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ON A LOCALIZATION OF THE SPECTRUM OF A COMPLEX VOLTERRA OPERATOR

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Abstract. A complex Volterra operator with the symbol $g = \log(1 + u(z))$, where u is an analytic self map of the unit disk $\mathbb{D}$ into itself is considered. We show that the spectrum of this operator on $H^p(\mathbb{D})$, $1 \le p < \infty$, is located in the disk $\{\lambda : |\lambda + p/2| \le p/2\}$.

1. Let g be a function holomorphic on the unit disc $\mathbb{D}$ of the complex plane. For $1 \le p < \infty$ we consider the following operator on $H^p(\mathbb{D})$:

$$(V_g f)(z) = \int_0^z f(\zeta) g'(\zeta) d\zeta \tag{1}$$

The operator V_g is called a complex Volterra operator with symbol g. It is well known (see, for example [1] for details and further references) that the operator V_g is bounded on $H^p(\mathbb{D})$ if and only if $g \in BMOA(\mathbb{D})$ and V_g is compact if and only if $g \in VMOA(\mathbb{D})$.

Suppose that the operator V_g is bounded and consider the resolvent operator

$$R(\lambda) = (V_g - \lambda I)^{-1}. \tag{2}$$

If $R(\lambda)f = h$, then $f = V_g h - \lambda h$, that is

$$\int_0^z h(\zeta) g'(\zeta) d\zeta - \lambda h = f. \tag{3}$$

In particular, $f(0) = -\lambda h(0)$. We differentiate both sides of (3) and obtain

$$h'(z) - \frac{1}{\lambda} g'(z) h(z) = -\frac{1}{\lambda} f'(z), \qquad f(0) = -\lambda h(0). \tag{4}$$

The solution of the differential equation above is given by

$$h(z) = Ce^{g(z)/\lambda} - \frac{1}{\lambda} e^{g(z)/\lambda} \int_0^z f'(\zeta) e^{-g(\zeta)/\lambda} d\zeta.$$

We integrate last term by parts and obtain

$$h(z) = Ce^{g(z)/\lambda} - \frac{1}{\lambda} e^{g(z)/\lambda} \left[f(\zeta) e^{-g(\zeta)/\lambda}|_0^z + \frac{1}{\lambda} \int_0^z e^{-g(\zeta)/\lambda} g'(\zeta) f(\zeta) d\zeta \right]$$

$$= Ce^{g(z)/\lambda} - \frac{1}{\lambda} f(z) + \frac{1}{\lambda} e^{(g(z)-g(0))/\lambda} f(0) - \frac{1}{\lambda^2} e^{g(z)/\lambda} \int_0^z e^{-g(\zeta)/\lambda} g'(\zeta) f(\zeta) d\zeta.$$

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In particular, $h(0) = Ce^{g(0)/\lambda}$. Taking into account the initial condition $f(0) = -\lambda h(0)$ one obtains $C = -f(0)e^{-g(0)/\lambda}/\lambda$. Therefore

$$(R(\lambda)f)(z) = -\frac{1}{\lambda}f(z) - \frac{1}{\lambda^2}e^{g(z)/\lambda}\int_0^z e^{-g(\zeta)/\lambda}g'(\zeta)f(\zeta)d\zeta. \tag{5}$$

Since for any $f \in H^p$ we have $(V_g f)(0) = 0$ it follows that $\lambda = 0$ is always in the spectrum of V_g. It is easy to see that if $g \in VMOA$ then the spectrum of V_g, $\sigma(V_g) = \{0\}$. It is also evident that $\sigma_p(V_g) = \emptyset$. From formula (5) it follows that $\mathrm{Ker}R(\lambda) = \{0\}$. Indeed, if $R(\lambda)f = 0$, then

$$-\lambda e^{-g(z)/\lambda}f(z) = \int_0^z g'(\zeta)\left[e^{-g(\zeta)/\lambda}f(\zeta)\right]d\zeta$$

from which the statement regarding the $\mathrm{Ker}R(\lambda)$ follows.

Also, for any $h \in H^p$ there is $f \in H^p$ such that $R(\lambda)f = h$. Indeed, $f = V_g h - \lambda h$ satisfies the requirement. Therefore a complex number $\lambda \in \mathbb{C}$ is in the resolvent set of V_g if the operator $R(\lambda)$ maps the whole H^p into H^p.

Note that

$$R(\lambda)\mathbb{1} = -\frac{1}{\lambda}e^{(g(z)-g(0))/\lambda} \tag{6}$$

where $\mathbb{1} \in H^p$ is the function which is equal to one for all $z \in \mathbb{D}$.

As a consequence of formula (6) we obtain the following statements:

1. *Suppose that $g \in BMOA$. Then for any $p \geq 1$ there exists $r_0 > 0$ which depends on g and p such that $e^{g(z)/\lambda} \in H^p$ $\forall|\lambda| > r_0$. If $g \in VMOA$ then $r_0 = 0$.*
2. *$\overline{\{\lambda : e^{g(z)/\lambda} \notin H^p\}} \subset \sigma(V_g)$.*

According to a conjecture of A. Aleman [2], the following equality is fulfilled

$$\sigma(V_g) = \overline{\{\lambda : e^{g(z)/\lambda} \notin H^p\}}.$$

The statement below is in favor of the Aleman's conjecture.

Proposition 1. *Let $\lambda \in \mathbb{C}$ be such that $\sup_{z\in\mathbb{D}}\{|\operatorname{Re} g(z)/\lambda|\} < \infty$. Then λ is in the resolvent set of the operator V_g.*

Indeed, under the assumption of the proposition the function $\exp[\pm g(z)/\lambda] \in H^\infty$. Therefore for any $h \in H^p$

$$e^{g(z)/\lambda}\int_0^z g'(\zeta)[e^{-g(\zeta)/\lambda}h(\zeta)]d\zeta = e^{g/\lambda}V_g[e^{-g/\lambda}h] \in H^p.$$

As particular cases of Proposition 1 we obtain that the following statements.

1. Suppose $g \in H^\infty$ (hence $g \in BMOA$). Then any nonzero λ is in the resolvent set of the V_g. Therefore $\sigma(V_g) = \{0\}$.

2. Suppose $g(z) = \log(1 + u(z))$, u is a Schur function, that is $u \in H^\infty$, $\|u\|_\infty \leq 1$. Clearly $g \in BMOA$. Then any $\lambda = i\mu$, $\mu \in \mathbb{R} \setminus \{0\}$ is in the resolvent set of the corresponding operator V_g.

The last example admits further detalization. Note that the function

$$\exp\{\log[1 + u(z)]/\lambda\} = [1 + u(z)]^{1/\lambda}$$

is subordinate to $(1+z)^{1/\lambda}$. (For a notion of subordination see, for example [3].) Therefore, if $[1+u(z)]^{1/\lambda} \notin H^p$ then $(1+z)^{1/\lambda} \notin H^p$. Now direct calculation shows that the last condition is fulfilled if and only if for $\lambda = \lambda_1 + i\lambda_2$ the following inequality is fulfilled

$$(\lambda_1 + p/2)^2 + \lambda_2^2 \leq 1.$$

Thus we obtained the following statement.

Theorem 1. *Let V_g be a complex Volterra operator operator with a symbol $g(z) = \log(1+u(z))$, $|z| < 1$, where u is a Schur function. Then the spectrum of V_g on the space $H^p(\mathbb{D})$ is located in the disk $|\lambda + p/2| \leq p/2$.*

Remark 1. It is known that if $h(z)$, $z \in \mathbb{D}$ is a univalent , zero free analytic function then $g(z) = \log h(z) \in BMOA$. This gives a large class of functions that can be tested for veracity of the Aleman conjecture.

References

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