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COMPLEX MOMENT PROBLEM AND RECURSIVE RELATIONS

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Abstract. We introduce a new methodology to solve the truncated complex moment problem. To this aim we investigate recursive doubly indexed sequences and their characteristic polynomials. A characterization of recursive doubly indexed *moment* sequences is given. A simple application gives a computable solution to the complex moment problem for cubic harmonic characteristic polynomials of the form $z^3 + az + b\overline{z}$, where a and b are arbitrary real numbers. We also recapture a recent result due to Curto-Yoo given for cubic column relations in $M(3)$ of the form $Z^3 = itZ + u\overline{Z}$ with t, u real numbers satisfying some suitable inequalities. Furthermore, we solve the truncated complex moment problem with column dependence relations of the form $Z^{k+1} = \sum\limits_{0\le n+m\le k} a_{nm}\overline{Z}^n Z^m$ $(a_{nm} \in \mathbb{C})$.

1. Introduction

Let $\gamma = \{\gamma_{ij}\}_{i,j\ge 0}$ be a doubly indexed sequence of complex numbers such that $\overline{\gamma}_{ij} = \gamma_{ji}$ and $\gamma_{00} > 0$. The truncated complex moment problem (TCMP for short) associated with $\{\gamma_{ij}\}_{0\le i,j\le r}$ entails finding a positive Borel measure μ supported in the complex plane $\mathbb{C}$ such that

$$\gamma_{ij} = \int \overline{z}^i z^j d\mu \quad (0 \le i, j \le r). \tag{1}$$

A sequence $\{\gamma_{ij}\}_{0\le i,j\le r}$ satisfying (1) will be called a truncated moment sequence and μ is said to be a representing measure for $\{\gamma_{ij}\}_{0\le i,j\le r}$. The full complex moment problem (FCMP) prescribes moments of all orders (see, for instance, [3, 30]); more precisely, an infinite doubly sequence $\{\gamma_{ij}\}_{i,j\ge 0}$, with $i, j \in \mathbb{Z}_+$, is a moment sequence provided that there exists a Borel measure μ supported in the complex plane $\mathbb{C}$ such that,

$$\gamma_{ij} = \int \overline{z}^i z^j d\mu \quad \text{for all} \quad i, j \in \mathbb{Z}_+. \tag{2}$$

J. Stochel [29] has shown that the truncated complex moment problem is more general than the full complex moment problem, in the following sense: a full moment sequence $\{\gamma_{ij}\}_{i,j\in\mathbb{Z}_+}$ admits a representing measure if and only if each truncation $\gamma^{(r)} \equiv \{\gamma_{ij}\}_{i+j\le r}$ admits a representing measure.

The truncated complex moment problem serves as a prototype for several other moment problems to which it is closely related. Its application can be found in the invariant subspace problem [1], subnormal operator theory [26, 28, 31], polynomial hyponormality [12] and joint hyponormality [5, 6] and arise in pure and applied mathematics and in the sciences in general. For example J.B. Lasserre developed, in [21, 22, 23], several applications of the moment problem in concrete optimization theory; see also [24, 25].

Therefore, the truncated complex moment problem has interested many authors, such as, for instance, [2, 7, 24, 25, 32]. Although interesting results were discovered, various

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basic situations are considered as open problems. For example, in the truncated complex moment problem associated with $\gamma \equiv \{\gamma_{ij}\}_{0\le i+j\le r}$, only the cases $r = 1, 2, 3, 4$ (the quadratic [19], the cubic [20] and the quartic [14] moment problem) have been (recently) completely achieved. All the other cases are open: cintic, sixtic, ...; as indicated in many recent papers (see, for instance, [15, 16, 17, 33]).

In [7, 8, 9] Curto-Fialkow introduced an approach to study the existence and uniqueness of solutions of the TCMP, $\gamma^{(2n)} := \{\gamma_{ij}\}_{0\le i+j\le 2n}$ with $\overline{\gamma}_{ij} = \gamma_{ji}$ and $\gamma_{00} > 0$, based on positivity and extensions of the moment matrix $M(n) \equiv M(n)(\gamma^{(2n)})$, built as follows:

$$M(n) = \begin{pmatrix} M[0,0] & M[0,1] & \dots & M[0,n] \\ M[1,0] & M[1,1] & \dots & M[1,n] \\ \vdots & \vdots & \ddots & \vdots \\ M[n,0] & M[n,1] & \dots & M[n,n] \end{pmatrix}, \tag{3}$$

where

$$M[i,j] = \begin{pmatrix} \gamma_{i,j} & \gamma_{i+1,j-1} & \cdots & \gamma_{i+j,0} \\ \gamma_{i-1,j+1} & \gamma_{i,j} & \cdots & \gamma_{i+j-1,1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{0,i+j} & \gamma_{1,i+j-1} & \cdots & \gamma_{j,i} \end{pmatrix}.$$

The matrix $M(n)$ detects the positivity of the Riesz functional

$$\Lambda_{\gamma^{(2n)}} : p(Z, \overline{Z}) \equiv \sum_{0\le i+j\le 2n} a_{ij}\overline{Z}^i Z^j \longrightarrow \sum_{0\le i+j\le 2n} a_{ij}\gamma_{ij} \tag{4}$$

on the cone generated by the collection $\{p\overline{p} : p \in \mathbb{C}_n[Z, \overline{Z}]\}$, where $\mathbb{C}_n[Z, \overline{Z}]$ is the vector space of polynomials in two variables with complex coefficients and total degree less than or equal to n. In the sequel, we will write, $d_z(P), d_{\bar{z}}(P)$ and $d_P \equiv \deg P$ for the degree in z, the degree in $\bar{z}$ and the total degree of P, respectively. Considering the lexicographic order

$$1, Z, \overline{Z}, Z^2, Z\overline{Z}, \overline{Z}^2, \dots, Z^n, Z^{n-1}\overline{Z}, \dots, Z\overline{Z}^{n-1}, \overline{Z}^n$$

to denote rows and columns of the moment matrix $M(n)$. It is immediate that the rows $\overline{Z}^k Z^l$, columns $\overline{Z}^i Z^j$ entry of the matrix $M(n)$ is equal to $\Lambda_{\gamma^{(2n)}}(\overline{z}^{i+l} z^{j+k}) = \gamma_{i+l,j+k}$. For reason of simplicity, we identify a polynomial $p(z; \overline{z}) \equiv \sum a_{ij}\overline{z}^i z^j$ with its coefficient vector $p = (a_{ij})$ with respect to the basis of monomials of $\mathbb{C}_n[z; \overline{z}]$ in degree-lexicographic order. Clearly, $M(n)$ acts on these coefficient vectors as follows:

$$< M(n)p, q >= \Lambda_{\gamma^{(2n)}}(p\overline{q}).$$

Similarly to (3), the infinite complex moment matrix is built as follows:

$$M(\gamma) \equiv M(\infty) = \begin{pmatrix} M[0,0] & M[0,1] & M[0,2] & \cdots \\ M[1,0] & M[1,1] & M[1,2] & \cdots \\ M[2,0] & M[2,1] & M[2,2] & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

A result of Curto-Fialkow [7] states that $\gamma^{(2n)} = \{\gamma_{ij}\}_{0\le i+j\le 2n}$ has a representing measure if and only if $M(n)$ admits a positive finite rank moment matrix extension $M(\gamma) \equiv M(\infty)$. In general, the existence of such extension is difficult to determine, but a complete solution to TCMP based on moment matrix extensions is known for $n \le 2$ and for $M(n)$ whenever the submatrix $M(2)$ is singular (cf. [10], [14], [19]).

Therefore, the first open case of TCMP concerns $n = 3$ with $M(2)$ positive definite. The Curto-Yoo paper [13] concerns part of this problem. We note that if $\gamma^{(2n)}$ has a representing measure, then $M(n)$ is positive and rank $M(n) \le$ card $\nu_{\gamma^{(2n)}}$, where $\nu_{\gamma^{(2n)}}$, the variety associated with $\gamma^{(2n)}$ as the intersection of the zero-sets of the polynomials

$p(z;\overline{z})$ such that $M(n)p = 0$. In Curto-Fialkow-Moller [11], TCMP is solved for the "extremal" case when rank $M(n) = $ card $\nu_{\gamma^{(2n)}}$. In this case, $\gamma^{(2n)}$ has a representing measure if and only if $M(n) \geq 0$ and $\gamma^{(2n)}$ is consistent (that is, if $p \in \mathbb{C}_{2n}[z;\overline{z}]$ and $p_{|\nu_{\gamma^{(2n)}}} \equiv 0$, then $\Lambda_{\gamma^{(2n)}}(p) = 0$). The proof of this fact does not require results on moment matrix extensions; it is based on elementary tools from vector space duality, convex analysis, and interpolation by polynomials.

As noted above, the simplest unsolved case of TCMP concerns $M(3)(\gamma^{(6)})$. Within this problem, [13] identifies a subproblem which is "extremal", with rank $M(3) = $ card $\nu_{\gamma^{(6)}} = 7$. Indeed, [13] focuses on the case when $M(3) \geq 0$ and $M(2) > 0$, and there is a column dependence relation $p(Z;\overline{Z}) = 0$ (and automatically, $\overline{p}(Z;\overline{Z}) = 0$), where p is a harmonic polynomial of the form $p(z,\overline{z}) = z^3 - itz - u\overline{z}$, with real parameters t and u. The $M(3)$ problem remains open and is only partially solved. Lemma 2.3 in [13] states that if $0 <\mid u \mid< t < 2 \mid u \mid$, then p has exactly 7 zeros in the complex plane. This later can be disproved by the next two examples.

Example 1. The equality $z^3 = 2iz - \frac{5}{4}\overline{z}$ admits only 3 zeros, 0 and $\pm\frac{\sqrt{13}}{2}e^{i\frac{\pi}{4}}$, although $t = 2$ and $u = -\frac{5}{4}$ verify the condition $0 < |u| < t < 2|u|$.

Example 2. The equality $z^3 = -2iz + \frac{5}{4}\overline{z}$ admits 7 zeros, 0 ; $\pm\frac{\sqrt{3}}{2}e^{-i\frac{\pi}{4}}$ and $(\pm\frac{3\sqrt{2}}{4} \pm i\frac{\sqrt{2}}{4})e^{-i\frac{\pi}{4}}$, although $t = -2$ and $u = \frac{5}{4}$ does not verify the condition $0 < |u| < t < 2|u|$.

This "inattention" led to the next incorrect version of the main theorem in [13],

Theorem 1. [13, Theorem 1.5] *Let $M(3) \geq 0$, with $M(2) > 0$ and $z^3 - itz - u\overline{z} = 0$, and let $\Lambda_{\gamma^{(6)}}$ be as in (4). For $u, t \in \mathbb{R}$, assume that $0 < |u| < t < 2|u|$. The following statements are equivalent.*

i) There exists a representing measure for $M(3)$.

ii)
$$\begin{cases} \Lambda_{\gamma^{(6)}}(q_{LC}) & = 0, \\ \Lambda_{\gamma^{(6)}}(zq_{LC}) & = 0. \end{cases}$$

iii)
$$\begin{cases} \mathfrak{Re}\gamma_{12} - \mathfrak{Im}\gamma_{12} & = u(\mathfrak{Re}\gamma_{01} - \mathfrak{Im}\gamma_{01}), \\ \gamma_{22} & = (t+u)\gamma_{11} - 2u\mathfrak{Im}\gamma_{02}. \end{cases}$$

iv) $q_{LC} := i(z - i\overline{z})(z\overline{z} - u) = 0$,

Example 1 shows that for $u < 0$ the last theorem is not valid, because $6 \leq$ rang $M(3)$ and card $\nu_{\gamma^{(6)}} \leq 3$ (recall that, if $\gamma^{(6)}$ is a moment sequence then rang $M(3) \leq$ card $\nu_{\gamma^{(6)}}$). Theorem 1 implies Corollary 4.3 in the same paper, which has inherited the same mistakes.

From the previous discussion, it appears natural to give a new solution to the TCMP for cubic column relations in $M(3)$ of the forms $Z^3 = itZ + u\overline{Z}$ and $Z^3 = aZ + b\overline{Z}$, where a, b, t and u are real numbers. This is the main goal of Section 5, using our methodology on recursive sequences.

We notice below that the recursiveness in the truncated moment problem is a natural concept and is totally inherent. It is obvious that the truncated moment problem is equivalent to the recursive full moment problem. Indeed, given a doubly indexed truncated moment sequence $\omega \equiv \{\gamma_{ij}\}_{0\leq i,j\leq n}$. A result of C. Bayer and J. Teichmann [2] states that if a finite double sequence of complex numbers has a representing measure, then it has a finitely atomic representing measure. It follows that ω admits a finite support representing measure μ, suppose that $\operatorname{supp}(\mu) \subset \nu = \{\lambda_1, \lambda_2, \ldots, \lambda_r\} \subset \mathbb{C}$. Let $p \equiv z^{s+1} - \sum_{i+j=0}^{s} a_{ij}\overline{z}^i z^j$ be a polynomial vanishing on ν, since $\int \overline{z}^n z^m p(z,\overline{z})d\mu = 0$ for all $n, m \in \mathbb{Z}_+$, then μ is a representing measure for the recursively generated sequence $\gamma \equiv \{\gamma_{ij}\}_{i,j\geq 0}$ $(\omega \subset \gamma)$ defined by the next relations:

i) For all $i, j \geq 0$,

$$\gamma_{ji} = \overline{\gamma}_{ij}. \tag{5}$$

ii) For all $n, m \geq 0$,

$$\gamma_{n,m+s+1} = \sum_{i+j=0}^{s} a_{ij}\gamma_{n+i,m+j}. \tag{6}$$

We shall refer to the polynomial $p(z,\overline{z}) = z^{s+1} - \sum_{i+j=0}^{s} a_{ij}\overline{z}^i z^j$ as a characteristic polynomial associated with γ. The sequence γ can be associated with several characteristic polynomials, see Section 2.

Thus every truncated complex moment sequence is a subsequence of a recursively moment sequence. We deduce that solving the TCMP is actually equivalent to solve the recursive full moment problem. The main goal in this paper is to investigate use the recursive double sequences (verifying (5) and (6)) to get an approach, based on the localization of zeros of the characteristic polynomials, for solving the TCMP.

This paper is organized as follows. We define in Section 2 recursive doubly indexed sequences. We show that for such sequences the TCMP and FCMP are equivalent. We devote Section 3 to the study of RDIS associated with analytic characteristic polynomial and we show that the family of analytic characteristic polynomials is a principal ideal of $\mathbb{C}[X]$. Section 4 is devoted to give a characterization of recursive doubly indexed moment sequences. In section 5, we apply our results to give an explicit solution for the TCMP with cubic column relations in $M(3)$ of the form $Z^3 = aZ + b\overline{Z}$ $(a, b \in \mathbb{R})$ and also to regain the correct form of Theorem 1. In the last section, we involve the RDIS to give a necessary and sufficient condition for the existence of a solution to the TCMP with column dependence relations of the form $Z^{k+1} = \sum_{0\leq n+m\leq k} a_{nm}\overline{Z}^n Z^m$ $(a_{nm} \in \mathbb{C})$.

2. Recursive double indexed sequences

Let $\{a_{lk}\}_{0\leq l,k\leq r}$ be some fixed complex numbers and let $\gamma \equiv \{\gamma_{ij}\}_{i,j\geq 0}$ be a doubly indexed sequence, with $\gamma_{00} > 0$, defined by the following relations:

i) For all $i, j \geq 0$,

$$\gamma_{ji} = \overline{\gamma}_{ij}. \tag{7}$$

ii) For all i and n with $0 \leq i$ and $r \leq n$,

$$\gamma_{i,n+1} = \sum_{0\leq l+k\leq r} a_{lk}\gamma_{l+i,n+k-r}. \tag{8}$$

Where $\omega \equiv \{\gamma_{ij}\}_{0\leq i\leq j\leq r}$ are given initial conditions.

In the sequel we shall refer to such sequence as Recursive Double Indexed Sequence, RDIS for short. The polynomial $P(z,\overline{z}) = z^{r+1} - \sum_{0\leq l+k\leq r} a_{lk}\overline{z}^l z^k$ is called a characteristic polynomial associated with γ, given by (7) and (8). This last polynomial has a finite roots set. More precisely P has at most $(r+1)^2$ roots (see Proposition 4.4 in [14]).

A RDIS can be defined in various ways using different characteristic polynomials as is shown in the following example. Let $\gamma = \{\gamma_{ij}\}_{i,j\geq 0}$ with $\gamma_{ij} = \frac{(-1)^{i+j}}{2} + \frac{1}{2}\mathfrak{Re}((1-2i)^i(1+2i)^j)$. Then γ may be defined by the following recursive relations,

- $\gamma_{n+2,m} = -2\gamma_{n,m+1} - \gamma_{n,m}$, with $\gamma_{n,m} = \overline{\gamma}_{m,n}$, for $\gamma_{00} = 1, \gamma_{01} = \gamma_{10} = 0, \gamma_{11} = 3$.
- $\gamma_{n+3,m} = \gamma_{n+2,m} - 3\gamma_{n+1,m} - 5\gamma_{n,m}$, with $\gamma_{n,m} = \overline{\gamma}_{m,n}$, for $\gamma_{00} = 1, \gamma_{01} = \gamma_{10} = 0, \gamma_{11} = 3, \gamma_{02} = \gamma_{20} = -1, \gamma_{21} = \gamma_{12} = 2, \gamma_{22} = 13$.

Therefore, $P_1(z,\overline{z}) = z^2 + 2\overline{z} + 1$ and $P_2(z,\overline{z}) = Q(z) = z^3 - z^2 + 3z + 5$ are two characteristic polynomials associated with γ.

Let $\mathcal{P}_\gamma$ be the set of characteristic polynomials associated with the sequence $\gamma \equiv \{\gamma_{ij}\}_{i,j\geq 0}$.

Remarks 2. *i) The subset $\mathcal{P}_\gamma$ is an ideal of $\mathbb{C}[z,\overline{z}]$.*

ii) The characteristic polynomial P, together with the initial conditions and the relations (7) and (8), are said to define the sequence γ.

iii) We notice that because of condition (7) , Equation (8) is equivalent to : For all n and j with $0 \leq j$ and $r \leq n$,

$$\gamma_{n+1,j} = \sum_{0\leq l+k\leq r} \overline{a}_{lk}\gamma_{n-r+k,l+j}. \tag{9}$$

The polynomial $Q(z,\overline{z}) = \overline{z}^{r+1} - \sum_{0\leq l+k\leq r} \overline{a}_{lk}\overline{z}^k z^l$ is a characteristic polynomial associated with $\{\gamma_{i,j}\}_{i,j\geq 0}$ given by (9), where $\{\gamma_{ij}\}_{0\leq j\leq i\leq r}$ are given initial conditions.

The following result is an immediate consequence of (8).

Lemma 3. *Let $\gamma \equiv \{\gamma_{ij}\}_{i,j\geq 0}$ be a doubly indexed sequence and let $p(z,\overline{z}) \in \mathbb{C}[z,\overline{z}]$. Then $p(z,\overline{z})$ is a characteristic of γ if and only if $M(\gamma)p = 0$.*

We use a structural properties of moment matrices to get the following interesting result.

Lemma 4. *Under the notations above, for every $f, g, h \in \mathbb{R}[x_1,\ldots,x_d]$, we have*

$$f^T M(\gamma)(gh) = (fg)^T M(\gamma)h. \tag{10}$$

Proof. Let $f, g, h \in \mathbb{R}[x_1,\ldots,x_d]$ be polynomials. We write $f = \sum_{\mathbf{i}} f_{\mathbf{i}}\mathbf{x}^{\mathbf{i}}$, $g = \sum_{\mathbf{j}} g_{\mathbf{j}}\mathbf{x}^{\mathbf{j}}$ and $h = \sum_{\mathbf{k}} h_{\mathbf{k}}\mathbf{x}^{\mathbf{k}}$. As the entry of the moment matrix corresponding to the column $\mathbf{x}^{\mathbf{i}}$ and the line $\mathbf{x}^{\mathbf{j}}$ is $\gamma_{\mathbf{i}+\mathbf{j}}$, we obtain

$$f^T M(\gamma)(gh) = (\sum_{\mathbf{i}} f_{\mathbf{i}}\mathbf{x}^{\mathbf{i}})^T M(\gamma)(\sum_{\mathbf{j},\mathbf{k}} g_{\mathbf{j}}h_{\mathbf{k}}\mathbf{x}^{\mathbf{j}+\mathbf{k}}) = \sum_{\mathbf{i},\mathbf{j},\mathbf{k}} f_{\mathbf{i}}g_{\mathbf{j}}h_{\mathbf{k}}\gamma_{\mathbf{i}+\mathbf{j}+\mathbf{k}} = (fg)^T M(\gamma)h.$$

The lemma is proved. □

It follows that

Proposition 5. *Let $\gamma \equiv \{\gamma_{ij}\}_{i,j\geq 0}$ be a doubly indexed sequence and let $M(\gamma)$ such that its associated moment matrix $M(\gamma)$ is semidefinite positive. Then, for any polynomial $p \in \mathbb{C}[z,\overline{z}]$ and any integer $n \geq 1$,*

$$M(\gamma)p^n = 0 \Longrightarrow M(\gamma)p = 0. \tag{11}$$

Proof. f $M(\gamma)p^2 = 0$, then $0 = M(\gamma)p^2 = 1^T M(\gamma)p^2 = p^T M(\gamma)p$, from (10); since $M(\gamma) \geq 0$, we obtain $M(\gamma)p = 0$ and hence (11) holds for $n = 2$. By induction, (11) remains valid for any power of 2. Now , if $M(\gamma)p^n = 0$ we choose r in such a way that $r + k$ is a power of 2 to ensure that

$$M(\gamma)p^{n+r} = (p^r)^{\perp} M(\gamma)p^n = 0.$$

Which gives $M(\gamma)p = 0$. □

In the next, we involve the celebrated Hilbert's Nullstellensatz to obtain a very useful result. Let $I_p \equiv (p)$ be the ideal of $\mathbb{C}[z, \overline{z}]$ generated by p. The set $V(I_p) := \{z \in \mathbb{C} \mid f(z) = 0 \text{ for every } z \in I_p\}$ is the (complex) variety associated with I_p. $I(V(I_p)) := \{f(z) \in \mathbb{C}[z, \overline{z}] \mid f(x) = 0 \text{ for every } z \in V(I_p)\}$ and $\sqrt{I_p} := \{f \in \mathbb{C}[z, \overline{z}] \mid f^k \in I_p \text{ for some integer} k \geq 1\}$, are again ideals in $\mathbb{C}[z, \overline{z}]$, that obviously contain the ideal I_p. The Hilbert's Nullstellensatz states that

$$I(V(I_p)) = \sqrt{I_p}.$$

Now let us consider a polynomial $q \in \mathbb{C}[z, \overline{z}]$ satisfying that $Z(p) := \{z \in \mathbb{C} \text{ such that } p(z, \overline{z}) = 0\} \subseteq Z(q)$. We have $q \in I(V(I_p))$ hence $q \in \sqrt{I_p}$, that is, there exists some integer $k \geq 1$ such that $q^k \in I_p$. Thus, by Remark 2-*i*) and Lemma 3, $M(\gamma)q^k = 0$. This implies, by Proposition 5, that $M(\gamma)q = 0$ and therefore, from Lemma 3, q is a characteristic polynomial of γ.

Proposition 6. *Let $\gamma \equiv \{\gamma_{ij}\}_{i,j \geq 0}$ be a RDIS, associated with the characteristic polynomial $p(z, \overline{z})$. Then every polynomial vanishing at all points of $Z(p) := \{z \in \mathbb{C}$ such that $p(z, \overline{z}) = 0\}$ is a characteristic polynomial of γ.*

In the following proposition, we show that for a given RDIS the TCMP and the FCMP are equivalent.

Proposition 7. *Let $\gamma \equiv \{\gamma_{ij}\}_{i,j \geq 0}$ be a RDIS whose initial conditions and characteristic polynomial are $\{\gamma_{ij}\}_{0 \leq i \leq j \leq r}$ and $P(z, \overline{z}) = z^{r+1} - \sum\limits_{0 \leq l+k \leq r} a_{lk} \overline{z}^l z^k$, respectively. The following are equivalent.*

i) There exists a positive Borel measure μ, solution of the FCMP for $\gamma \equiv \{\gamma_{ij}\}_{i,j \geq 0}$.
ii) There exists a positive Borel measure μ, solution of the TCMP for $\omega \equiv \{\gamma_{ij}\}_{0 \leq i \leq j \leq r}$ with

$$supp(\mu) \subset Z(P) := \{z \in \mathbb{C} \text{ such that } P(z, \overline{z}) = 0\}.$$

Proof. *i*) ⇒ *ii*) It suffices to prove that $supp(\mu) \subset Z(P)$.
Write $\gamma_{ij} = \int \overline{z}^i z^j d\mu$, for $i, j \geq 0$. Since, for every $i \geq 0$ and $r \leq n$, we have

$$\gamma_{i,n+1} - \sum_{0 \leq l+k \leq r} a_{lk} \gamma_{l+i,n+k-r} = 0,$$

we get

$$\int \overline{z}^i z^{n+1} - \sum_{0 \leq l+k \leq r} a_{lk} \overline{z}^{l+i} z^{n+k-r} d\mu = 0.$$

Hence

$$\int \overline{z}^i z^{n-r} P(z, \overline{z})\, d\mu = 0 \quad \text{for every} \quad i \geq 0 \quad \text{and} \quad r \leq n.$$

Taking an adequate combination, we manage to obtain

$$\int \overline{P}(z, \overline{z}) P(z, \overline{z})\, d\mu = \int \mid P(z, \overline{z}) \mid^2 d\mu = 0.$$

It follows that $P.\mu = 0$, and thus $supp(\mu) \subset Z(P)$.

ii) $\Rightarrow$ *i)* Suppose that $\gamma_{ij} = \int \overline{z}^i z^j d\mu$, for all integers i, j such that $0 \le j \le i \le r$, and that $\mathrm{supp}(\mu) \subset Z(P)$. Since $\gamma_{ij} = \overline{\gamma}_{ji}$, we also have $\gamma_{ij} = \int \overline{z}^i z^j d\mu$, for every i, j such that $0 \le i, j \le r$. Now

$$\gamma_{i,r+1} = \sum_{0 \le l+k \le r} a_{lk} \gamma_{l+i,k} = \int \overline{z}^i (\sum_{0 \le l+k \le r} a_{lk} \overline{z}^l z^k)\, d\mu,$$

and since $\mathrm{supp}(\mu) \subset Z(z^{r+1} - \sum\limits_{0 \le l+k \le r} a_{lk} \overline{z}^l z^k)$, we get $\gamma_{i,r+1} = \int \overline{z}^i z^{r+1} d\mu$.
By induction we obtain $\gamma_{ij} = \int \overline{z}^i z^j d\mu$ (for all $i, j \ge 0$) and consequently, μ is a solution of the FCMP for $\{\gamma_{ij}\}_{i,j \ge 0}$. $\square$

3. Recursive sequences of Fibonacci type

In this section, we focus ourself on a particular case of RDIS, which will play a crucial role in the sequel.

We shall refer to a RDIS with analytic characteristic polynomial as Recursive Sequences of Fibonacci Type (RSFT for short). In other words, a sequence $\gamma \equiv \{\gamma_{ij}\}_{i,j \ge 0}$, with $\gamma_{00} > 0$, is said to be RSFT, associated with the characteristic polynomial $P(x) = x^r - a_0 x^{r-1} - \ldots - a_{r-2} x - a_{r-1}$, if it verifies the following relations:

i) For all $i, j \ge 0$,

$$\gamma_{ji} = \overline{\gamma}_{ij}. \tag{12}$$

ii) For all i and n such that $0 \le i \le r-1 \le n$,

$$\gamma_{i,n+1} = a_0 \gamma_{i,n} + a_1 \gamma_{i,n-1} + \cdots + a_{r-1} \gamma_{i,n-r+1}. \tag{13}$$

We denote by $\mathcal{A}_\gamma$ the family of analytic characteristic, monic, polynomial associated with γ.

Remarks 8.
- $\mathcal{A}_\gamma \subseteq \mathcal{P}_\gamma$.
- *A sequence* $\gamma \equiv \{\gamma_{ij}\}_{i,j \ge 0}$ *is a RSFT if and only if* $\mathcal{A}_\gamma \ne \emptyset$.

Proposition 9. *Let* $\gamma \equiv \{\gamma_{ij}\}_{i,j \ge 0}$ *be a RSFT, satisfying the relations* (12) *and* (13). *Then* $\mathcal{A}_\gamma$ *is a principal ideal of* $\mathbb{C}[X]$.

Proof. It's obvious that $\mathcal{A}_\gamma$ is an ideal of $\mathbb{C}[X]$. It suffices to show that there exists a unique analytic characteristic polynomial, $P_\gamma \in \mathcal{A}_\gamma$, with minimal degree and that every characteristic polynomial is a multiple of P_γ. Since, for every $j \ge 0$, the polynomial $P(x) = x^r - a_0 x^{r-1} - \ldots - a_{r-2} x - a_{r-1} = \prod\limits_{k=1}^{n} (x - \lambda_i)^{d_i}$ is a characteristic polynomial associated with the singly indexed Fibonacci sequences $\gamma_j = \{\gamma_{ij}\}_{i \ge 0}$, then (by [4, Proposition 3.1]) there exists a unique characteristic polynomial of minimal degree $P_{\gamma_j} = \prod\limits_{i=0}^{n} (x - \lambda_i)^{\alpha_{ij}}$, that divides $P(x)$, associated with γ_j. Since $0 \le \alpha_{ij} \le d_i$, for every $j \ge 0$, then $P_\gamma = \bigwedge\limits_{j \ge 0} P_{\gamma_j} = \prod\limits_{i=0}^{n} (x - \lambda_i)^{\alpha_i}$, where $\alpha_i = \max_{j \ge 0} \alpha_{ij}$, is the smallest common multiple of P_{γ_j} divides P and provides a positive answer to the proposition. $\square$

Let $\gamma \equiv \{\gamma_{ij}\}_{i,j \ge 0}$ be a RSFT, we call P_γ the minimal polynomial associated with γ. Below, we associate with every RSFT its minimal polynomial.

Corollary 10. *Let* $\gamma \equiv \{\gamma_{ij}\}_{i,j \ge 0}$ *be a RSFT, associated with the minimal analytic characteristic polynomial* $P_\gamma \in \mathbb{C}[z]$. *If* $M(\gamma) \ge 0$, *then* P_γ *has distinct roots.*

Proof. Setting $P_\gamma(z) = \prod_{i=1}^{r}(z-\lambda_i)^{n_i}$ for the characteristic polynomial of (γ), we get $M(\gamma)\prod_{i=1}^{r}(z-\lambda_i)^{n_i} = 0$ (by Lemma 3). It follows that $M(\gamma)\prod_{i=1}^{r}(z-\lambda_i)^{m} = 0$, where $m = \max_{i=1}^{r} n^i$, and then from Proposition 5, we have $M(\gamma)\prod_{i=1}^{r}(z-\lambda_i) = 0$. Therefore, again by Lemma 3, the polynomial $\prod_{i=1}^{r}(z-\lambda_i)$ is a characteristic polynomial of γ and divides P_γ. Since P_γ is minimal, then $P_\gamma = \prod_{i=1}^{r}(z-\lambda_i)$. □

We next solve the complex moment problem for a RSFT. Consider the following quadratic forms

$$\begin{array}{rll} & \mathbb{C}[z,\overline{z}] & \to \mathbb{C} \\ \varphi^\gamma : & \displaystyle\sum_{0\le i+j\le n} a_{ij}\overline{Z}^i Z^j & \to \displaystyle\sum_{0\le i+j,h+k\le n} a_{ij}a_{hk}\gamma_{i+k,j+h} \end{array}$$

and

$$\varphi_n^\gamma \equiv \varphi^\gamma_{|\mathbb{C}_n[\overline{Z},Z]}.$$

Let $M(\gamma)$ and $M(n)(\gamma)$ be the matrices associated with φ^γ and φ_n^γ, respectively. We denote $M(n)(\gamma) \in M_m(\mathbb{C})$, where $m = m(n) = \frac{(n+1)(n+2)}{2}$. Let also $\{e_{ij}\}_{0\le i+j\le n}$ be the canonical basis of $\mathbb{C}^m$, that is, e_{ij} is the vector with 1 in the $\overline{Z}^i Z^j$ entry and 0 all other positions.

The next proposition establishes a link between the positivity of φ^γ and that of φ_n^γ.

Proposition 11. *Let γ be a RSFT such that* $\deg P_\gamma = r$ *and let $n \ge 2r-2$. Then φ_n^γ is positive semi-definite if and only if φ^γ is positive semi-definite. Moreover rank φ_n^γ = rank φ^γ.*

Proof. We only need to show the direct implication. To this aim, we construct a matrix $W \in M_{m,n+2}(\mathbb{C})$ such that the successive rows are defined by

$$\overline{Z}^i Z^{2r-1-i} = \sum_{j=0}^{r-1} a_j e_{(i,2r-2-i-j)}, \quad \text{for all} \quad 0 \le i \le r-1,$$

and

$$\overline{Z}^i Z^{2r-1-i} = \sum_{j=0}^{r-1} \overline{a_j} e_{(i-j-1,2r-1-i)}, \quad \text{for all} \quad r \le i \le 2r-1.$$

Clearly $M(n+1)(\gamma)$ has the form $\begin{pmatrix} M(n)(\gamma) & B \\ B^* & C \end{pmatrix}$, with $B = M(n)(\gamma)W$ and $C = B^*W$. Since $M(n)(\gamma) \ge 0$ then, by the Smul'jan's theorem (see [27]), $M(n+1)(\gamma) \ge 0$ and *rank* $M(n)(\gamma) =$ *rank* $M(n+1)(\gamma)$. By induction we obtain $M(\gamma) \equiv M(\infty) \ge 0$ and thus φ^γ is positive semi-definite, as desired. □

We are able now to give a necessary and sufficient condition for a RSFT to be a moment sequence.

Theorem 12. *Let $\gamma \equiv \{\gamma_{ij}\}_{i,j\ge 0}$ be a RSFT, and P_γ of degree r be its minimal analytic characteristic polynomial. Then γ admits a representing measure μ if and only if φ_{2r-2}^γ is positive semi-definite. Moreover*

$$supp(\mu) = Z(P_\gamma).$$

Proof. If γ admits a representing measure μ; then, for any $p \in \mathbb{C}[z,\overline{z}]$, $p^T M(\gamma)p = \int \mid p \mid^2 d\mu \geq 0$. Thus it follows that φ^{γ} is positive semidefinite, and hence φ^{γ}_{2r-2} is positive semi-definite. Conversely, if $M(2r-2)(\gamma) \geq 0$, then using Proposition 11, we get $M(\gamma) \equiv M(\infty)$ is a positive semidefinite. Now let $P_\gamma(z) = \prod_{i=1}^{r}(z-\lambda_i)$ and let $L_{\lambda_j} = \prod_{\substack{1 \leq i \leq r \\ i \neq j}} \frac{z-\lambda_i}{\lambda_j-\lambda_i} \in \mathbb{C}[z,\overline{z}]$ $(i = 1\cdots,r)$ be the interpolation polynomials at the points of $Z(P_\gamma)$. Since the polynomials $Q(z,\overline{z}) = z^n - \sum_{i=1}^{r}\lambda_i^n L_{\lambda_i}$ and $H(z,\overline{z}) = \overline{z}^m - \sum_{j=1}^{r}\overline{\lambda}_j^m \overline{L_{\lambda_j}}$ vanish at all points of $Z(P_\gamma)$, we derive from Lemma 6 that, $Q(z,\overline{z})$ and $H(z,\overline{z})$ are characteristic polynomials of γ. Hence Lemma 3 ensures that $M(\gamma)Q(z,\overline{z}) = M(\gamma)H(z,\overline{z}) = 0$, thus γ_{mn} $(m,n \in \mathbb{Z}_+)$ can be expressed as follows:

$$\begin{aligned}
\gamma_{mn} &= (z^m)^T M(\gamma) z^n \\
&= 1^T M(\gamma)(z^n\overline{z}^m) && \text{by applying (10)} \\
&= 1^T M(\gamma)(\sum_{i=1}^{r}\lambda_i^n L_{\lambda_i}\overline{z}^m) && \text{from Remark 2-i) and Lemma 3} \\
&= 1^T M(\gamma)(\sum_{i=1}^{r}\lambda_i^n L_{\lambda_i}\sum_{j=1}^{r}\overline{\lambda}_j^m \overline{L_{\lambda_j}}) \\
&= 1^T M(\gamma)(\sum_{i,j=1}^{r}\lambda_i^n\overline{\lambda}_j^m L_{\lambda_i}\overline{L_{\lambda_j}}) \\
&= \sum_{i,j=1}^{r}\lambda_i^n\overline{\lambda}_j^m 1^T M(\gamma)(L_{\lambda_i}\overline{L_{\lambda_j}}).
\end{aligned}$$

If $i \neq j$, then $L_{\lambda_i}L_{\lambda_j}$ vanishing at all points of $Z(P_\gamma)$, hence (in this case and from Lemma 6) $L_{\lambda_i}L_{\lambda_j}$ is a characteristic polynomial of γ, that is, $M(\gamma)(L_{\lambda_i}\overline{L_{\lambda_j}}) = 0$. Thus

$$\gamma_{mn} = \sum_{i=1}^{r}\overline{\lambda}_i^m\lambda_i^n 1^T M(\gamma)(L_{\lambda_i}\overline{L_{\lambda_i}}) = \sum_{i=1}^{r}\lambda_i^n\overline{\lambda}_i^m L_{\lambda_i}^T M(\gamma)L_{\lambda_i}.$$

Since $M(\gamma) \geq 0$, we get $c_i = L_{\lambda_i}^T M(\gamma)L_{\lambda_i} \geq 0$ for $i = 0,\ldots,r$. Therefore the measure $\mu = \sum_{i=1}^{r} c_i\delta_{\lambda_i}$ gives a positive answer to the problem (1) associated with γ.

It remains to show that $supp(\mu) = Z(P_\gamma)$, that is, $c_i \neq 0$ for all $i = 0,\ldots,r$. Indeed, if $c_i = 0$ for some $i \in \{1,\ldots,r\}$, then $L_{\lambda_i}^T M(\gamma)L_{\lambda_i} = 0$. Since $M(\gamma) \geq 0$, then $M(\gamma)L_{\lambda_i} = 0$ and from Lemma 3, L_{λ_i} is an analytic characteristic polynomial of γ, and this is a contradiction, because the degree of the polynomial L_{λ_i} is less strictly than the degree of the minimal polynomial P_γ. □

4. Solving the complex moment problem for RDIS

Let $\gamma = \{\gamma_{ij}\}_{i,j\geq 0}$ be a double indexed recursive *moment* sequence, associated with the characteristic polynomial $P(z,\overline{z}) = z^r - \sum_{0\leq l+k\leq r-1} a_{lk}\overline{z}^l z^k$. Proposition 7 ensures the existence of a representing measure μ associated with γ such that $supp(\mu) \subseteq Z(P) := \{\lambda_0,\ldots,\lambda_n\}$. It follows that $Q(z) = \prod_{i=1}^{n}(z-z_i) = z^n - a_1z^{n-1} - \cdots - a_n$ is a characteristic polynomial associated with γ, and since for every $i \geq 0$ and $n \geq r$, we have

$$\gamma_{i,n} - a_1\gamma_{i,n-1} - a_2\gamma_{i,n-2} - \cdots - a_r\gamma_{i,n-r} = \int \overline{z}^i z^{n-r}Q(z)\,d\mu = 0.$$

We conclude that a double indexed recursive *moment* sequence is a RSFT.

In order to obtain necessary and sufficient condition for a RDIS to be moment sequence, we need to find the smallest n which verifies the following equivalence:

$$M(n) \geq 0 \Longleftrightarrow M(\gamma) \equiv M(\infty) \geq 0.$$

Since γ admits a characteristic polynomial of the form $z^r - P_{r-1}(z,\overline{z})$ (with $\deg P_{r-1} \leq r-1$) the best known result in this direction is the one given by Curto-Fialkow [9, Theorem 3.1] which guarantees the equivalence for every n satisfying the inequality $r \leq [\frac{n}{2}] + 1$. In the next theorem we involve the characteristic polynomials $P(z,\overline{z})$ and $Q(z)$ (associated with γ) to refine this result.

Theorem 13. *Let $\gamma \equiv \{\gamma_{ij}\}_{i,j\geq 0}$ be a RDIS associated with the characteristic polynomials $P(z,\overline{z}) = z^r - \sum_{0\leq l+k\leq r-1} a_{lk}\overline{z}^l z^k$ and $Q(z) = \prod_{i=1}^{n}(z-\lambda_i)$, where $Z(P) = \{\lambda_1,\ldots,\lambda_n\}$, then:*

- *i) There exists a characteristic polynomial, $h(z,\overline{z}) \in \mathbb{C}[z,\overline{z}]$, associated with γ such that $Q(z) = h(z,\overline{z}) + f_1(z,\overline{z})P(z,\overline{z}) + f_2(z,\overline{z})\overline{P}(z,\overline{z})$, where $f_1, f_2 \in \mathbb{C}[z,\overline{z}]$, with $d_z(h) < r$ and $d_{\overline{z}}(h) < r$.*
- *ii) Let A_h be the set of monomials $\overline{z}^i z^j$ in h such that $i+j = d_h$, and denote $c_1 := \max\{k \mid \overline{z}^l z^k \in A_h\}$ and $c_1' := \max\{l \mid \overline{z}^l z^k \in A_h\}$. We also define $c_2 := \max\{k \mid \overline{z}^l z^k \in A_h \setminus \{\overline{z}^{d_h - c_1} z^{c_1}\}\}$ and $c_2' := \max\{l \mid \overline{z}^l z^k \in A_h \setminus \{\overline{z}^{c_1'} z^{d_h - c' - 1}\}\}$, if card $A_h \geq 2$. We put for convenience $c_2 = c_2' = -\infty$, in the case where $card(A_h) = 1$. We finally denote $c = sup(c_1, c_1')$, $\alpha_{c_1} = inf(r - c_1, c_1 - c_2)$ and $\alpha_{c_1'} = inf(r - c_1', c_1' - c_2')$. Then*

$$M(\infty) \geq 0 \Longleftrightarrow M(2r-2-\alpha_c) \geq 0.$$

Proof. *i)* It obvious to show that there exists $h(z,\overline{z})$ such that

$$Q(z) = h(z,\overline{z}) + f_1(z,\overline{z})P(z,\overline{z}) + f_2(z,\overline{z})\overline{P}(z,\overline{z})$$

for some $f_1, f_2 \in \mathbb{C}[z,\overline{z}]$, with $d_z(h) < r$ and $d_{\overline{z}}(h) < r$. Since $P, \overline{P}$ and Q are characteristic polynomials then $h(z,\overline{z})$ is also a characteristic polynomial associated with (γ).

ii) Since the two conditions are symmetric, we only need to prove the case $c_1 \geq c_1'$. Recall first that for $m \geq 0$, $M_m(\mathbb{C})$ denotes the algebra of $m \times m$ complex matrices and for $n \geq 0$, let $m \equiv m(n) := (n+1)(n+2)/2$ and $M(n)(\gamma) \in M_m(\mathbb{C})$, as in the introduction. We define a basis $\{e_{ij}\}_{0\leq i+j\leq n}$ of $\mathbb{C}^m$ as follows: e_{ij} is the vector with 1 in the $\overline{Z}^i Z^j$ entry and 0 all other positions.

The main idea is to write, from the characteristic polynomials associated with γ, monomials of order $2r-1-\alpha_c+e-1$ ($e \in \mathbb{N}$) as linear combination of monomials of order strictly less than $2r-1-\alpha_c+e$ modulo $P, \overline{P}, h$ and $\overline{h}$, that is,

$$\overline{z}^i z^{2r-1-\alpha_c+e-i} = H_{(i,2r-1-\alpha_c+e-i)}(z,\overline{z}) \bmod \{P, \overline{P}, h, \overline{h}\},$$

with $0 \leq i \leq 2r-1-\alpha_c+e$ and $\deg H_{(i,2r-1-\alpha_c+e-i)}(z,\overline{z}) \leq 2r-2-\alpha_c$.

We construct for every $e \in \mathbb{Z}_+$ a matrix $W_e \in M_{m(2r-2-\alpha_c+e),2r-1-\alpha_c+e}(\mathbb{C})$ such that the coefficients of the column $\overline{Z}^i Z^{2r-1-\alpha_c+e-i}$, $0 \leq i \leq 2r-1-\alpha_c+e$, are that of the polynomial $H_{(i,2r-1-\alpha_c+e-i)}$ (considering the lexicographic order cited in the introduction).

Since $P, \overline{P}, h$ and $\overline{h}$ are characteristic polynomials associated with γ, the above discussion leads, in view of Lemma 3, to the following equality:

$$M(2r-1-\alpha_c+e)(\gamma) = \begin{pmatrix} M(2r-2-\alpha_c+e)(\gamma) & B \\ B^* & C \end{pmatrix},$$

such that $B = M(2r-2-\alpha_c+e)(\gamma)W_e$ and $C = B^*W_e$, for all $e \in \mathbb{N}$. According to Smul'jan's theorem we have

$$M(2r-2-\alpha_c+e)(\gamma) \geq 0 \Longleftrightarrow M(2r-1-\alpha_c+e)(\gamma) \geq 0,$$

and hence it follows by induction that

$$M(2r-2-\alpha_c)(\gamma) \geq 0 \Longleftrightarrow M(\infty)(\gamma) \geq 0.$$

We distinguish 3 cases,

- $r \leq i \leq 2r-1-\alpha_c+e$. Since $\overline{z}^r = \overline{P} + \sum_{0 \leq l+k \leq r-1} \overline{a}_{lk}\overline{z}^k z^l$, we obtain

$$\overline{z}^i z^{2r-1-\alpha_c+e-i} = \overline{z}^{i-r} z^{2r-1-\alpha_c+e-i}\overline{P} + \sum_{0 \leq l+k \leq r-1} \overline{a}_{lk}\overline{z}^{k+i-r} z^{l+2r-1-\alpha_c+e-i}.$$

  Hence

$$\overline{Z}^i Z^{2r-1-\alpha_c+e-i} = \sum_{0 \leq l+k \leq r-1} \overline{a}_{lk} e_{(k+i-r,l+2r-1-\alpha_c+e-i)}.$$

- $0 \leq i \leq r-1-\alpha_c$. As $z^r = P + \sum_{0 \leq l+k \leq r-1} a_{lk}\overline{z}^l z^k$, we get

$$\overline{z}^i z^{2r-1-\alpha_c+e-i} = \overline{z}^i z^{r-1-\alpha_c+e-i} P + \sum_{0 \leq l+k \leq r-1} a_{lk}\overline{z}^{l+i} z^{k+r-1-\alpha_c+e-i},$$

  and thus,

$$\overline{Z}^i Z^{2r-1-\alpha_c+e-i} = \sum_{0 \leq l+k \leq r-1} a_{lk} e_{(l+i,k+r-1-\alpha_c+e-i)}.$$

- $r-\alpha_c \leq i \leq r-1$. This third case requires more work. We will distinguish two subcases.

  a) Card $A = 1$. (In this case we have $\alpha_c = r - c_1$). Let

$$h(z,\overline{z}) = \overline{z}^{d_h-c_1} z^{c_1} - \sum_{0 \leq l+k \leq d_h-1} a_{lk}\overline{z}^l z^k,$$

  we have

$$\overline{z}^{d_h-c_1} z^{c_1} = h(z,\overline{z}) + \sum_{0 \leq l+k \leq d_h-1} a_{lk}\overline{z}^l z^k$$

  and since $d_h \leq c_1 + c_1' \leq 2c_1$, we get

$$\begin{array}{rcl} \overline{z}^{r+c_1+e-1-i} z^i & = & \overline{z}^{r+2c_1+e-1-i-d_h} z^{i-c_1}(\overline{z}^{d_h-c_1} z^{c_1}), \\ & = & \overline{z}^{r+2c_1+e-1-i-d_h} z^{i-c_1} h(z,\overline{z}) \\ & & + \sum_{0 \leq l+k \leq d_h-1} a_{lk}\overline{z}^{l+r+2c_1+e-1-i-d_h} z^{k+i-c_1}. \end{array}$$

  Hence,

$$\overline{Z}^{r+c_1+e-1-i} Z^i = \sum_{0 \leq l+k \leq d_h-1} a_{lk} e_{(l+r+2c_1+e-1-i-d_h,k+i-c_1)},$$

  and it follows that,

$$\overline{Z}^{2r-1-\alpha_c+e-i} Z^i = \sum_{0 \leq l+k \leq d_h-1} a_{lk} e_{(l+3r-2\alpha_c+e-1-i-d_h,k+i-r+\alpha_c)}.$$

  b) Card $A \geq 2$. Let

$$\overline{z}^{d_h-c_1} z^{c_1} = h(z,\overline{z}) + \sum_{l+k=d_h} \alpha_{lk}\overline{z}^l z^k + \sum_{0 \leq l+k \leq d_h-1} a_{lk}\overline{z}^l z^k,$$

we deduce that,

$$\begin{aligned}\overline{z}^{c_2-c_1+r}z^{c_1} &= (\overline{z}^{r-d_h+c_2})(\overline{z}^{d_h-c_1}z^{c_1})\\ &= \overline{z}^{r-d_h+c_2}h + \sum_{l+k=d_h}\alpha_{lk}\overline{z}^{l+r-d_h+c_2}z^k\\ &+ \sum_{0\leq l+k\leq d_h-1} a_{lk}\overline{z}^{l+r-d_h+c_2}z^k.\end{aligned}$$

Since in the above equality, all monomials in the sum $\sum_{l+k=d_h}\alpha_{lk}\overline{z}^{r+l-d_h+c_2}z^k$ satisfy $d_{\overline{z}}(\overline{z}^{l+r-d_h+c_2}z^k)\geq r$ (since $k\leq c_2$ and $l+k=d_h$, then $l-d_h+c_2\geq 0$), we get

$$\sum_{l+k=d_h}\alpha_{lk}\overline{z}^{l+r-d_h+c_2}z^k = \overline{P}\sum_{l+k=d_h}\overline{z}^{l-d_h+c_2}z^k - \sum_{0\leq l'+k'\leq r+c_2-1} a_{l'k'}\overline{z}^{l'}z^{k'},$$

where $\{a_{l'k'}\}_{0\leq l'+k'\leq r+c_2-1}$ are complex numbers. Thus, there exists $\alpha_{l''k''}\in\mathbb{C}$ such that

$$\overline{z}^{c_2-c_1+r}z^{c_1} = \sum_{0\leq l''+k''\leq r+c_2-1}\alpha_{l''k''}\overline{z}^{l''}z^{k''} mod(\overline{P},h). \tag{14}$$

Here again we discuss two situations,
*) If $r-c_1\leq c_1-c_2$ (that is, $\alpha_c = r-c_1$).
Then $0\leq c_1-r+c_1-c_2\leq i-r+c_1-c_2$ and $0\leq c_2-c_1+r$ (since $c_2\leq c_1\leq i\leq r$), hence (14) yields

$$\begin{aligned}\overline{z}^{i}z^{r+c_1+e-1-i} &= (\overline{z}^{i-c_2+c_1-r}z^{r+e-1-i})(\overline{z}^{c_2-c_1+r}z^{c_1})\\ &= \sum_{0\leq l''+k''\leq r+c_2-1}\alpha_{l''k''}\overline{z}^{l''+i-c_2+c_1-r}z^{k''+r+e-1-i} mod(\overline{P},h).\end{aligned}$$

Then

$$\overline{Z}^{i}Z^{r+c_1+e-1-i} = \sum_{0\leq l''+k''\leq r+c_2-1}\alpha_{l''k''}e_{(l''+i-c_2+c_1-r,k''+r+e-1-i)},$$

that is,

$$\overline{Z}^{i}Z^{2r-1-\alpha_c+e-i} = \sum_{0\leq l''+k''\leq r+c_2-1}\alpha_{l''k''}e_{(l''+i-c_2+c_1-r,k''+r+e-1-i)}.$$

**) If $r-c_1 > c_1-c_2$ (that is, $\alpha_c = c_1-c_2$).
By (14), we get

$$\begin{aligned}\overline{z}^{2r-c_1+c_2+e-1-i}z^{i} &= (\overline{z}^{r+e-1-i}z^{i-c_1})(\overline{z}^{r-c_1+c_2}z^{c_1})\\ &= \sum_{0\leq l''+k''\leq r+c_2-1}\alpha_{l''k''}\overline{z}^{l''+r+e-1-i}z^{k''+i-c_1} mod(\overline{P},h),\end{aligned}$$

and then

$$\overline{z}^{i}z^{2r-1-\alpha_c+e-i} = \sum_{0\leq l''+k''\leq r+c_2-1}\overline{\alpha}_{l''k''}\overline{z}^{k''+i-c_1}z^{l''+r+e-1-i} mod(P,\overline{h}).$$

This finally gives

$$\overline{Z}^{i}Z^{2r-1-\alpha_c+e-i} = \sum_{0\leq l''+k''\leq r+c_2-1}\overline{\alpha}_{l''k''}e_{(k''+i-c_1,l''+r+e-1-i)},$$

as required.

$\square$

Corollary 14. *Let* $\gamma \equiv \{\gamma_{ij}\}_{i,j\geq 0}$ *be a RDIS associated with the characteristic polynomial* $P(z,\overline{z}) = z^r - \sum\limits_{0\leq l+k\leq r-1} a_{lk}\overline{z}^l z^k$, *then*

$$M(\infty)(\gamma) \geq 0 \Longleftrightarrow M(2r-2)(\gamma) \geq 0.$$

Proof. It suffices to rewrite the first and second cases, in the above theorem's proof, with $\alpha_c = 0$. □

In the next theorem and in the sequel, we set $\xi \equiv \xi_\gamma = 2r - 2 - \alpha_c$.

Theorem 15. *Let* $\gamma \equiv \{\gamma_{ij}\}_{i,j\geq 0}$ *be a RDIS associated with the characteristic polynomial* $P(z,\overline{z}) = z^r - \sum\limits_{0\leq l+k\leq r-1} a_{lk}\overline{z}^l z^k$ *and let* $Z(P) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$. *The following are equivalent:*

- γ *is a moment sequence;*
- $Q(z) = \Pi_{i=1}^n (z - \lambda_i) \in \mathcal{P}_\gamma$ *and* $M(\xi) \geq 0$.

Proof. Suppose that γ admits a representing measure μ.

Then, from Proposition 7, $\mathrm{supp}(\mu) \subset Z(P) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$. Setting $Q(z) = \Pi_{i=1}^n (z - \lambda_i) = z^n - \alpha_1 z^{n-1} - \cdots - \alpha_n$, we have

$$\begin{aligned}\gamma_{im} &= \int \overline{z}^i z^m d\mu \qquad\qquad \text{where} \quad 0 \leq i \quad \text{and} \quad n \leq m \\ &= \int \overline{z}^i (\alpha_1 z^{m-1} + \alpha_2 z^{m-2} + \cdots + \alpha_n z^{m-n}) d\mu \\ &= \alpha_1 \gamma_{i,m-1} + \alpha_2 \gamma_{i,m-2} + \cdots + \alpha_n \gamma_{i,m-n}.\end{aligned}$$

Hence $Q(z)$ is a characteristic polynomial associated with γ. The condition $M(\xi) \geq 0$ is obvious. Conversely, since $M(\gamma)(\xi) \geq 0$ then, from Theorem 13, $M(\gamma)(\infty) \geq 0$ and thus, from Theorem 12, the sequence γ owns a representing measure. □

Corollary 16. *Let* $\gamma \equiv \{\gamma_{ij}\}_{i,j\geq 0}$ *be a RDIS associated with the characteristic polynomial* $P(z,\overline{z}) = z^r - \sum\limits_{0\leq l+k\leq r-1} a_{lk}\overline{z}^l z^k$, *then* γ *is a moment sequence if and only if* $M(2r-2)(\gamma) \geq 0$.

Proof. We only need to show the converse implication. As noted in the introduction the polynomial $P(z,\overline{z})$ has finite number of roots, say $Z(P) := \{\lambda_1, \lambda_2, \dots, \lambda_n\}$. Since $M(2r-2)(\gamma) \geq 0$, Corollary 14 yields $M(\infty)(\gamma) \geq 0$. Let $Q(z) = \Pi_{i=1}^n (z - \lambda_i)$, applying Proposition 6, we obtain $Q(z,\overline{z}) \in \mathcal{P}_\gamma$. Therefore, Theorem 15 implies that γ is a moment sequence. □

5. The case of cubic moment problem

5.1. **The TCMP with cubic relation of the form $z^3 + az + b\overline{z} = 0$.** Whenever we have the zeros of a characteristic polynomial associated with a RDIS, Theorem 15 allows us to give a concrete, computable, necessary and sufficient conditions for the existence of a representing measure associated with this sequence. In this section we apply the above mentioned theorem to solve the complex moment problem for a RDIS, associated with a harmonic characteristic polynomial of the form $z^3 + az + b\overline{z}$, where $a, b \in \mathbb{R}$. First, we start by giving the number of zeros of the harmonic polynomial $P(z,\overline{z}) = z^3 + az + b\overline{z}$. We solve the equation $z^3 + az + b\overline{z} = 0$, for completeness. Writing $z = x + iy$, we get $(x+iy)^3 + a(x+iy) + b(x-iy) = 0$, and then $x^3 - 3xy^2 + (a+b)x - i(y^3 - (3x^2 + a + d)y) = 0$. It follows that

$$\begin{cases} x(x^2 - 3y^2 + a + b) &= 0, \\ y(y^2 - (3x^2 + a - b)) &= 0. \end{cases}$$

- If $y = 0$, then $x = 0$ or $x^2 = -a - b$.
- If $x = 0$, then $y = 0$ or $y^2 = a - b$.
- If $xy \neq 0$, then $x^2 = \frac{-a+2b}{4}$ and $y^2 = \frac{a+2b}{4}$.

We deduce from the above cases that $z^3 + az + b\overline{z}$ has at most 7 roots and it has exactly 7 roots if and only if $b < |a| < 2b$.

Let $\lambda_1, \lambda_2, \ldots, \lambda_7$ be the roots of the polynomial $P(z, \overline{z}) = z^3 + az + b\overline{z}$, with $b < -a < 2b$. Direct computations lead to the expression

$$\begin{aligned}
Q(z) &= \Pi_{i=1}^{7}(z - \lambda_i) \\
&= z^7 + (2a + b)z^5 + (a^2 + b^2 + ab)z^3 + (b^3 + ab^2)z \\
&= (z^4 + (a + b)z^2 - b\overline{z}z + b^2)P(z, \overline{z}) - b^2(\overline{z}z^2 - \overline{z}^2 z - bz + b\overline{z}) \\
&= (z^4 + (a + b)z^2 - b\overline{z}z + b^2)P(z, \overline{z}) - b^2 h(z, \overline{z}).
\end{aligned}$$

Now we are able to solve the moment problem of this section, in the case $b < -a < 2b$. Recall again that, $1, Z, \overline{Z}, \ldots, Z^n, \ldots, \overline{Z}^n$ denote the successive columns of $M(\omega)(n)$.

Theorem 17. *Let $\omega \equiv \{\gamma_{ij}\}_{0\leq i+j\leq 6}$, with $\overline{\gamma}_{ij} = \gamma_{ji}$ and $\gamma_{00} > 0$, be a truncated complex sequence, let $M(\omega)(3)$ be its associated moment matrix and let Λ_ω be as in (4). If $M(\omega)(3) \geq 0$ and has cubic column relation of the form $Z^3 = -aZ - b\overline{Z}$, with $a, b \in \mathbb{R}$ and $b < -a < 2b$. Then the following statements are equivalent:*

i) *There exists a representing measure for ω.*

ii) *There exists a representing measure for the RDIS $\gamma \equiv \{\gamma_{ij}\}_{i,j\geq 0}$, whose initial conditions and characteristic polynomial are $\{\gamma_{ij}\}_{0\leq i\leq j\leq 2}$ and $P(z, \overline{z}) = z^3 + az + b\overline{z}$, respectively .*

iii)
$$\begin{cases} \Lambda_\omega(h) &= 0, \\ \Lambda_\omega(zh) &= 0. \end{cases}$$

iv)
$$\begin{cases} \mathfrak{Im}\gamma_{12} &= b\mathfrak{Im}\gamma_{01}, \\ \gamma_{22} + 2b\mathfrak{Re}\gamma_{20} + (a - b)\gamma_{11} &= 0. \end{cases}$$

v) *$h(z, \overline{z}) = \overline{z}z^2 - \overline{z}^2 z - bz + b\overline{z} \in \mathcal{P}_\gamma$, where γ is the RDIS defined in ii).*

vi) $\overline{Z}Z^2 - \overline{Z}^2 Z - bZ + b\overline{Z} = 0$.

Proof. It is straightforward to see that $iii) \Leftrightarrow iv)$, $ii) \Rightarrow i)$ and $v) \Rightarrow vi) \Rightarrow iii)$. Thus, it is enough to show $i) \Rightarrow ii)$, $iii) \Rightarrow v)$ and $v) \Leftrightarrow ii)$.

$i) \Longrightarrow ii)$. Suppose that ω admits a representing measure μ. Because the cubic column in $M(\omega)(3)$ verifies $Z^3 = -aZ - b\overline{Z}$, we get the following relations:

$$\begin{aligned}
\int \overline{z}^3 P(z, \overline{z})\, d\mu = \int \overline{z}^3(z^3 + az + b\overline{z})\, d\mu &= \gamma_{3,3} + a\gamma_{3,1} + b\gamma_{4,0} = 0, \\
\int z P(z, \overline{z})\, d\mu = \int z(z^3 + az + b\overline{z})\, d\mu &= \gamma_{0,4} + a\gamma_{0,2} + b\gamma_{1,1} = 0, \\
\int \overline{z} P(z, \overline{z})\, d\mu = \int \overline{z}(z^3 + az + b\overline{z})\, d\mu &= \gamma_{1,3} + a\gamma_{1,1} + b\gamma_{2,0} = 0,
\end{aligned}$$

thus $\int |\ P\ |^2\ d\mu = \int(\overline{z}^3 + a\overline{z} + bz)P\, d\mu = 0$ hence $supp(\mu) \subseteq Z(P)$. It follows, from Proposition 7, that μ is a representing measure for γ.

$iii) \Rightarrow v)$. It suffices to show that $\Lambda_\gamma(\overline{z}^i z^j h(z,\overline{z})) = 0$, for all $i, j \geq 0$.
Remark that, whenever $z^3 = -az - b\overline{z}$, we have

$$\begin{aligned} z^2 h &= (b-a)h, \\ \overline{z}zh &= (a-b)h, \\ \overline{h} &= -h. \end{aligned}$$

Then

$$\begin{aligned} &\Lambda_\gamma(z^2h) = (b-a)\Lambda_\omega(h) = 0, \\ &\Lambda_\gamma(\overline{z}zh) = (a-b)\Lambda_\omega(h) = 0, \\ &\Lambda_\gamma(\overline{h}) = -\Lambda_\omega(h) = 0, \\ &\Lambda_\gamma(\overline{z}h) = \overline{\Lambda_\omega(z\overline{h})} = -\overline{\Lambda_\omega(zh)} = 0, \\ &\Lambda_\gamma(\overline{z}z^2h) = (b-a)\Lambda_\omega(\overline{z}h) = -(b-a)\overline{\Lambda_\omega(zh)} = 0. \end{aligned}$$

Since γ is a RDIS associated with the characteristic polynomial $z^3 + az + b\overline{z}$, then, for every $i, j \in \mathbb{Z}_+$, we have

$$\Lambda_\gamma(\overline{z}^i z^j h(z,\overline{z})) = \sum_{0\leq l,k\leq 2} a_{lk}\Lambda_\omega(\overline{z}^l z^k h(z,\overline{z})) = 0,$$

where $\{a_{lk}\}_{0\leq l,k\leq 2}$ are real numbers.

$v) \Rightarrow ii)$. We know that $Q(z) = (z^4 + (a+b)z^2 - b\overline{z}z + b^2)P(z,\overline{z}) - b^2h(z,\overline{z})$. Since $P(z,\overline{z})$ and $h(z,\overline{z})$ are characteristic polynomials, $Q(z)$ is also a characteristic polynomial associated with γ. As $M(\xi)(\gamma) \equiv M(3) \geq 0$ (observe that $\xi = 2 \times 3 - 2 + 1 = 3$) and $Q(z) \in \mathcal{P}_\gamma$, then Theorem 15 yields that γ admits a representing measure.

$ii) \Rightarrow v)$. We have

$$h(z,\overline{z}) = \frac{1}{b^2}(z^4 + (a+b)z^2 - b\overline{z}z + b^2)P(z,\overline{z}) - \frac{1}{b^2}Q(z),$$

since (by Theorem 15) $Q(z)$ is a characteristic polynomial of γ, as well as the polynomial $P(z,\overline{z})$, then $h(z,\overline{z})$ is a characteristic polynomial associated with γ (by Remark 2-i)). □

Let us now suppose that $b < a < 2b$, then

$$\begin{aligned} Q(z) &= \Pi_{i=1}^7 (z - \lambda_i) \\ &= z^7 + (2a-b)z^5 + (a^2 + b^2 - ab)z^3 + (ab^2 - b^3)z \\ &= (z^4 + (a-b)z^2 - b\overline{z}z + b^2)P(z,\overline{z}) + b^2(z+\overline{z})(\overline{z}z - b). \end{aligned}$$

Let $h(z,\overline{z}) = (z+\overline{z})(\overline{z}z - b)$, we have

$$\begin{aligned} z^2h(\overline{z},z) &= (\overline{z}z + \overline{z}^2 - b)P(z,\overline{z}) - b\overline{P}(\overline{z},z) - (a+b)h(\overline{z},z), \\ z\overline{z}h(\overline{z},z) &= (z^2 - b)\overline{P}(\overline{z},z) + (\overline{z}^2 - b)P(\overline{z},z) - (a+b)h(\overline{z},z), \\ \overline{h} &= h. \end{aligned}$$

If $\Lambda_\omega(h) = \Lambda_\omega(zh) = 0$, we get

$$\begin{aligned} \Lambda_\gamma(\overline{h}) &= \Lambda_\omega(h) &&= 0, \\ \Lambda_\gamma(\overline{z}h) &= \overline{\Lambda_\gamma(z\overline{h})} = \overline{\Lambda_\omega(zh)} &&= 0, \\ \Lambda_\gamma(z\overline{z}h) &= -(a+b)\Lambda_\omega(h) &&= 0, \\ \Lambda_\gamma(z^2h) &= -(a+b)\Lambda_\omega(h) &&= 0, \\ \Lambda_\gamma(\overline{z}z^2h) &= -(a+b)\overline{\Lambda_\omega(zh)} &&= 0, \\ \Lambda_\gamma(\overline{z}^2zh) &= -(a+b)\Lambda_\omega(zh) &&= 0. \end{aligned}$$

The above equalities and cubic relation allow us to prove that $\Lambda_\gamma(z^i\overline{z}^j h) = 0$, for all $i,j \in \mathbb{Z}_+$; that is, $h(\overline{z},z) \in \mathcal{P}_\gamma$. From this observation and similarly to the above theorem's proof, we are able to state the following theorem.

Theorem 18. *Let $\omega \equiv \{\gamma_{ij}\}_{0\leq i+j\leq 6}$, with $\overline{\gamma}_{ij} = \gamma_{ji}$ and $\gamma_{00} > 0$, be a given truncated complex sequence, let $M(\omega)(3)$ be its associated moment matrix and let Λ_ω be as in (4). If $M(\omega)(3) \geq 0$ and has cubic column relation of the form $Z^3 = -aZ - b\overline{Z}$, with $a,b \in \mathbb{R}$ and $b < a < 2b$. Then the following statements are equivalent:*

i) *There exists a representing measure for ω.*

ii) *There exists a representing measure for the RDIS $\gamma \equiv \{\gamma_{ij}\}_{i,j\geq 0}$, whose initial conditions and characteristic polynomial are $\{\gamma_{ij}\}_{0\leq i\leq j\leq 2}$ and $P(z,\overline{z}) = z^3 + az + b\overline{z}$, respectively.*

iii)

$$\begin{cases} \Lambda_\omega(h) &= 0, \\ \Lambda_\omega(zh) &= 0. \end{cases}$$

iv)

$$\begin{cases} \mathfrak{Re}\gamma_{12} &= b\mathfrak{Re}\gamma_{01}, \\ \gamma_{22} &= 2b\mathfrak{Re}\gamma_{20} + (a+b)\gamma_{11}. \end{cases}$$

v) *$h(z,\overline{z}) = (z+\overline{z})(z\overline{z}-b) \in \mathcal{P}_\gamma$, where γ is the RDIS defined in ii).*

vi) *$(Z+\overline{Z})(Z\overline{Z}-b) = 0$.*

By the same technique we treat the other cases, we find the following results.

$b>0$	N	Zeros	$h(z,\overline{z})$	Necessary and sufficient conditions	
$2b \leq a$	3	$\pm i\sqrt{a-b}$ 0	$z+\overline{z}$	$M(2)\geq 0$ $\Lambda_\gamma(h)=0$ $\Lambda_\gamma(zh)=0$ $\Lambda_\gamma(z^2h)=0$	$M(2)\geq 0$ $\mathfrak{Im}\gamma_{01}=0$ $\gamma_{11}+\gamma_{02}=0$ $\gamma_{12}=a\gamma_{01}+b\gamma_{10}$
$b<a<2b$	7	$\pm i\sqrt{a-b}$; $\pm\frac{\sqrt{-a+2b}}{2}$ $\pm i\frac{\sqrt{a+2b}}{2}$; 0.	$(z+\overline{z})(z\overline{z}-u)$	$M(3)\geq 0$ $\Lambda_\gamma(h)=0$ $\Lambda_\gamma(zh)=0$	$M(3)\geq 0$ $\mathfrak{Re}\gamma_{12}=b\mathfrak{Re}\gamma_{01}$ $\gamma_{22}=2b\mathfrak{Re}\gamma_{20}+(a+b)\gamma_{11}$
$-b\leq a\leq b$	1	0			$\gamma_{00}>0$ and $\gamma_{ij}=0$ for all $0\leq i\leq j\leq 2$
$b<-a<2b$	7	$\pm\sqrt{-a-b}$; $\pm\frac{\sqrt{-a+2b}}{2}$ $\pm i\frac{\sqrt{a+2b}}{2}$; 0.	$(z-\overline{z})(z\overline{z}-u)$	$M(3)\geq 0$ $\Lambda_\gamma(h)=0$ $\Lambda_\gamma(zh)=0$	$M(3)\geq 0$ $\mathfrak{Im}\gamma_{12}=b\mathfrak{Im}\gamma_{01}$ $\gamma_{22}+2b\mathfrak{Re}\gamma_{20}=(b-a)\gamma_{11}$
$a\leq -2b$	3	$\pm\sqrt{-a-b}$ 0	$z-\overline{z}$	$M(2)\geq 0$ $\Lambda_\gamma(h)=0$ $\Lambda_\gamma(zh)=0$ $\Lambda_\gamma(z^2h)=0$	$M(2)\geq 0$ $\gamma_{01}=\gamma_{10}$ $\gamma_{02}=\gamma_{11}$ $a\gamma_{01}+b\gamma_{10}+\gamma_{12}=0$

TABLE 1. b positive

$b<0$	N	Zeros	$h(z,\overline{z})$	Necessary and sufficient conditions	
$-b \leq a$	3	$\pm i\sqrt{a-b}$ 0	$z+\overline{z}$	$M(2)\geq 0$ $\Lambda_\gamma(h)=0$ $\Lambda_\gamma(zh)=0$ $\Lambda_\gamma(z^2h)=0$	$M(2)\geq 0$ $\mathfrak{Im}\gamma_{01}=0$ $\gamma_{11}+\gamma_{02}=0$ $\gamma_{12}=a\gamma_{01}+b\gamma_{10}$
$\lvert a\rvert<-b$	5	$\pm i\sqrt{a-b}$ $\pm\sqrt{-a-b}$ 0	$z^2\overline{z}+a\overline{z}+bz$	$M(3)\geq 0$ $\Lambda_\gamma(h)=0$ $\Lambda_\gamma(zh)=0$ $\Lambda_\gamma(\overline{z}h)=0$	$M(3)\geq 0$ $\gamma_{21}+a\gamma_{01}+b\gamma_{10}=0$ $\gamma_{20}=\gamma_{02}$ $\gamma_{22}+a\gamma_{01}+b\gamma_{10}=0$
$a\leq b$	3	$\pm\sqrt{-a-b}$ 0	$z-\overline{z}$	$M(2)\geq 0$ $\Lambda_\gamma(h)=0$ $\Lambda_\gamma(zh)=0$ $\Lambda_\gamma(z^2h)=0$	$M(2)\geq 0$ $\gamma_{01}=\gamma_{10}$ $\gamma_{02}=\gamma_{11}$ $a\gamma_{01}+b\gamma_{10}+\gamma_{12}=0$

TABLE 2. b negative

5.2. **The TCMP with cubic relation in $M(3)$ of the form $Z^3 = itZ + u\overline{Z}$.** We end this section by considering another class of cubic column relations in truncated moment problems.

Set $w = e^{-i\frac{\pi}{4}}z$, the form $w^3 = itw + u\overline{w}$ became $z^3 + tz + u\overline{z} = 0$.

We take $t = a$ and $u = b$, then $w^3 = itw + u\overline{w}$ owns exactly 7 roots if and only if $u < |t| < 2u$, see Table (1). Hence if $u \leq 0$ then rank $M(3) < 7$ (more precisely rank $M(3) \leq 5$); as noted in the introduction this case is not interesting.

In view of Theorems 17 and 18 we deduce the solution of the TCMP for cubic column relations in $M(3)$ of the form $Z^3 = itZ + u\overline{Z}$, where $u < |t| < 2u$.

If $0 < u < t < 2u$, then

$$\begin{aligned} &h(z,\overline{z}) = 0,\\ &(z+\overline{z})(z\overline{z}-u) = 0,\\ &(e^{\frac{\pi}{4}i}w + e^{-\frac{\pi}{4}i}\overline{w})(\overline{w}w-u) = 0,\\ &i(w-i\overline{w})(\overline{w}w-u) = 0.\end{aligned}$$

If $0 < u < -t < 2u$, then

$$\begin{aligned} &h(z,\overline{z}) = 0,\\ &(z-\overline{z})(z\overline{z}-u) = 0,\\ &(e^{\frac{\pi}{4}i}w - e^{-\frac{\pi}{4}i}\overline{w})(\overline{w}w-u) = 0,\\ &i(w+i\overline{w})(\overline{w}w-u) = 0.\end{aligned}$$

Now we are able to state the main theorem in [13].

Theorem 19. *Let $\omega \equiv \{\gamma_{ij}\}_{0\leq i+j\leq 6}$, with $\overline{\gamma}_{ij} = \gamma_{ji}$ and $\gamma_{00} > 0$, be a truncated complex sequence, let $M(\omega)(3)$ be its associated moment matrix and let $\Lambda_\omega(h)$ be as in (4). If $M(\omega)(3) \geq 0$ and has cubic column relation of the form $Z^3 = itZ - u\overline{Z}$, with $t, u \in \mathbb{R}$ and $u < t < 2u$. Then the following statements are equivalent:*

i) There exists a representing measure for ω.

ii) There exists a representing measure for the RDIS $\gamma \equiv \{\gamma_{ij}\}_{i,j\geq 0}$, whose initial conditions and characteristic polynomial are $\{\gamma_{ij}\}_{0\leq i\leq j\leq 2}$ and $P(z,\overline{z}) = z^3 - itz - u\overline{z}$, respectively .

iii)

$$\begin{cases} \Lambda_\omega(h) &= 0, \\ \Lambda_\omega(zh) &= 0. \end{cases}$$

iv)

$$\begin{cases} \mathfrak{Re}\gamma_{12} - \mathfrak{Im}\gamma_{12} &= u(\mathfrak{Re}\gamma_{01} - \mathfrak{Im}\gamma_{01}), \\ \gamma_{22} &= (t+u)\gamma_{11} - 2u\mathfrak{Im}\gamma_{02}. \end{cases}$$

v) $h(z,\overline{z}) = i(z - i\overline{z})(z\overline{z} - u) \in \mathcal{P}_\gamma$, *where* γ *is the RDIS defined in ii).*

vi) $Z^2\overline{Z} - iZ\overline{Z}^2 - uZ + iu\overline{Z} = 0.$

Theorem 20. *Let* $\omega \equiv \{\gamma_{ij}\}_{0\le i+j\le 6}$, *with* $\overline{\gamma}_{ij} = \gamma_{ji}$ *and* $\gamma_{00} > 0$, *be a truncated complex sequence, let* $M(\omega)(3)$ *be its associated moment matrix and let* Λ_ω *be as in* Λ_ω. *If* $M(\omega)(3) \ge 0$ *and has cubic column relation of the form* $Z^3 = itZ - u\overline{Z}$, *with* $t,u \in \mathbb{R}$ *and* $u < -t < 2u$. *Then the following statements are equivalent:*

i) *There exists a representing measure for* ω.

ii) *There exists a representing measure for the RDIS* $\gamma \equiv \{\gamma_{ij}\}_{i,j\ge 0}$, *whose initial conditions and characteristic polynomial are* $\{\gamma_{ij}\}_{0\le i\le j\le 2}$ *and* $P(z,\overline{z}) = z^3 - itz - u\overline{z}$, *respectively .*

iii)

$$\begin{cases} \Lambda_\omega(h) &= 0, \\ \Lambda_\omega(zh) &= 0. \end{cases}$$

iv)

$$\begin{cases} \mathfrak{Re}\gamma_{12} + \mathfrak{Im}\gamma_{12} &= u(\mathfrak{Re}\gamma_{01} + \mathfrak{Im}\gamma_{01}), \\ \gamma_{22} &= (u-t)\gamma_{11} + 2u\mathfrak{Im}\gamma_{02}. \end{cases}$$

v) $h(z,\overline{z}) = i(z + i\overline{z})(z\overline{z} - u) \in \mathcal{P}_\gamma$, *where* γ *is the RDIS defined in ii).*

vi) $Z^2\overline{Z} + iZ\overline{Z}^2 - uZ - iu\overline{Z} = 0.$

6. Solving the TCMP with column dependence relations of the form $Z^{k+1} = \sum_{0\le n+m\le k} a_{nm}\overline{Z}^n Z^m$ $(a_{nm} \in \mathbb{C})$

In this section, we involve the RDIS to solve the TCMP associated with the truncated sequence $\gamma \equiv \{\gamma_{ij}\}_{0\le i+j\le 2k+2}$, with $\gamma_{ij} = \gamma_{ji}$ and $M(k+1)(\gamma)$ has a column relation of the form

$$Z^{k+1} = \sum_{0\le n+m\le k} a_{nm}\overline{Z}^n Z^m \qquad (a_{nm} \in \mathbb{C},\ \text{for all } n,m \in \mathbb{Z}_+ \text{ and } n+m \le k). \tag{15}$$

According to (15), we have $\gamma_{i+k+1,j} = \sum_{0\le n+m\le k} a_{nm}\gamma_{n+i,m+j}$, for all $i,j \in \mathbb{Z}_+$ such that $i+j \le k+1$. Hence γ is a subsequence of the RDIS $\widetilde{\gamma}$, defined by the initial conditions $\{\gamma_{ij}\}_{0\le i\le j\le k}$ and by the characteristic polynomial $p(z,\overline{z}) = z^{k+1} - \sum_{0\le n+m\le k} a_{nm}\overline{z}^n z^m$.

We give now the main result of this section.

Theorem 21. *Let* $M(k+1)(\gamma)$ *has a column dependence relations of the form* $Z^{k+1} = \sum_{0\le n+m\le k} a_{nm}\overline{Z}^n Z^m$ *and let* $\widetilde{\gamma}$ *be a RDIS defined as above. Then* $M(k+1)(\gamma)$ *admits a representing measure if and only if* $M(2k)(\widetilde{\gamma})$ *is positive semidefinite.*

Proof. Suppose that $\gamma \equiv \{\gamma_{ij}\}_{0\le i+j\le 2k+2}$ is a moment sequence, then there exists a positive Borel measure μ verifies the relation

$$\gamma_{ij} = \int \overline{z}^i z^j \, d\mu \qquad (i+j \le 2k+2).$$

Set $p(\overline{z},z) = z^{k+1} - \sum\limits_{0\leq n+m\leq k} a_{nm}\overline{z}^n z^m$. Since $M(k+1)(\gamma)$ has a column dependence relations of the form $p(\overline{Z},Z)=0$, then

$$\int \overline{z}^i z^j p(\overline{z},z)\,d\mu = \gamma_{i+k+1,j} - \sum_{0\leq n+m\leq k} a_{nm}\gamma_{i+n,j+m} = 0,$$

for every $i,j\in\mathbb{Z}_+$ such that $i+j\leq k+1$. Hence

$$\begin{aligned}\int \mid p(\overline{z},z)\mid^2 d\mu &= \int \overline{p(\overline{z},z)}p(\overline{z},z)\,d\mu\\ &= \int \overline{z}^{k+1}p(\overline{z},z)\,d\mu - \sum_{0\leq n+m\leq k}\overline{a}_{nm}\int \overline{z}^n z^m p(\overline{z},z)\,d\mu\\ &= 0,\end{aligned}$$

thus $supp\mu \subset Z(p)$. It follows, from Proposition (7), that $\widetilde{\gamma}$ is a moment sequence, and obviously $M(2k)(\widetilde{\gamma})$ is positive semidefinite. Conversely, if $M(2k)(\widetilde{\gamma})\geq 0$, then Corollary (16) yields that $\widetilde{\gamma}$ has a representing measure, and thus $\gamma\equiv\{\gamma_{ij}\}_{0\leq i+j\leq 2k+2}$ is a moment sequence. □

On account of Theorem (21) we can formulate the following corollary, which proved a complete solution to the truncated moment problems with cubic column relations.

Corollary 22. *Let $\gamma\equiv\{\gamma_{ij}\}_{0\leq i+j\leq 6}$ (with $\overline{\gamma_{ij}}=\gamma_{ji}$) be a complex numbers, let $M(3)(\gamma)$ admits a cubic column relations of the form $Z^3 = \sum\limits_{0\leq i+j\leq 2} a_{ij}\overline{Z}^i Z^j$ and let $\widetilde{\gamma}$ be the RDIS defined by the initial condition $\{\gamma_{ij}\}_{0\leq i,j\leq 2}$ and by the characteristic polynomial $z^3 - \sum\limits_{0\leq i+j\leq 2} a_{ij}\overline{z}^i z^j$. Then γ admits a representing measure if and only if $M(4)(\widetilde{\gamma})\geq 0$.*

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