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BOUNDARY TRIPLES FOR INTEGRAL SYSTEMS ON THE HALF-LINE

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Abstract. Let P, Q and W be real functions of locally bounded variation on $[0,\infty)$ and let W be non-decreasing. In the case of absolutely continuous functions P, Q and W the following Sturm-Liouville type integral system:

$$J\vec{f}(x) - J\vec{a} = \int_0^x \begin{pmatrix} \lambda dW - dQ & 0 \\ 0 & dP \end{pmatrix} \vec{f}(t), \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{1}$$

(see [5]) is a special case of so-called canonical differential system (see [16, 20, 24]). In [27] a maximal $A_{\max}$ and a minimal $A_{\min}$ linear relations associated with system (1) have been studied on a compact interval. This paper is a continuation of [27], it focuses on a study of $A_{\max}$ and $A_{\min}$ on the half-line. Boundary triples for $A_{\max}$ on $[0,\infty)$ are constructed and the corresponding Weyl functions are calculated in both limit point and limit circle cases at ∞.

1. Introduction

This paper deals with the following integral system

$$J\vec{f}(x) - J\vec{a} = \int_0^x dS(t) \cdot \vec{f}(t), \tag{2}$$

where J and dS are 2×2 matrices of the form

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad dS = \begin{pmatrix} \lambda dW - dQ & 0 \\ 0 & dP \end{pmatrix}, \tag{3}$$

$\lambda \in \mathbb{C}$, all the functions P, Q and W are real of locally bounded variation on $[0,\infty)$ and W is non-decreasing. Such systems were studied in [3, 5, 2]. System (2) contains Sturm-Liouville systems, Stieltjes string and Krein-Feller string [13, 18] as special cases.

We associate with system (2) a minimal $A_{\min}$ and a maximal $A_{\max}$ linear relations in the Hilbert space $L^2(W)$. In this paper both $A_{\min}$ and $A_{\max}$ are not supposed to be single-valued, therefore we use for them a term *linear relation* (see [1]). In Theorem 3.12 it is shown that $A_{\max} = A_{\min}^*$.

The notions of the boundary triple and Weyl function introduced in [7, 19, 6] and [9], respectively, were proved to be useful in the study of spectral problems and extension theory problems for symmetric operators, see [14, 10, 11]. Boundary triples for various differential and difference operators were constructed in [25, 14, 9, 21, 10, 22, 4].

In [27] the boundary triples for $A_{\max}$ (and for some its intermediate extensions) were constructed on a compact interval. The results of [27] are extended to the so-called quasiregular case, which is characterized by the condition that P, Q, and W are of bounded variation on $[0,\infty)$ (see Theorem 3.17). In this case the limit $\lim_{x\to\infty} \vec{f}(x)$ exists for every element of $A_{\max}$ and the boundary triple for $A_{\max}$ is determined with the help of these limiting values. In the general case the Weyl classification for singular point at ∞ is presented. Boundary triples for the linear relation $A_{\max}$ are constructed

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both in the limit point case (Theorem 4.4) and in the limit circle case (Theorem 4.6). Notice that in the limit circle case the limits $\lim_{x\to\infty}\vec{f}(x)$ may not exist for some element of $A_{\max}$ and the boundary triple is defined in terms of generalized Wronskians at ∞. Expressions for the corresponding Weyl functions and γ-fields are also found.

2. Preliminaries

2.1. **Linear relations.** Let $\mathfrak{H}$ be a Hilbert space. Any linear subspace of $\mathfrak{H}\times\mathfrak{H}$ is called a *linear relation* on $\mathfrak{H}$, [1]. The *domain*, the *range*, the *kernel*, and the *multivalued part* of a linear relation T are defined by the following equalities (see [1]):

$$\operatorname{dom} T := \left\{ f\colon \begin{pmatrix} f\\ g\end{pmatrix}\in T\right\}, \qquad \operatorname{ran} T := \left\{ g\colon \begin{pmatrix} f\\ g\end{pmatrix}\in T\right\}, \tag{4}$$

$$\ker T := \left\{ f\colon \begin{pmatrix} f\\ 0\end{pmatrix}\in T\right\}, \qquad \operatorname{mul} T := \left\{ g\colon \begin{pmatrix} 0\\ g\end{pmatrix}\in T\right\}. \tag{5}$$

The *adjoint* linear relation T^* is defined as

$$T^* := \left\{ \begin{pmatrix} u\\ v\end{pmatrix}\in \mathfrak{H}\times\mathfrak{H}\colon (v,f)_{\mathfrak{H}} = (u,g)_{\mathfrak{H}} \text{ for any } \begin{pmatrix} f\\ g\end{pmatrix}\in T\right\}. \tag{6}$$

A linear relation T in $\mathfrak{H}$ is called *closed* if T is closed as a subspace of $\mathfrak{H}\times\mathfrak{H}$. The set of all closed linear operators (relations) is denoted by $\mathcal{C}(\mathfrak{H})$ ($\widetilde{\mathcal{C}}(\mathfrak{H})$). Identifying a linear operator $T\in\mathcal{C}(\mathfrak{H})$ with its graph one can consider $\mathcal{C}(\mathfrak{H})$ as a part of $\widetilde{\mathcal{C}}(\mathfrak{H})$.

Definition 2.1. Suppose T is a linear relation, $\lambda\in\mathbb{C}$, then

$$T-\lambda I := \left\{ \begin{pmatrix} f\\ g-\lambda f\end{pmatrix} : \begin{pmatrix} f\\ g\end{pmatrix}\in T\right\}. \tag{7}$$

A point $\lambda\in\mathbb{C}$ such that $\ker(T-\lambda I)=\{0\}$ and $\operatorname{ran}(T-\lambda I)=\mathfrak{H}$ is called a *regular point* of the linear relation T and is written $\lambda\in\rho(T)$. The *point spectrum* $\sigma_p(T)$ of the linear relation T is defined by

$$\sigma_p(T) := \{\lambda\in\mathbb{C}\colon \ker(T-\lambda I)\neq\{0\}\}, \tag{8}$$

A linear relation A is called *symmetric* if $A\subseteq A^*$. A point $\lambda\in\mathbb{C}$ is called a *point of regular type* (and is written $\lambda\in\widehat{\rho}(A)$) for a closed symmetric linear relation A, if $\lambda\notin\sigma_p(A)$ and the subspace $\operatorname{ran}(A-\lambda I)$ is closed in H. For $\lambda\in\widehat{\rho}(A)$ let us set $\mathfrak{N}_\lambda := \ker(A^*-\lambda I)$ and

$$\widehat{\mathfrak{N}}_\lambda := \left\{ \begin{pmatrix} f_\lambda\\ \lambda f_\lambda\end{pmatrix} : f_\lambda\in\mathfrak{N}_\lambda\right\}. \tag{9}$$

The *deficiency indices* of a symmetric linear relation A are defined as

$$n_\pm(A) := \dim\ker(A^*\mp iI). \tag{10}$$

2.2. **Boundary triples.** Let A be a symmetric linear relation. In the case of densely defined operators a boundary triple notion was introduced in [7, 6, 19, 14] (in different forms). Following the paper [21, 10] we shall give a definition of a boundary triple for the linear relation A^*.

Definition 2.2. A tuple $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$, where $\mathcal{H}$ is a Hilbert space, Γ_0 and Γ_1 are linear mappings from A^* to $\mathcal{H}$, is called a *boundary triple* for the linear relation A^*, if the following conditions hold:

(i) generalized Green's identity

$$(g,u)_{\mathfrak{H}} - (f,v)_{\mathfrak{H}} = \left(\Gamma_1\begin{pmatrix} f\\ g\end{pmatrix}, \Gamma_0\begin{pmatrix} u\\ v\end{pmatrix}\right)_{\mathcal{H}} - \left(\Gamma_0\begin{pmatrix} f\\ g\end{pmatrix}, \Gamma_1\begin{pmatrix} u\\ v\end{pmatrix}\right)_{\mathcal{H}} \tag{11}$$

holds for all $\begin{pmatrix} f\\ g\end{pmatrix}, \begin{pmatrix} u\\ v\end{pmatrix}\in A^*$;

(ii) the mapping $\Gamma = \begin{pmatrix}\Gamma_0\\ \Gamma_1\end{pmatrix} : A^* \to \mathcal{H} \times \mathcal{H}$ is surjective.

Notice that in contrast to [21] the linear relation A is not supposed to be single-valued. A boundary triple for A^* exists if and only if the deficiency indices of A coincide ($n_+(A) = n_-(A)$), see [19, 21, 10].

An extension $\widetilde{A}$ of a symmetric linear relation A is called *proper* if $A \subsetneq \widetilde{A} \subsetneq A^*$. The class of all proper extensions of the linear relation A completed with relations A and A^* is denoted by $\operatorname{Ext}(A)$. Denote also

(12) $$A_\Theta := \left\{ \begin{pmatrix} f\\ g\end{pmatrix} \in A^* : \Gamma \begin{pmatrix} f\\ g\end{pmatrix} \in \Theta \right\}.$$

Proposition 2.3 ([10])**.** *Let A be a symmetric linear relation, $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for the adjoint linear relation A^*. Then the mapping $\Gamma \colon \widetilde{A} = A_\Theta \to \Theta = \Gamma\widetilde{A}$ is one-to-one from* $\operatorname{Ext}(A)$ *to* $\widetilde{\mathcal{C}}(\mathfrak{H})$. *Notice also that A_Θ is selfadjoint if and only if the linear relation Θ is selfadjoint.*

In particular, linear relations

(13) $$A_0 := \ker \Gamma_0, \quad A_1 := \ker \Gamma_1$$

are disjoint, i.e., $A_0 \cap A_1 = A$, and they are selfadjoint extensions of the symmetric linear relation A (see [10]).

Definition 2.4 ([9, 10])**.** Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for the linear relation A^*. Operator valued functions $M(\cdot)$, $\gamma(\cdot)$ defined by

(14) $$M(\lambda)\Gamma_0 \hat{f}_\lambda = \Gamma_1 \hat{f}_\lambda, \quad \gamma(\lambda)\Gamma_0\hat{f}_\lambda = f_\lambda, \quad \hat{f}_\lambda \in \hat{\mathfrak{N}}_\lambda, \quad \lambda \in \rho(A_0)$$

are called the Weyl function and the γ-field of the symmetric linear relation A with respect to the boundary triple Π.

The Weyl function and the γ-field are connected with the next identity (see. [10])

(15) $$M(\lambda) - M(\zeta)^* = (\lambda - \overline{\zeta})\gamma(\zeta)^*\gamma(\lambda), \quad \lambda, \zeta \in \rho(A_0).$$

Definition 2.5 ([17, 11])**.** An operator valued function $F \colon \mathbb{C}_+ \cup \mathbb{C}_- \to \mathcal{B}(\mathcal{H})$ is said to belong to the class $R[\mathcal{H}]$ if the following conditions hold:

(i) F is holomorphic in $\mathbb{C}_+ \cup \mathbb{C}_-$;
(ii) $\operatorname{Im} F(\lambda) \geq 0$ as $\lambda \in \mathbb{C}_+$;
(iii) $F(\overline{\lambda}) = F^*(\lambda)$ for $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$.

It is known that the Weyl function $M(\lambda)$ of a linear relation A from Definition 2.4 belongs to the class $R[\mathcal{H}]$. If $\mathcal{H} = \mathbb{C}$ then $R[\mathcal{H}]$ is denoted by R and turns out to be the well-known Pick-Nevanlinna class.

The next proposition gives a description of the spectrum of a linear relation $\widetilde{A} \in \operatorname{Ext}(A)$.

Proposition 2.6 ([10])**.** *Let A be a symmetric linear relation in $\mathfrak{H}$, $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for A^*, $M(\lambda)$ be the corresponding Weyl function of A, $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$, and $\lambda \in \rho(A_0)$. Then*

(i) $\lambda \in \rho(\widetilde{A}_\Theta) \iff 0 \in \rho(\Theta - M(\lambda))$;
(ii) $\lambda \in \sigma_p(\widetilde{A}_\Theta) \iff 0 \in \sigma_p(\Theta - M(\lambda))$.

2.3. **Integral systems.** Denote by $BV_{loc}[0,\infty)$ the class of functions that have bounded variation on every compact interval $j \subset [0,\infty)$. Let us consider on $[0,\infty)$ an integral system (2), where $\vec{f}$ maps $[0,\infty)$ to $\mathbb{C}^2$, $\vec{a} \in \mathbb{C}^2$ is a fixed vector (or a function from $BV_{loc}[0,\infty)$), λ is a complex parameter, P, Q, and W are functions from the class $BV_{loc}[0,\infty)$ satisfying the condition

$$P(0) = Q(0) = W(0) = 0 \tag{16}$$

and W is non-decreasing.

Remark 2.7. In equality (2) and further in the text we mean that the integration set is a half-open interval $[0,x)$ (under this convention integrals as the functions of upper limits are always left-continuous).

Definition 2.8. We say that a vector valued function $\vec{f}$ is a solution to integral system (2) (with a fixed function $\vec{a}$) if (each component of) $\vec{f}$ is of locally bounded variation on $[0,\infty)$ and the equality (2) holds for every point x from $[0,\infty)$.

An existence and uniqueness theorem for system (2) has been proved in [5].

Theorem 2.9 ([5]). *For any left-continuous vector-function $\vec{a}(x)$ from the class $BV_{loc}[0,\infty)$ there is a unique solution of system* (2).

Everywhere in the following, we suppose that

Assumption 2.10. *The functions Q and W have no discontinuities in common with P.*

3. Green's identity and linear relation $A_{\max}$

3.1. **Green's identity.** Let us denote by $\mathcal{L}_{loc}(W)$ and $\mathcal{L}^2_{loc}(W)$ the sets of functions such that

$$\int_j |f(t)|\, dW(t) < \infty \quad \text{and} \quad \int_j |f(t)|^2\, dW(t) < \infty, \tag{17}$$

respectively, for every compact interval $j \subset [0,\infty)$. In the case where the integrals in (17) are finite for $j = [0,\infty)$, we should write $\mathcal{L}(W)$ and $\mathcal{L}^2(W)$, respectively. An inner product in $\mathcal{L}^2(W)$ is defined by

$$(f,g)_W := \int_0^\infty f(t)\overline{g(t)}dW(t). \tag{18}$$

Denote by $L^2(W)$ the corresponding quotient space, which consists of equivalence classes with respect to the measure dW. To avoid confusion we will denote elements of the space $L^2(W)$ with Gothic letters $\mathfrak{f}, \mathfrak{g}$ etc.

Let us consider on $[0,\infty)$ the nonhomogeneous system:

$$J\begin{pmatrix} f \\ f^{[1]} \end{pmatrix}\Bigg|_0^x = \int_0^x \begin{pmatrix} -dQ & 0 \\ 0 & dP \end{pmatrix}\begin{pmatrix} f \\ f^{[1]} \end{pmatrix} + \int_0^x \begin{pmatrix} dW & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} g \\ 0 \end{pmatrix}. \tag{19}$$

Definition 3.1. A pair $\{\vec{f}, g\}$ that consists of a vector-function $\vec{f} = \begin{pmatrix} f \\ f^{[1]} \end{pmatrix}$ and a scalar function g is said to satisfy system (19) (or $\vec{f}$ is a solution to this system with fixed g), if the following conditions hold:

(i) $g \in \mathcal{L}_{loc}(W)$;
(ii) $\vec{f} \in BV_{loc}[0,\infty)$;
(iii) equality (19) holds for each $x \in [0,\infty)$.

For a pair of vector valued functions $\vec{f} = \begin{pmatrix} f \\ f^{[1]} \end{pmatrix}$ and $\vec{u} = \begin{pmatrix} u \\ u^{[1]} \end{pmatrix}$ we define the *generalized Wronskian* by

$$\left[\vec{f}, \vec{u}\right] := \left(f u^{[1]} - f^{[1]} u\right). \tag{20}$$

In the case of a finite interval the following theorem has been proved in [27, Theorem 3.3], however in the case of the half-line the proof is similar.

Theorem 3.2 (The second Green's identity)**.** *Suppose Assumption 2.10 holds, pairs* $\left\{\vec{f}, g\right\}$*,* $\{\vec{u}, v\}$ *satisfy system* (19) *(see Definition 3.1) and* $0 \leqslant \alpha < \beta < \infty$*. Then the next equality holds:*

$$\int_\alpha^\beta (g\overline{u} - f\overline{v}) dW = \left[\vec{f}, \overline{\vec{u}}\right]\Big|_\alpha^\beta. \tag{21}$$

3.2. **Linear relation $A_{\max}$.**

Definition 3.3. We shall say that a pair of classes $\begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in L^2(W) \times L^2(W)$ belongs to the linear relation $A_{\max}$ if there exist functions f, $f^{[1]}$, and g such that

(i) the pair $\left\{\vec{f}, g\right\}$, where $\vec{f} = \begin{pmatrix} f \\ f^{[1]} \end{pmatrix}$, satisfies (19) (in the sense of Definition 3.1);

(ii) $f \in \mathfrak{f}$, $g \in \mathfrak{g}$.

In the succeeding we require the following

Assumption 3.4. *There exists a compact interval* $[\alpha, \beta] \subset [0, \infty)$ *such that system* (19) *is surjective on it, i.e., for any* $a, b, a_1, b_1 \in \mathbb{C}$ *one can choose a pair* $\{\vec{f}, g\}$ *that satisfies* (19) *and the next boundary conditions hold:*

$$f(\alpha) = a, \quad f(\beta) = b, \quad f^{[1]}(\alpha) = a_1, \quad f^{[1]}(\beta) = b_1. \tag{22}$$

Remark 3.5. If all the functions P, Q, and W are absolutely continuous, then the *definiteness* (see e.g. [20]) of system (19) implies its surjectivity. In case of arbitrary coefficients the Assumption 3.4 does not hold, however in a special case the sufficient condition for system (19) to be surjective is provided by the following proposition.

Proposition 3.6 ([27])**.** *If* $dQ \equiv 0$ *and there exist closed on the left disjoint intervals* $\imath_1, \imath_2 \subset [\alpha, \beta]$ *such that*

$$\dim L^2(W, \imath_k) > 0, \quad k \in \{1, 2\}, \tag{23}$$

$$\frac{1}{dW(\imath_2)} \int_{\imath_2} P(t) dW(t) > \frac{1}{dW(\imath_1)} \int_{\imath_1} P(t) dW(t), \tag{24}$$

then Assumption 3.4 holds.

Proposition 3.7. *If Assumption 3.4 holds for system* (19) *on some compact interval* $[\alpha, \beta]$ *then it also holds on an interval* $[\widetilde{\alpha}, \widetilde{\beta}]$*, where* $[\alpha, \beta] \subseteq [\widetilde{\alpha}, \widetilde{\beta}] \subset [0, \infty)$*.*

Proof. Let $\widetilde{a}, \widetilde{a}_1, \widetilde{b}, \widetilde{b}_1$ be fixed values from $\mathbb{C}$. We build the function $\vec{f}$ as a (unique) solution of the next systems

$$J\vec{f}(x) - J\begin{pmatrix} \widetilde{a} \\ \widetilde{a}_1 \end{pmatrix} = \int_{\widetilde{\alpha}}^x \begin{pmatrix} -dQ & 0 \\ 0 & dP \end{pmatrix} \vec{f} \tag{25}$$

and

$$J\begin{pmatrix} \widetilde{b} \\ \widetilde{b}_1 \end{pmatrix} - J\vec{f}(x) = \int_x^{\widetilde{\beta}} \begin{pmatrix} -dQ & 0 \\ 0 & dP \end{pmatrix} \vec{f} \tag{26}$$

on $[\widetilde{\alpha}, \alpha]$ and $[\beta, \widetilde{\beta}]$, respectively. Thus, we have

$$\vec{f}(\alpha) = \vec{f}(\widetilde{\alpha}) + J^{-1} \int_{\widetilde{\alpha}}^{\alpha} \begin{pmatrix} -dQ & 0 \\ 0 & dP \end{pmatrix} \vec{f}, \tag{27}$$

$$\vec{f}(\beta) = \vec{f}(\widetilde{\beta}) + J^{-1} \int_{\beta}^{\widetilde{\beta}} \begin{pmatrix} -dQ & 0 \\ 0 & dP \end{pmatrix} \vec{f}. \tag{28}$$

As it follows from Assumption 3.4, there exists a function $g \in \mathcal{L}(W, [\alpha, \beta])$ such that the corresponding (unique) solution $\vec{f}$ of system (19) on $[\alpha, \beta]$ satisfies (27) and (28). Finally, assuming $g \equiv 0$ on $[\widetilde{\alpha}, \widetilde{\beta}] \setminus [\alpha, \beta]$, one can see that the solution $\vec{f}$ satisfies the conditions of the Proposition. □

It follows from Assumption 3.4 and Proposition 3.7 that system (19) with $l \geqslant \beta$ is surjective on $[0, l]$. In this case, the following Proposition holds, see [27, Theorem 3.8].

Proposition 3.8. *Suppose Assumption 3.4 holds for system* (19) *on* $[0, l]$, $\begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in A_{\max}$, *pairs* $\{\vec{f}_1, g_1\}$ *and* $\{\vec{f}_2, g_2\}$ *satisfy system* (19), $f_1, f_2 \in \mathfrak{f}$, $g_1, g_2 \in \mathfrak{g}$. *Then*

$$f_1(0) = f_2(0), \qquad f_1^{[1]}(0) = f_2^{[1]}(0), \qquad f_1(l) = f_2(l), \qquad f_1^{[1]}(l) = f_2^{[1]}(l). \tag{29}$$

Proposition 3.9. *If* $\begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix}, \begin{pmatrix} \mathfrak{u} \\ \mathfrak{v} \end{pmatrix} \in A_{\max}$ *then there exists a finite limit*

$$\left[\vec{f}, \vec{\overline{u}}\right]_{\infty} := \lim_{l \to \infty} \left[\vec{f}, \vec{\overline{u}}\right]_{l}. \tag{30}$$

Proof. It follows from Theorem 3.2 that

$$\int_0^{\beta} (g\overline{u} - f\overline{v})\, dW = \left[\vec{f}, \vec{\overline{u}}\right]_0^{\beta}. \tag{31}$$

Let us pass to the limit in the last equality as $\beta \to \infty$. The finiteness of the limit on the left hand side follows from the conditions of this Proposition. And the finiteness of $\left[\vec{f}, \vec{\overline{u}}\right]_0$ follows from Proposition 3.8, which completes the proof. □

In the case of densely defined symmetric linear operator S the next lemma has been proven in [11].

Lemma 3.10. *Let* S *be a symmetric linear relation in* $\mathfrak{H}$, P_n $(n \in \mathbb{N})$ *be a sequence of orthogonal projections in* $\mathfrak{H}$ *such that* $P_n \xrightarrow{s} I_{\mathfrak{H}}$, $S = \cup_{n \in \mathbb{N}} S_n$ *where* $S_n = (P_n \times P_n)S$. *Then*

$$\begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in S^* \Longleftrightarrow \begin{pmatrix} P_n\mathfrak{f} \\ P_n\mathfrak{g} \end{pmatrix} \in S_n^* \quad \text{for any} \quad n \in \mathbb{N}. \tag{32}$$

Proof. Suppose $\begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in S^*$, then for any pair $\begin{pmatrix} \mathfrak{u} \\ \mathfrak{v} \end{pmatrix} \in S_n$ one has

$$(\mathfrak{v}, P_n\mathfrak{f}) = (\mathfrak{v}, \mathfrak{f}) = (\mathfrak{u}, \mathfrak{g}) = (\mathfrak{u}, P_n\mathfrak{g}) \tag{33}$$

This implies $\begin{pmatrix} P_n\mathfrak{f} \\ P_n\mathfrak{g} \end{pmatrix} \in S_n^*$. Conversely, let $\begin{pmatrix} P_n\mathfrak{f} \\ P_n\mathfrak{g} \end{pmatrix} \in S_n^*$ for any $n \in \mathbb{N}$. For any $\begin{pmatrix} \mathfrak{u} \\ \mathfrak{v} \end{pmatrix} \in S$ there exists $n \in \mathbb{N}$ such that $\begin{pmatrix} \mathfrak{u} \\ \mathfrak{v} \end{pmatrix} \in S_n$ and equality (33) holds, hence $\begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in S^*$. □

Definition 3.11. We define a linear relation $A_{\min}$ as

$$A_{\min} := \left\{ \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in A_{\max} \colon f(0) = f^{[1]}(0) = \left[\vec{f}, \vec{u}\right]_{\infty} = 0 \text{ for all } \begin{pmatrix} \mathfrak{u} \\ \mathfrak{v} \end{pmatrix} \in A_{\max} \right\}, \tag{34}$$

where $f \in \mathfrak{f}$, $g \in \mathfrak{g}$, $u \in \mathfrak{u}$, $v \in \mathfrak{v}$ and the pairs $\{f, g\}$, $\{u, v\}$ satisfy system (19).

Theorem 3.12. *The linear relation $A_{\min}$ is symmetric and $A_{\min}^* = A_{\max}$.*

Proof. Note that by Proposition 3.9 the linear relation $A_{\min}$ in (34) is well defined. The symmetry property of $A_{\min}$ is implied by the Green formula (see Theorem 3.2).

Let $\Delta_n = [0, l_n]$ be a sequence of nested intervals which exhaust the interval $[0, \infty)$, and let P_n be the orthogonal projections from $L^2(W)$ to $L^2(W, \Delta_n)$ such that the interval $[\alpha, \beta]$ from Assumption 3.4 is contained in Δ_1. Consider the sequence of minimal and maximal linear relations $A_{n,\min}$, $A_{n,\max}$ generated by system (19) in $L^2(W, \Delta_n)$, which in view of [27, Theorem 3.12] are connected by $A_{n,\min}^* = A_{n,\max}$.

Let S be the linear relation in $L^2(W)$ defined by $S = \cup_{n \in \mathbb{N}} A_{n,\min}$. Obviously, S is symmetric. Since $A_{\max}$ has the property

$$\begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in A_{\max} \Leftrightarrow \begin{pmatrix} P_n \mathfrak{f} \\ P_n \mathfrak{g} \end{pmatrix} \in A_{n,\max} \quad \text{for any} \quad n \in \mathbb{N} \tag{35}$$

one obtains from Lemma 3.10 that $A_{\max} = S^*$.

Let us show that $\overline{S} = A_{\min}$. Indeed, if $\begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in A_{\min}$ then by Theorem 3.2 $\begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in A_{\max}^* = \overline{S}$. Conversely, if $\begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in \overline{S} = A_{\max}^*$ then for every $\begin{pmatrix} \mathfrak{u} \\ \mathfrak{v} \end{pmatrix} \in A_{\max}$ one has

$$\left[\vec{f}, \vec{\overline{u}}\right]_{\infty} - \left[\vec{f}, \vec{\overline{u}}\right]_{0} = 0. \tag{36}$$

It follows from Assumption 3.4 (and Proposition 3.7) that for any $n \in \mathbb{N}$ there exists pairs $\begin{pmatrix} \mathfrak{u}_1 \\ \mathfrak{v}_1 \end{pmatrix}$ and $\begin{pmatrix} \mathfrak{u}_2 \\ \mathfrak{v}_2 \end{pmatrix}$ such that

$$u_1(0) = 1, \quad u_1^{[1]}(0) = 0, \quad u_1(x) = 0, \quad u_1^{[1]}(x) = 0, \tag{37}$$

$$u_2(0) = 0, \quad u_2^{[1]}(0) = 1, \quad u_2(x) = 0, \quad u_2^{[1]}(x) = 0, \tag{38}$$

as $x \geqslant l_n$. Substituting these pairs into (36) one obtains

$$f(0) = \left[\vec{f}, \vec{\overline{u_2}}\right]_0 = 0, \quad f^{[1]}(0) = -\left[\vec{f}, \vec{\overline{u_1}}\right]_0 = 0, \quad \left[\vec{f}, \vec{\overline{u}}\right]_{\infty} = \left[\vec{f}, \vec{\overline{u}}\right]_0 = 0. \tag{39}$$

Hence $\begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in A_{\min}$. This proves that $\overline{S} = A_{\min}$ and thus $A_{\min}^* = A_{\max}$. $\square$

Lemma 3.13 (a decomposition of generalized Wronskian)**.** *Let $\vec{y}_1$ and $\vec{y}_2$ be real vector valued functions satisfying condition*

$$[\vec{y}_1, \vec{y}_2]_x \equiv 1, \quad x \in [0, \infty]. \tag{40}$$

Then for every vector valued functions $\vec{f}, \vec{u}$ defined on $[0, \infty)$, the next equality holds

$$\left[\vec{f}, \vec{\overline{u}}\right]_x = \left[\vec{f}, \vec{y}_1\right]_x \overline{[\vec{u}, \vec{y}_2]_x} - \left[\vec{f}, \vec{y}_2\right]_x \overline{[\vec{u}, \vec{y}_1]_x}, \quad x \in [0, \infty). \tag{41}$$

Proof. Applying formula (20) we obtain

$$\left[\vec{f}, \vec{y}_1\right] \overline{[\vec{u}, \vec{y}_2]} - \left[\vec{f}, \vec{y}_2\right] \overline{[\vec{u}, \vec{y}_1]} = \left(f\overline{u^{[1]}} - f^{[1]}\overline{u}\right)\left(y_1 y_2^{[1]} - y_1^{[1]} y_2\right) = \left[\vec{f}, \vec{\overline{u}}\right]. \tag{42}$$

$\square$

3.3. **Quasiregular case.**

Definition 3.14. The endpoint ∞ is called *quasiregular* for system (19) if all the functions P, Q and W are of bounded variation on $[0,\infty)$.

Next, we need generalized Gronwall's lemma.

Lemma 3.15 ([5]). *Let function u be locally integrable on $[0,\infty)$ w.r.t. a positive measure df, A be a positive constant and*

$$0 \leqslant u(x) \leqslant A + \int_0^x u\ df, \quad x \in [0,\infty) \tag{43}$$

then $u(x) \leqslant A \exp \int_0^x df$.

The following theorem is an analogue of [4, Proposition 2.6].

Theorem 3.16. *Suppose the endpoint ∞ is quasiregular for system (19) and $g \in \mathcal{L}(W)$. Then:*

(i) *each solution $\vec{f}$ of system (19) belongs to $\mathcal{L}^2(W)$;*
(ii) *there exists a finite limit $\vec{f}(\infty) := \lim_{x\to\infty} \vec{f}(x)$;*
(iii) *for any fixed $\vec{b} \in \mathbb{C}^2$ there exists a unique solution of system (19) such that $\lim_{x\to\infty} \vec{f}(x) = \vec{b}$.*

Proof. (i) Let us rewrite system (19) as follows

$$\vec{f}(x) = \vec{f}(0) + \int_0^x J^{-1}\begin{pmatrix} -dQ & 0 \\ 0 & dP \end{pmatrix}\vec{f} + \int_0^x J^{-1}\begin{pmatrix} dW & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} g \\ 0 \end{pmatrix}. \tag{44}$$

Using the uniform norm in $\mathbb{C}^2$ and the corresponding norm for matrices $\|A\| = \max_j \sum_k |a_{jk}|$, we pass to the inequality in (44):

$$\|\vec{f}(x)\| \leqslant \left(\|\vec{f}(0)\| + \int_0^x |g|\ dW\right) + \int_0^x \|\vec{f}(s)\| \max\{|dP|, |dQ|\}. \tag{45}$$

By the conditions of the theorem we have $\int_0^x |g|\ dW \leqslant \|g\|_{\mathcal{L}(W)} < \infty$. Applying Lemma 3.15 we obtain an estimate

$$\|\vec{f}(x)\| \leqslant \left(\|\vec{f}(0)\| + \|g\|_{\mathcal{L}(W)}\right) \exp\left(\max\{V_0^\infty(P), V_0^\infty(Q)\}\right). \tag{46}$$

It follows from the last inequality that solution $\vec{f}$ is bounded by the norm, and taking into account that $W \in BV[0,\infty)$ we get $\vec{f} \in \mathcal{L}^2(W)$.

(ii) Passing to the limit in (44), we get

$$\lim_{x\to\infty} \vec{f}(x) = \vec{f}(0) + \int_0^\infty J^{-1}\begin{pmatrix} -dQ & 0 \\ 0 & dP \end{pmatrix}\vec{f} + \int_0^\infty J^{-1}\begin{pmatrix} dW & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} g \\ 0 \end{pmatrix}. \tag{47}$$

(iii) There exists a finite limit in (47), therefore

$$\vec{f}(x) = \lim_{x\to\infty} \vec{f}(x) + \int_x^\infty J^{-1}\begin{pmatrix} -dQ & 0 \\ 0 & dP \end{pmatrix}\vec{f} + \int_x^\infty J^{-1}\begin{pmatrix} dW & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} g \\ 0 \end{pmatrix}, \tag{48}$$

and further

$$\|\vec{f}(x)\| \leqslant \left(\|\lim_{x\to\infty} \vec{f}(x)\| + \|g\|_{\mathcal{L}(W)}\right) \exp\left(\max\{V_x^\infty(P), V_x^\infty(Q)\}\right). \tag{49}$$

It follows from (49) that for any solution $\vec{f}$ to the system (19) (as $g = 0$) the linear mapping $\vec{f} \mapsto \lim_{x\to\infty} \vec{f}(x)$ is injective, and hence surjective. This concludes the proof. $\square$

Theorem 3.17. *Suppose Assumption 3.4 holds, endpoint ∞ is quasiregular for system* (19), *and mappings $\Gamma_0, \Gamma_1 \colon A_{\max} \mapsto C^2$ are defined as*

$$\Gamma_0 \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} := \begin{pmatrix} f(0) \\ f(\infty) \end{pmatrix}, \quad \Gamma_1 \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} := \begin{pmatrix} f^{[1]}(0) \\ -f^{[1]}(\infty) \end{pmatrix}, \tag{50}$$

where the pair $\{\vec{f}, g\}$ satisfies system (19), *$f \in \mathfrak{f}$, $g \in \mathfrak{g}$, and $f(\infty) := \lim_{x\to\infty} f(x)$, $f^{[1]}(\infty) := \lim_{x\to\infty} f^{[1]}(x)$. Then*

(i) *the mappings Γ_0 and Γ_1 in* (50) *are well defined;*
(ii) *the tuple $\{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$ is a boundary triple for the linear relation $A_{\max}$.*

Proof. (i) Notice that by Theorem 3.16 the limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} f^{[1]}(x)$ are well defined for any solution $\vec{f}$ to the system (19) (with an arbitrary g). Since Assumption 3.4 holds, the values $\Gamma_0 \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix}$ and $\Gamma_1 \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix}$ are independent of the choice of the classes $f \in \mathfrak{f}$, $g \in \mathfrak{g}$ (see the proof of Theorem 3.7 in [27]).

(ii) The fulfillment of the requirements of Definition 2.2 follows directly from Theorem 3.2 and Theorem 3.16. ☐

4. Weyl classification for the linear relation $A_{\max}$

4.1. **Weyl classification.** Suppose Assumption 3.4 holds. Let $c(x,\lambda)$ and $s(x,\lambda)$ be the solutions to the spectral problem

$$J \begin{pmatrix} f \\ f^{[1]} \end{pmatrix} \Bigg|_0^x = \int_0^x \begin{pmatrix} \lambda dW - dQ & 0 \\ 0 & dP \end{pmatrix} \begin{pmatrix} f \\ f^{[1]} \end{pmatrix}, \tag{51}$$

that satisfy the initial conditions

$$c(0,\lambda) = 1, \quad c^{[1]}(0,\lambda) = 0, \quad s(0,\lambda) = 0, \quad s^{[1]}(0,\lambda) = 1 \tag{52}$$

(their existence and uniqueness follow from Theorem 2.9). Notice, that for any $\lambda \in \mathbb{C}$ these functions satisfy the conditions of Lemma 3.13, see [27, Theorem 3.14].

Since the linear relation $A_{\max}$ is not self-adjoint and for any $\lambda \in \mathbb{C}$ there exist exactly two linearly independent solutions to (51), the deficiency indices $n_\pm(A_{\min})$ are equal to either 2 or 1. For further references we fix this as the following Assertion.

Assertion 4.1. *For any $\lambda \in \mathbb{C} \setminus \mathbb{R}$ at least one solution to system* (51) *belongs to $\mathcal{L}^2(W)$ on $[0,\infty)$.*

Definition 4.2. System (51) is said to be in

(i) the *limit point* case at ∞, if $n_\pm(A_{\min}) = 1$;
(ii) the *limit circle* case at ∞, if $n_\pm(A_{\min}) = 2$.

In the limit point case $\widehat{\mathfrak{N}}_\lambda(A_{\min})$ contains a unique element $\widehat{\mathfrak{d}}_\lambda$ for such that $d(0,\lambda) = 1$ for any instance $d(\cdot,\lambda) \in \mathfrak{d}_\lambda$ satisfying (51); the solution $d(\cdot,\lambda)$ is called *the Weyl solution* to (51). In the limit circle case, denote by $\mathfrak{c}_\lambda$ and $\mathfrak{s}_\lambda$ the equivalence classes in $L^2(W)$ generated by the fundamental solutions $c(\cdot,\lambda)$ and $s(\cdot,\lambda)$, see [27, Section 3.3].

4.2. **Limit point case.**

Theorem 4.3. *Let system* (2) *be in the limit point case, then*

(i) *for any $\begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix}, \begin{pmatrix} \mathfrak{u} \\ \mathfrak{v} \end{pmatrix} \in A_{\max}$ the following equality holds:*

$$\lim_{x\to\infty} \left[\vec{f}, \vec{u}\right]_x = 0, \tag{53}$$

here the pairs $\{\vec{f}, g\}$, $\{\vec{u}, v\}$ satisfy system (19) *and the following inclusions hold $f \in \mathfrak{f}$, $g \in \mathfrak{g}$, $u \in \mathfrak{u}$, $v \in \mathfrak{v}$;*

(ii) *the minimal linear relation* $A_{\min}$ *defined already in Definition 3.11 coincides with the linear relation*

$$A := \left\{ \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in A_{\max} \colon f(0) = f^{[1]}(0) = 0 \right\}. \tag{54}$$

Proof. Since $n_{\pm}(A_{\min}) = 1$, then it follows from the first Neumann's formula that $\dim(A_{\max}/A_{\min}) = 2$. By the Assumption 3.4 there exist two pairs of functions $\{\vec{f}_1, g_1\}$ and $\{\vec{f}_2, g_2\}$ that satisfy system (19) and the next boundary conditions

$$\begin{aligned} &f_1(0) = 1, \quad && f_1^{[1]}(0) = 0, \quad && f_1(\beta) = 0, \quad && f_1^{[1]}(\beta) = 0 \\ &f_2(0) = 0, && f_2^{[1]}(0) = 1, && f_2(\beta) = 0, && f_2^{[1]}(\beta) = 0. \end{aligned} \tag{55}$$

Extending functions g_1 and g_2 to $[\beta, \infty)$ with zero, let us build on the half-line the corresponding finite solutions $\vec{f}_1$ and $\vec{f}_2$. Each element $\begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in A_{\max}$ can be written as

$$\begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} = \begin{pmatrix} \mathfrak{f}_0 \\ \mathfrak{g}_0 \end{pmatrix} + \begin{pmatrix} \mathfrak{f}_1 \\ \mathfrak{g}_1 \end{pmatrix} + \begin{pmatrix} \mathfrak{f}_2 \\ \mathfrak{g}_2 \end{pmatrix}, \quad \begin{pmatrix} \mathfrak{f}_0 \\ \mathfrak{g}_0 \end{pmatrix} \in A_{\min}, \quad \begin{pmatrix} \mathfrak{f}_1 \\ \mathfrak{g}_1 \end{pmatrix}, \begin{pmatrix} \mathfrak{f}_2 \\ \mathfrak{g}_2 \end{pmatrix} \in A_{\max}, \tag{56}$$

where $f_1 \in \mathfrak{f}_1$, $f_2 \in \mathfrak{f}_2$, $g_1 \in \mathfrak{g}_1$, $g_2 \in \mathfrak{g}_2$.

Then $[\vec{f}_1, \vec{u}]_\infty = [\vec{f}_2, \vec{u}]_\infty = 0$ and from Definition 3.11 we have $[\vec{f}_0, \vec{u}]_\infty = 0$. The theorem assertions follow now from decomposition (56). □

Theorem 4.4. *Suppose Assumption 3.4 holds, system* (2) *is in the limit point case,* $\widehat{\mathfrak{d}}_\lambda$ *is the defect element of* $A_{\max}$*, and the mappings* $\Gamma_0, \Gamma_1 \colon A_{\max} \mapsto \mathbb{C}$ *are defined as*

$$\Gamma_0 \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} := f^{[1]}(0), \quad \Gamma_1 \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} := -f(0) \tag{57}$$

where the pair $\{\vec{f}, g\}$ *satisfies system* (19)*,* $f \in \mathfrak{f}$*,* $g \in \mathfrak{g}$*. Then*

(i) *the mappings* Γ_0*,* Γ_1 *in* (57) *are well defined;*

(ii) *the tuple* $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ *with the mappings* Γ_0*,* Γ_1 *from* (57) *is a boundary triple for the linear relation* $A_{\max}$*;*

(iii) *the corresponding Weyl function and* γ*-field have the forms*

$$M(\lambda) = d(0, \lambda)/d^{[1]}(0, \lambda), \quad \gamma(\lambda) = \mathfrak{d}(\lambda), \quad d(\cdot, \lambda) \in \mathfrak{d}_\lambda. \tag{58}$$

Proof. (ii) It follows immediately from Proposition 3.8 that Γ_0, Γ_1 in (57) are well defined.

(iii) The generalized Green's identity from Definition 2.2 may be verified directly and the surjectivity of the mapping $\Gamma := \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix}$ follows from Theorem 2.9.

(iv) The formulas (58) follow directly from Definition 2.4. □

4.3. Limit circle case.

Theorem 4.5. *Suppose Assumption 3.4 holds, system* (2) *is in the limit circle case. Then* $\dim \mathfrak{N}_\lambda(A_{\min}) = 2$ *for any* $\lambda \in \mathbb{C}$.

Proof. By the assumptions, $c(x, \lambda), s(x, \lambda) \in \mathcal{L}^2(W)$ for any $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Now let $a \in \mathbb{R}$. It follows from Theorem 3.2 that

$$[\vec{c}(x, a), \vec{c}(x, \lambda)] = (a - \lambda) \int_0^x c(t, a) c(t, \lambda)\, dW, \tag{59}$$

$$[\vec{c}(x, a), \vec{s}(x, \lambda)] = 1 + (a - \lambda) \int_0^x c(t, a) s(t, \lambda)\, dW. \tag{60}$$

Multiplying (59) by $s(x, \lambda)$ and subtracting it from (60) multiplied by $s(x, \lambda)$, we obtain

$$c(x, a) = c(x, \lambda) + (a - \lambda) \int_0^x c(t, a) \left\{ c(x, \lambda) s(t, \lambda) - s(x, \lambda) c(t, \lambda) \right\} dW. \tag{61}$$

Using the well-known procedure (see e.g. [3, Theorem 5.6.1]) one can show that $c(x, a)$ belongs to $\mathcal{L}^2(W)$. For $s(x, a)$ the proof is similar. □

Theorem 4.6. *Suppose Assumption 3.4 holds, system* (2) *is in the limit circle case, and the mappings* Γ_0, Γ_1 *are defined as*

$$\Gamma_0 \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} := \begin{pmatrix} f(0) \\ \left[\vec{f}, \vec{s}_0\right]_\infty \end{pmatrix}, \quad \Gamma_1 \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} := \begin{pmatrix} f^{[1]}(0) \\ \left[\vec{f}, \vec{c}_0\right]_\infty \end{pmatrix}, \tag{62}$$

where $\vec{c}_0 := \vec{c}(x,0)$, $\vec{s}_0 := \vec{s}(x,0)$, *pair* $\vec{f}$ *and* g *satisfies system* (19), $f \in \mathfrak{f}$, $g \in \mathfrak{g}$. *Then*

(i) *the mappings in* (62) *are well defined;*
(ii) *the tuple* $\{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$ *from* (62) *is a boundary triple for the linear relation* $A_{\max}$*;*
(iii) *the corresponding Weyl function and the* γ*-field have the form*

$$M(\lambda) = \frac{1}{[\vec{s}(\cdot,\lambda), \vec{s}_0]_\infty} \begin{pmatrix} -[\vec{c}(\cdot,\lambda), \vec{s}_0]_\infty & 1 \\ 1 & [\vec{s}(\cdot,\lambda), \vec{c}_0]_\infty \end{pmatrix}, \tag{63}$$

$$\gamma(\lambda) = \frac{1}{[\vec{s}(\cdot,\lambda), \vec{s}_0]_\infty} (\mathfrak{c}_\lambda \;\; \mathfrak{s}_\lambda) \begin{pmatrix} [\vec{s}(\cdot,\lambda), \vec{s}_0]_\infty & 0 \\ -[\vec{c}(\cdot,\lambda), \vec{s}_0]_\infty & 1 \end{pmatrix}. \tag{64}$$

Proof. (i) This statement follows from Propositions 3.8, 3.9 and Theorem 4.5

(ii) Lemma 3.13 implies that the generalized Green's identity from Definition 2.2 holds. Let us show that the mapping $\Gamma := \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix}$ is surjective. According to the Assumption 3.4 there exist two pairs $\{\vec{f}_1, g_1\}$ and $\{\vec{f}_2, g_2\}$ satisfying system (19) with the boundary conditions (55). Extend the functions g_1 and g_2 to $[\beta,\infty)$ with zero, then the corresponding solutions $\vec{f}_1$ and $\vec{f}_2$ are trivial on $[\beta,\infty)$ and belong to $L^2(W)$. Applying the mapping Γ to the elements $\begin{pmatrix} \mathfrak{f}_1 \\ \mathfrak{g}_1 \end{pmatrix}$, $\begin{pmatrix} \mathfrak{f}_2 \\ \mathfrak{g}_2 \end{pmatrix}$, $\begin{pmatrix} \mathfrak{c}_0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} \mathfrak{s}_0 \\ 0 \end{pmatrix}$ of the linear relation $A_{\max}$, one will have linearly independent vectors

$$\Gamma \begin{pmatrix} \mathfrak{f}_1 \\ \mathfrak{g}_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \Gamma \begin{pmatrix} \mathfrak{f}_2 \\ \mathfrak{g}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \Gamma \begin{pmatrix} \mathfrak{c}_0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \Gamma \begin{pmatrix} \mathfrak{s}_0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}. \tag{65}$$

This proves surjectivity of Γ.

(iii) Let $\mathfrak{f}_\lambda \in \mathfrak{N}_\lambda(A_{\min})$, then $\mathfrak{f}_\lambda = \xi_1 \mathfrak{c}_\lambda + \xi_2 \mathfrak{s}_\lambda$. Herewith

$$\Gamma_0 \begin{pmatrix} \mathfrak{f}_\lambda \\ \lambda \mathfrak{f}_\lambda \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ [\vec{c}(\cdot,\lambda), \vec{s}_0]_\infty & [\vec{s}(\cdot,\lambda), \vec{s}_0]_\infty \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} =: Y_0 \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \tag{66}$$

$$\Gamma_1 \begin{pmatrix} \mathfrak{f}_\lambda \\ \lambda \mathfrak{f}_\lambda \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ [\vec{c}(\cdot,\lambda), \vec{c}_0]_\infty & [\vec{s}(\cdot,\lambda), \vec{c}_0]_\infty \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} =: Y_1 \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}. \tag{67}$$

It follows from the Weyl function definition 2.4 and Lemma 3.13, that

$$\begin{aligned} M(\lambda) = Y_1 Y_0^{-1} &= \frac{1}{[\vec{s}(\cdot,\lambda), \vec{s}_0]_\infty} \begin{pmatrix} -[\vec{c}(\cdot,\lambda), \vec{s}_0]_\infty & 1 \\ [\vec{c}(\cdot,\lambda), \vec{s}(\cdot,\lambda)]_\infty & [\vec{s}(\cdot,\lambda), \vec{c}_0]_\infty \end{pmatrix} \\ &= \frac{1}{[\vec{s}(\cdot,\lambda), \vec{s}_0]_\infty} \begin{pmatrix} -[\vec{c}(\cdot,\lambda), \vec{s}_0]_\infty & 1 \\ 1 & [\vec{s}(\cdot,\lambda), \vec{c}_0]_\infty \end{pmatrix}. \end{aligned} \tag{68}$$

By the definition of the γ-field, we have

$$\gamma(\lambda) = (\mathfrak{c}_\lambda \;\; \mathfrak{s}_\lambda) Y_0^{-1} = \frac{1}{[\vec{s}(\cdot,\lambda), \vec{s}_0]_\infty} (c(\cdot,\lambda) \;\; s(\cdot,\lambda)) \begin{pmatrix} [\vec{s}(\cdot,\lambda), \vec{s}_0]_\infty & 0 \\ -[\vec{c}(\cdot,\lambda), \vec{s}_0]_\infty & 1 \end{pmatrix}. \tag{69}$$

□

Remark 4.7. Noticing that

$$\begin{aligned} [\vec{s}(\cdot,\lambda), \vec{s}_0]_\infty = -\lambda \mathfrak{s}_\lambda \mathfrak{s}_0|_\infty, \quad [\vec{c}(\cdot,\lambda), \vec{s}_0]_\infty = -\lambda \mathfrak{c}_\lambda \mathfrak{s}_0|_\infty, \\ [\vec{s}(\cdot,\lambda), \vec{c}_0]_\infty = -\lambda \mathfrak{s}_\lambda \mathfrak{c}_0|_\infty, \end{aligned} \tag{70}$$

one can clarify the formulas (63) and (64).

5. Special cases

5.1. **Absolutely continuous case.** Let functions P, Q and W be absolutely continuous on $[0,\infty)$, i.e. there exist functions p, q and w from $L^1[0,\infty)$ such that

$$P(x)=\int_0^x p(t)dt,\quad Q(x)=\int_0^x q(t)dt,\quad W(t)=\int_0^x w(t)dt, \tag{71}$$

$p(t)\neq 0$ and $w(t)\geqslant 0$ almost everywhere with respect to Lebesgue measure on $[0,\infty)$.

In this case system (51) may be written as a special Hamiltonian system

$$J\vec{f}'(x)=\lambda\Delta(x)\vec{f}(x)+H(x)\vec{f}(x),\quad \vec{f}(0)=\vec{a}(0), \tag{72}$$

where

$$\Delta(x)=\begin{pmatrix} w(x) & 0\\ 0 & 0\end{pmatrix},\quad H(x)=\begin{pmatrix} -q(x) & 0\\ 0 & p(x)\end{pmatrix},\quad \vec{f}(x)=\begin{pmatrix} f(x)\\ f^{[1]}(x)\end{pmatrix},$$

which is also equivalent to the Sturm-Liouville equation in the most general form

$$-\frac{d}{dx}\left(\frac{1}{p(x)}\frac{d}{dx}f(x)\right)+q(x)f(x)=\lambda w(x)f(x),\quad f(0)=a,\quad f^{[1]}(0)=a_1. \tag{73}$$

Analogues of the Titchmarsh-Weyl coefficient for general canonical systems with matrix valued coefficients $\Delta(x)$ and $H(x)$ were given in [12, 26, 2]. Boundary triple approach to general canonical systems was developed in [20, 4, 23]. Spectral and pseudospectral functions of regular (resp. singular) systems were described in [26, 2] (resp. [23]). Notice that our results of Theorems 3.17, 4.4, and 4.6 in the absolutely continuous case are contained in the corresponding statements of [4].

5.2. **The Krein-Feller operator.** Suppose $dQ\equiv 0$, $dP=dx$ is the Lebesgue measure on $[0,\infty)$, and function W is arbitrary increasing such that $P(0)=0$. In this special case the function f is absolutely continuous and $f^{[1]}$ coincides with the derivative f' a.e. on $[0,\infty)$. The system (19) may be written as

$$f(x)=f(0)+xf^{[1]}(0)-\int_0^x (x-s)g(s)\,dW(s). \tag{74}$$

The differential operation (74) was investigated by I. Kats and M. Krein in [18]. There was shown that equation (74) is in the limit circle case precisely if the integral $\int_0^\infty x^2\,dW(x)$ diverges. In this case the Weyl function $M(\lambda)$ introduced in Theorem 4.4 coincides with the Stieltjes function $\Gamma(\lambda)$ associated with the orthogonal spectral function of a singular string in [18, Theorem 10.1]

In [5, Proposition 2.4] the last criteria was extended to a more general case. With an arbitrary function P equation (74) takes the form

$$f(x)=f(0)+P(x)f^{[1]}(0)-\int_0^x (P(x)-P(s))\,g(s)\,dW(s). \tag{75}$$

It has been shown in [5] that (75) is in the limit point case if and only if the integral $\int_0^\infty \left(1+|P(x)|^2\right)\,dW(x)$ diverges.

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