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ON EIGENVALUES OF BANDED MATRICES

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Abstract. In the paper, asymptotics for eigenvalues of Hermitian, compact operators, generated by infinite, banded matrices is obtained in terms of the asymptotics of their matrix entries. Analogues for banded matrices of Gershgorin's disks theory are discussed.

1. Introduction

An infinite matrix $A = \|a_{ij}\|_{i,j\ge 1}$ is called *banded* or *finite band-width* if there is a positive integer N so that

$$a_{ij} = 0, \quad |i-j| > N, \quad a_{ij} \not\equiv 0, \quad |i-j| = N. \tag{1.1}$$

In this case the number N is referred to as the *order* of A. Hermitian, banded matrices are banded matrices A with $a_{ji} = \overline{a}_{ij}$.

It is well known, see, e.g., [2, Chapter VII], [3, Theorem 8.6.6], that the banded matrices with uniformly bounded entries, i.e., $\sup_{i,j}|a_{ij}| < \infty$, generate bounded operators on the Hilbert space $\ell^2 = \ell^2(\mathbb{N})$. We use the same symbol A for such operators and call them the *banded operators*. Such operators are Hermitian as long as such are the corresponding matrices.

Banded matrices arise in various problems of analysis and spectral theory of linear operators [6, 7, 8, 9]. One of the main problems here is the connection between the spectrum of an operator A (its location and asymptotics) and the behavior of its matrix entries a_{ij}. In particular, if A is an Hermitian, compact operator, the problem relates the asymptotic behavior of the eigenvalues of A, and the asymptotics of a_{ij} as $i,j\to\infty$.

A banded matrix of order $N = 1$ (a *Jacobi matrix*) is one of the main objects in the theory of orthogonal polynomials on the real line. In this setting the matrix entries (also known as the Jacobi parameters) appear in the three-term recurrence relation for the orthogonal polynomials. There is a one-to-one correspondence between such matrices and orthogonality measures on the real line. The above problem concerns the description of the support of this orthogonality measure in terms of the behavior of its Jacobi parameters.

We proceed as follows. In Section 2 we obtain a two-sided bound for Hermitian, banded matrices.

Theorem 1.1. *Let $A = \|a_{ij}\|_{i,j\ge 1}$ be an Hermitian, banded matrix of order N with uniformly bounded entries. Then*

$$\begin{gathered} C_- \le A \le C_+, \quad C_\pm = \operatorname{diag}\big(c_j^\pm\big)_{j=1}^\infty, \\ c_j^\pm = a_{jj} \pm \sum_{m=1}^N \big(|a_{j,j-m}| + |a_{j,j+m}|\big), \quad j = 1,2,\dots . \end{gathered} \tag{1.2}$$

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As a consequence, the asymptotics for the spectrum of such a matrix is obtained in the case where A is compact, and its main diagonal dominates the other ones.

In Section 3 we prove an analogue of the well-known Gershgorin's theorem for banded matrices, and study the location of their spectrum (Gershgorin's disks theory).

2. Two-sided bounds for Hermitian, banded matrices

As usual, a matrix inequality $A \le B$ is understood in the sense of quadratic forms

$$A \le B \quad \Leftrightarrow \quad (Ax, x) \le (Bx, x), \quad \forall x \in \ell^2.$$

Proof of Theorem 1.1.
Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1,m+1} & \dots & a_{1,N+1} & 0 & 0 & \dots \\ \overline{a}_{12} & a_{22} & \dots & a_{2,m+1} & a_{2,m+2} & \dots & a_{2,N+2} & 0 & 0 \\ \vdots & \vdots & \vdots & & & & & & \\ \overline{a}_{1,N+1} & \overline{a}_{2,N+1} & \dots & & & & & & \\ 0 & \overline{a}_{2,N+2} & \overline{a}_{3,N+2} & & & & & & \\ 0 & 0 & \overline{a}_{3,N+3} & & & & & & \\ \vdots & \vdots & \vdots & & & & & & \end{bmatrix}.$$

Put $A_0 := \operatorname{diag}(a_{jj})_{j\ge 1}$, and define, for $m = 1, 2, \dots, N$, Hermitian, two-diagonal matrices

$$A_m = \begin{bmatrix} 0 & 0 & \dots & a_{1,m+1} & 0 & \dots \\ 0 & 0 & \dots & \dots & a_{2,m+2} & 0 \\ \vdots & \vdots & \vdots & & & \\ \overline{a}_{1,m+1} & 0 & \dots & & & \\ 0 & \overline{a}_{2,m+2} & 0 & & & \\ 0 & 0 & \vdots & & & \\ \vdots & \vdots & & & & \end{bmatrix} = \|a_{ij}^{(m)}\|_{i,j=1}^{\infty},$$

where

$$a_{ij}^{(m)} := \begin{cases} 0, & |i-j| \ne m; \\ a_{ij}, & |i-j| = m. \end{cases}$$

So we have the expansion of A in the sum of the matrices

$$A = A_0 + \sum_{m=1}^{N} A_m.$$

We are looking for nonnegative, diagonal matrices $B_m = \operatorname{diag}\big(b_j^{(m)}\big)_{j\ge 1}$ so that

$$-B_m \le A_m \le B_m, \quad m = 1, \dots, N.$$

We have

$$B_m - A_m = \begin{bmatrix} b_1^{(m)} & 0 & \dots & -a_{1,m+1} & 0 & \dots \\ 0 & b_2^{(m)} & \dots & \dots & -a_{2,m+2} & 0 \\ \vdots & \vdots & \vdots & & & \\ -\overline{a}_{1,m+1} & 0 & \dots & & & \\ 0 & -\overline{a}_{2,m+2} & 0 & & & \\ 0 & 0 & \vdots & & & \\ \vdots & \vdots & & & & \end{bmatrix}.$$

For $f = (f_j)_{j\geq 1}$ define

$$g^{(m)} = \big(g_j^{(m)}\big)_{j\geq 1} := (B_m - A_m)f, \quad m = 1, \dots, N,$$

where

$$\begin{aligned} g_j^{(m)} &= b_j^{(m)} f_j - a_{j,m+j} f_{m+j}, \quad j = 1, \dots, m, \\ g_j^{(m)} &= -\overline{a}_{j-m,j} f_{j-m} + b_j^{(m)} f_j - a_{j,m+j} f_{m+j}, \quad j = m+1, \dots . \end{aligned}$$

It is easy to compute the quadratic form

$$\begin{aligned} \big((B_m - A_m)f, f\big) &= \sum_{j=1}^{m} \big(b_j^{(m)} f_j - a_{j,m+j} f_{m+j}\big)\overline{f}_j \\ &+ \sum_{j=m+1}^{\infty} \big(-\overline{a}_{j-m,j} f_{j-m} + b_j^{(m)} f_j - a_{j,m+j} f_{m+j}\big)\overline{f}_j \\ &= \sum_{j=1}^{\infty} b_j^{(m)} |f_j|^2 - \sum_{j=1}^{\infty} a_{j,m+j} f_{m+j} \overline{f}_j - \sum_{j=1}^{\infty} \overline{a}_{j,m+j} f_j \overline{f}_{j+m}, \end{aligned}$$

so, finally,

$$\big((B_m - A_m)f, f\big) = \sum_{j=1}^{\infty} b_j^{(m)} |f_j|^2 - 2\mathrm{Re} \sum_{j=1}^{\infty} a_{j,m+j} f_{m+j} \overline{f}_j.$$

In exactly the same way we find

$$\big((B_m + A_m)f, f\big) = \sum_{j=1}^{\infty} b_j^{(m)} |f_j|^2 + 2\mathrm{Re} \sum_{j=1}^{\infty} a_{j,m+j} f_{m+j} \overline{f}_j.$$

Hence, in view of $b_j^{(m)} \geq 0$, we come to the bound from below

$$\begin{aligned} \big((B_m \pm A_m)f, f\big) &\geq \sum_{j=1}^{\infty} b_j^{(m)} |f_j|^2 - \sum_{j=1}^{\infty} |a_{j,m+j}| \big(|f_j|^2 + |f_{m+j}|^2\big) \\ &= \sum_{j=1}^{\infty} \big(b_j^{(m)} - \beta_j^{(m)}\big) |f_j|^2 \end{aligned}$$

with

$$\beta_j^{(m)} = \begin{cases} |a_{j,m+j}|, & j = 1, \dots, m; \\ |a_{j,m+j}| + |a_{j-m,j}|, & j = m+1, \dots. \end{cases}$$

Under the standard convention $a_{pq} = 0$ for $\min(p,q) \leq 0$, we can take

$$b_j^{(m)} := |a_{j,m+j}| + |a_{j-m,j}|, \quad j = 1, 2, \dots .$$

For such a choice for $b_j^{(m)}$, the inequalities $-B_m \leq A_m \leq B_m$ hold, and so

$$A_0 - \sum_{m=1}^{N} B_m \leq A \leq A_0 + \sum_{m=1}^{N} B_m.$$

The proof is complete. □

Remark 2.1. Two-sided, banded matrices $A = \|a_{ij}\|_{i,j=-\infty}^{\infty}$ are defined by the same condition (1.1). The result (1.2) holds for such matrices as well.

Assume that A is an Hermitian, banded matrix of order N, which generates a compact operator A on ℓ^2. A standard result from the spectral theory [1, Section 31] provides a necessary and sufficient condition for

$$\lim_{n\to\infty} a_{n,n+k} = 0, \quad k = \pm 1, \pm 2, \dots, \pm N. \tag{2.1}$$

In this case the majorants $C_\pm$ are also Hermitian and compact, diagonal operators. The spectrum $\sigma(A)$ consists of two sequences of eigenvalues of finite multiplicity (either of them may be finite or missing), which tend to zero from above and from below, respectively,

$$\sigma(A) = \{\lambda_n^\pm\}_{n\ge 1}: \quad \lambda_1^+ \ge \lambda_2^+ \ge \cdots \ge 0, \quad \lambda_1^- \le \lambda_2^- \le \cdots \le 0.$$

The result below is an immediate consequence of Theorem 1.1.

Corollary 2.2. *In the hypothesis of Theorem 1.1, let A be a compact operator, let $\nu_1^+ \ge \nu_2^+ \ge \cdots$ be a rearrangement in a non-increasing order of the sequence of nonnegative numbers from (c_j^+), and $\nu_1^- \le \nu_2^- \le \cdots$ a rearrangement in a non-decreasing order of the sequence of non-positive numbers from (c_j^-). Then*

$$\nu_n^- \le \lambda_n^- \le \lambda_n^+ \le \nu_n^+, \quad n = 1, 2, \dots.$$

Indeed, now λ_j^+ (λ_j^-) is the j-th eigenvalue of A from the top (resp., from the bottom), and so are ν_j^+ (ν_j^-) for the operators $C_\pm$. The Min-Max Principle for the eigenvalues, see [10, Theorem XIII-1], completes the proof.

Example 2.3. Let A be an Hermitian, banded matrix so that $a_{nn} \to 0$ as $n \to \infty$, and

$$a_{n,n+k} = o(|a_{nn}|), \quad n \to \infty, \quad k = \pm 1, \pm 2, \dots, \pm N. \tag{2.2}$$

In other words, the main diagonal dominates the other ones. Then the asymptotics of the eigenvalues $(\lambda_{nn})_{n\ge 1}$ is the same as that of $(a_{nn})_{n\ge 1}$. For example, a particular case of (2.2) is

$$\begin{aligned} a_{nn} &= \frac{c}{n^\alpha}\big(1 + o(1)\big), \quad n \to \infty, \quad c, \alpha > 0; \\ a_{n,n+j} &= o\Big(\frac{1}{n^\alpha}\Big), \quad n \to \infty, \quad j = \pm 1, \pm 2, \dots, \pm N. \end{aligned} \tag{2.3}$$

Then the number of eigenvalues λ_n^- is at most finite, and

$$\lambda_n^+ = \frac{c}{n^\alpha}\big(1 + o(1)\big), \quad n \to \infty. \tag{2.4}$$

Note that the results close to (2.3), (2.4) for banded, not necessarily Hermitian matrices (in a more pricese form) are obtained recently by A. Pushnitski, see [9, Theorem 1.4].

3. Gershgorin's disks for banded matrices

There is a well-known result due to S. Gershgorin [5] concerning the domains that contain the spectra of finite matrices (Gershgorin's disks theory), see, e.g., [4, Section XIV.5]. As it turns out, a version of this result holds for certain banded matrices as well.

Given a bounded, linear operator T on the Hilbert space, we denote by $\sigma(T)$ its spectrum, and by $\sigma_p(T)$ its point spectrum, that is, the set of all eigenvalues.

Theorem 3.1. *Let $A = \|a_{ij}\|_{i,j\ge 1}$ be a banded, not necessarily Hermitian matrix of order N, with uniformly bounded entries. Then the point spectrum $\sigma_p(A)$ is contained in the union of Gershgorin's disks for A*

$$\sigma_p(A) \subset \bigcup_{k=1}^\infty G_k, \quad G_k := \Big\{z \in \mathbb{C}: \ |z - a_{kk}| \le \sum_{j\ne k} |a_{jk}|\Big\}, \quad k = 1, 2, \dots. \tag{3.1}$$

Proof. Note that G_k is well defined for banded matrices, as the number of nonzero a_{jk} with $j \neq k$ does not exceed $2N+1$, so the sum in (3.1) is finite. The argument below follows the original one.

Let $\mu \in \sigma_p(A)$, and $f = (f_j)_{j\geq 1}$ be the corresponding eigenvector. Since $f_n \to 0$ as $n \to \infty$, there is a positive integer k so that $|f_k| \geq |f_j|$ for all positive integers j. The eigenvalue equation $Af = \mu f$ implies

$$\mu f_k = \sum_{j=k-N}^{k+N} a_{kj} f_j = a_{kk} f_k + \sum_{j\neq k} a_{kj} f_j,$$

and so

$$|\mu - a_{kk}|\,|f_k| \leq \sum_{j\neq k} |a_{kj}||f_j| \leq |f_k| \sum_{j\neq k} |a_{kj}|.$$

Since the vector f is nonzero, $|f_k| > 0$, and we come to the inequality

$$|\mu - a_{kk}| \leq \sum_{j\neq k} |a_{kj}| \quad \Rightarrow \quad \mu \in G_k,$$

as claimed. □

Remark 3.2. Assume that (2.1) holds, that is, the banded operator A is compact in ℓ^2. According to the spectral theorem for compact operators on the Hilbert spaces, the spectrum $\sigma(A)$ is the set of eigenvalues of finite algebraic multiplicity with the only possible accumulation point at the origin [3, Theorem 9.4.1]

$$\sigma(A) = \sigma_p(A) \cup \{0\} = \{\mu_n\}_{n\geq 1}, \quad \lim_{n\to\infty} \mu_n = 0. \tag{3.2}$$

The above Theorem 3.1 provides some information about the location of the spectrum $\sigma(A)$.

It is not hard to calculate the number of eigenvalues of A in Gershgorin's disk G_q, at least in the case of compact operators in question. We restrict ourselves with operators of infinite rank (the case of finite rank operators is actually the original one of Gershgorin).

Theorem 3.3. *Let A be a compact, banded operator of infinite rank. Assume that the disk G_q is disjoint from the closure of the union of the rest,*

$$G_q \cap F_q = \emptyset, \quad F_q := \overline{\Big(\bigcup_{j\neq q} G_j\Big)}. \tag{3.3}$$

Let the number a_{qq} occur in the diagonal $(a_{jj})_{j\geq 1}$ n times. Then the disk G_q contains exactly n, counting algebraic multiplicity, eigenvalues of A.

Proof. Put $A_0 := \operatorname{diag}(a_{jj})_{j\geq 1}$, and for $0 \leq t \leq 1$ define a family of operators

$$A(t) := (1-t)A_0 + tA = \|a_{ij}(t)\|_{i,j\geq 1}, \quad a_{ij}(t) = \begin{cases} a_{ii}, & i = j; \\ t a_{ij}, & i \neq j. \end{cases}$$

It is clear from (2.1) that $A(t)$ is compact for each t.

Since A is of infinite rank, there are infinitely many eigenvalues μ_n in (3.2). By Theorem 3.1, $\mu_n \in F_q$ for $n \geq n_0$, and so $0 \in F_q$.

For $0 \leq t \leq 1$ we define Gershgorin's disks $G_m(t)$ and the closed sets $F_m(t)$ for the banded matrix $A(t)$ as in (3.3). Clearly,

$$G_q(t) \subset G_q, \quad F_q(t) \subset F_q,$$

so, by Theorem 3.1, $\sigma(A(t)) \subset G_q \cup F_q$ for all t.

Next, let γ be a closed contour enclosing the disk G_q so that F_q is outside γ. We can write the spectral projection

$$P_q(t) := \frac{1}{2\pi i}\int_\gamma (\lambda - A(t))^{-1}\,d\lambda.$$

Its range is a direct sum of the algebraic eigenspaces, corresponding to the eigenvalues from G_q, and its dimension $\dim P_q(\cdot)$ equals the sum of algebraic multiplicities of those eigenvalues. The latter function is continuous on $[0,1]$, and takes integer values. Therefore, $\dim P_q$ is constant. But, by the hypothesis of the theorem, $\dim P_q(0) = n$, so $\dim P_q(1) = n$, as claimed.

The proof is complete. □

Remark 3.4. Instead of one disk G_q one can take a union of a finite (or even infinite) number of disks. The condition (3.3) now looks

$$G(N)\cap F(N) = \emptyset, \quad G(N) := \bigcup_{q=1}^{N} G_q \quad F(N) := \overline{\Big(\bigcup_{q=N+1}^{\infty} G_q\Big)}. \tag{3.4}$$

The set $G(N)$ contains exactly n, counting multiplicity, eigenvalues of A, where n is the number of total occurrences of the entries $\{a_{qq}\}_{q=1}^{N}$ in the main diagonal.

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