, 383–406.
19. M. P. H. Wolff, *Discrete approximation of unbounded operators and approximation of their spectra*, J. Approx. Theory **113** (2001), 229–244.

Department of Mathematics, University of Sfax, Faculty of Sciences of Sfax, Route de soukra Km 3.5, B.P. 1171, 3000, Sfax, Tunisia
E-mail address: ammar_aymen84@yahoo.fr

Department of Mathematics, University of Sfax, Faculty of Sciences of Sfax, Route de soukra Km 3.5, B.P. 1171, 3000, Sfax, Tunisia
E-mail address: Aref.Jeribi@fss.rnu.tn

Department of Mathematics, University of Sfax, Faculty of Sciences of Sfax, Route de soukra Km 3.5, B.P. 1171, 3000, Sfax, Tunisia
E-mail address: kamelmahfoudhi72@yahoo.com

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