Methods of Functional Analysis and Topology
Vol. 25 (2019), no. 2, pp. 104–117

ON MAXIMAL MULTIPLICITY OF EIGENVALUES OF FINITE-DIMENSIONAL SPECTRAL PROBLEM ON A GRAPH

OLGA BOIKO, OLGA MARTYNYUK, AND VYACHESLAV PIVOVARCHIK

Abstract. Recurrence relations of the second order on the edges of a metric connected graph together with boundary and matching conditions at the vertices generate a spectral problem for a self-adjoint finite-dimensional operator. This spectral problem describes small transverse vibrations of a graph of Stieltjes strings. It is shown that if the graph is cyclically connected and the number of masses on each edge is not less than 3 then the maximal multiplicity of an eigenvalue is $\mu+1$ where μ is the cyclomatic number of the graph. If the graph is not cyclically connected and each edge of it bears at least one point mass then the maximal multiplicity of an eigenvalue is expressed via μ, the number of edges and the number of interior vertices in the tree obtained by contracting all the cycles of the graph into vertices.

1. Introduction

Second order difference equations (1) appear in different fields of physics (synthesis of electrical circuits [7], p. 129, transversal vibrations of the so-called Stieltjes strings [13], [2], and longitudinal vibrations of point masses connected by springs [16]).

A massless elastic thread bearing point masses is called a Stieltjes string [2]. Small transverse vibrations of such a string are described by the equation

$$\frac{v_k(t)-v_{k+1}(t)}{l_k}+\frac{v_k(t)-v_{k-1}(t)}{l_{k-1}}+m_kv_k''(t)=0 \quad (k=1,2,\dots,n), \tag{1}$$

where $v_k(t)$ is the transverse displacement of the mass m_k and l_k is the distance between m_{k-1} and m_k. We assume the number of masses n to be finite. Let the ends of this string be fixed, i.e. $v_0(t)=v_{n+1}(t)=0$ (Dirichlet conditions). Substituting $v_k(t)=u_ke^{i\lambda t}$ and $z=\lambda^2$ we obtain

$$\frac{u_k-u_{k+1}}{l_k}+\frac{u_k-u_{k-1}}{l_{k-1}}-zm_ku_k=0 \quad (k=1,2,\dots,n), \tag{2}$$

$$u_0=u_{n+1}=0. \tag{3}$$

It is known (see [2]) that the eigenvalues of this problem are simple and for any sequence of distinct positive numbers $\{z_k\}_{k=1}^n$ there exist sequences $\{m_k\}_{k=1}^n$, $\{l_k\}_{k=0}^n$ of positive numbers such that $\{z_k\}_{k=1}^n$ is the spectrum of the corresponding Dirichlet problem (2), (3). Also it is known [2] that the data necessary and sufficient to solve the inverse problem of recovering the sequences $\{m_k\}_{k=1}^n$, $\{l_k\}_{k=0}^n$ consist of two spectra of Dirichlet-Dirichlet spectral problem (2), (3) and Dirichlet-Neumann spectral problem which is (2) with conditions $u_0=u_{n+1}-u_n=0$ and the total length of the string. Generalizations to the case of damped vibrations are given in [23], [17].

A natural generalization of problem (2), (3) are the problems generated by equations of the type (2) on tree domains [3], [4], [5]. An unexpected application can be found in [6]. The simplest tree is a star graph and the simplest generalization of the two

2010 *Mathematics Subject Classification.* Primary 05C50; Secondary 39A06, 15A18, 39A70.
Key words and phrases. Tree, cycle, eigenvalue.

spectra inverse problem of [2] is given in [1] for the case of simple eigenvalues and for the general case in [19]. More complicated case appears if the numbers of the masses on the edges of the star graph are prescribed. It causes restrictions on possible multiplicities of eigenvalues. Such inverse problem was solved in [20]. It should be mentioned that those restrictions on the multiplicities of the eigenvalues are similar to the ones obtained in [14] for the so-called tree-patterned matrices. The difference is that the point masses in [20] can be considered as the vertices of the corresponding star-patterned matrices but the central vertex may be free of masses.

It should be mentioned that the inverse problem by two spectra with simple eigenvalues for an arbitrary tree was solved in [18].

Unfortunately, any general answer about restrictions on eigenvalue multiplicities for a spectral problem on an arbitrary tree as well as for an arbitrary tree patterned matrices is not known in spite of many particular results for tree-patterned matrices in [9], [10], [11].

The problem of maximal possible multiplicity of eigenvalues for the Sturm-Liouville spectral problem on graph was solved in [15] and [12]. In present paper we show that in the case where each edge of the graph of Stieltjes strings bears not less than three point masses the results of [12] remain true. However, if the number of masses is less than 3, the results are not true for cyclically connected graphs (see Remark 3.3 below).

In Section 2 we give the corresponding definitions and describe the spectral problem generated by the Stieltjes string recurrence relations on a connected graph. In Section 3 we consider cyclically connected graphs and show that if the number of point masses is not less than 3 on each edge then the maximal possible multiplicity of an eigenvalue $\omega = \mu + 1$ where μ is the number of linearly independent cycles in the graph. In Section 4 we consider quasi-trees and show that if the number of masses on each edge is greater or equal to 1 then $\omega = \mu + p_{pen} - 1$ where p_{pen} is the number of pendant vertices. In Section 5 we consider an arbitrary connected graph which is not cyclically connected. We prove that $\omega = \mu_0 + g^T - p_{in}^T = \mu_0 + p_{pen}^T - 1$ where μ_0 is the number of linearly independent cycles, g^T, p_{in}^T and p_{pen}^T are the numbers of edges, of interior and pendant vertices, respectively, in the tree obtained by contracting each cycle of the graph into a vertex.

2. Formulation of the problem

For a graph G we denote its vertices by v_i, $i = 1, 2, \ldots, p$, where p (or p^G) is the number of the vertices of G, its edges by e_j, $j = 1, 2, \ldots, g$, where g (or g^G) is the number of edges of G. For each i denote by $d(v_i)$ the degree of the vertex v_i and for each j we denote by l_j the length of the edge e_j.

Assumption. We notice that presence of vertices of degree 2 does not change the maximal multiplicity of the eigenvalues. Therefore, we assume all through the paper that each interior vertex is of degree higher than 2.

Definition 2.1. A walk in a graph is a sequence of edges $v_0v_1, v_1v_2, \ldots, v_{k-1}v_k$. A walk in which all the edges are distinct is called trail. A trail in which all vertices $v_0, v_1, \ldots, v_k$ are distinct (except possibly for $v_k = v_0$) is called a path; a path with $v_k = v_0$ is called a cycle. Any edge joining a vertex v_i to itself is called a loop.

Definition 2.2. Two vertices v and w of a connected graph G are said to be cyclically connected if a finite set of cycles $C_1, C_2, \ldots, C_k$ ($C_j \subset G$, $j = 1, 2, \ldots, k$) exists such that $v \in C_1$, $w \in C_k$ and each neighboring pair of cycles possesses at least one common vertex.

Definition 2.3. A graph is said to be cyclically connected if each pair of vertices in it is cyclically connected (see Fig. 1).

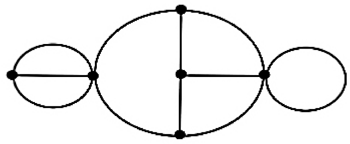

FIGURE 1. A cyclically connected graph.

Definition 2.4. A cyclically connected subgraph G' of a graph G is said to be maximal cyclically connected if it is not a nontrivial subgraph of another cyclically connected subgraph of G.

Definition 2.5. A connected graph with $g \geq 1$ is said to be a tree if it has no cycles. The edges of a tree are said to be its branches.

Definition 2.6. A connected graph with $g \geq 1$ is said to be a quasi-tree if it is not cyclically connected and each (if any) cyclically connected subgraph of it has more than one vertex in common with the complement of this subgraph. (An example see at Fig. 2 below).

If we direct the edges of G to obtain an oriented graph then in addition to the degree $d(v_i)$ of a vertex v_i we introduce $d^-(v_i)$ the *indegree*, the number of edges directed towards the vertex and $d^+(v_i)$, the *outdegree*, the number of edges directed away from the vertex v_i.

The local coordinate on G identifies a directed edge e_j ($j = 1, 2, \ldots, g$) of G with the interval $[0, l_j]$ and the coordinate x_j increases in the direction of the edge.

To every cycle we ascribe any of the two possible directions. It is clear that the direction of an edge can be opposite to the direction of the cycle.

Definition 2.7. The matrix $M = \{M_{k,j}\}$, $k = 1, 2, \ldots, s$, $j = 1, 2, \ldots, g$, where s is the number of cycles, is said to be the matrix of cycles for an oriented graph G if

1) for an edge e_j which does not belong to the k-th cycle $M_{k,j} = 0$,

2) for an edge e_j which belongs to the k-th cycle and whose direction coincides with the direction of the cycle $M_{j,k} = 1$,

3) for an edge e_j which belongs to the k-th cycle and whose direction is opposite to the direction of the cycle $M_{j,k} = -1$.

Definition 2.8. A set of cycles in an oriented graph G is said to be linearly independent if the corresponding set of rows in the matrix of cycles is linearly independent. The rank $\mu = \mu^G$ of this matrix is said to be the cyclomatic number of the graph G. Each set of μ linearly independent cycles is said to be fundamental.

It is known (see, e.g. [22]) that $\mu = g - p + 1$.

Let G be a plane metric graph with q edges and denote by l_j the length of the edge e_j. Each edge e_j is divided into $n_j + 1$ subintervals of the lengths $l_0^{(j)}$, $l_1^{(j)}$,..., $l_{n_j}^{(j)}$ by point masses $m_1^{(j)}$, $m_2^{(j)}$, ..., $m_{n_j}^{(j)}$ ($l_k^{(j)} > 0$, $m_k^{(j)} > 0$, $l_j = \sum_{k=0}^{n_j} l_k^{(j)}$). An interior vertex v_i has outgoing edges e_j starting with a subinterval of length $l_0^{(j)}$, while each incoming edge e_r ends at v with an interval of lengths $l_{n_r}^{(r)}$. It is assumed that the graph is stretched and the pendant vertices are fixed. The graph can vibrate in the direction orthogonal to the equilibrium position of the strings. We denote by $v_k^{(j)}(t)$ the transverse displacement of the mass $m_k^{(j)}$. If an edge e_j is ingoing for an interior vertex v then the displacement of the ingoing end of the edge is denoted by $v_{n_j+1}^{(j)}(t)$, while if an edge e_r is outgoing for a vertex v then the displacement of the outgoing end of the edge is denoted $v_0^{(r)}(t)$. Using such notation vibrations of the graph can be described by the system of equations

$$\frac{v_k^{(j)}(t)-v_{k+1}^{(j)}(t)}{l_k^{(j)}}+\frac{v_k^{(j)}(t)-v_{k-1}^{(j)}(t)}{l_{k-1}^{(j)}}+m_k^{(j)}\frac{\partial^2 v_k^{(j)}}{\partial t^2}(t)=0 \tag{4}$$
$$(k=1,2,\dots,n_j;\quad n_j\geq 1,\quad j=1,2,\dots,g).$$

Let J be the set of numbers of the edges incident with pendant vertices, K be the set of numbers of interior vertices, W_i^+ the set of numbers of edges outgoing away from the vertex v_i and W_i^- the set numbers of edges incoming into the vertex v_i $(i=1,2,\dots,p)$.

For each interior vertex with ingoing edges e_j $(j\in W_i^-)$ and outgoing edges e_r $(r\in W_i^+)$ we impose the continuity conditions

$$v_0^{(r)}(t)=v_{n_j+1}^{(j)}(t). \tag{5}$$

If $W_i^-=\emptyset$ $(W_i^+=\emptyset)$ then instead of (5) we have $v_{n_j+1}^{(j)}(t)=v_{n_s+1}^{(s)}(t)$ $(v_0^{(j)}(t)=v_0^{(s)}(t))$ for all $j,s\in W_i^-$ $(j,s\in W_i^+)$.

The balance of forces at such a vertex implies

$$\sum_{r\in W_i^+}\frac{v_1^{(r)}(t)-v_0^{(r)}(t)}{l_0^{(r)}}=\sum_{j\in W_i^-}\frac{v_{n_j+1}^{(j)}(t)-v_{n_j}^{(j)}(t)}{l_{n_j}^{(j)}}. \tag{6}$$

Here the the right-hand side (left-hand side) of (6) must be taken zero in case of $W_i^+=\emptyset$ $(W_i^-=\emptyset)$. For an edge e_j incident with a pendant vertex we impose Dirichlet boundary condition

$$v_0^{(j)}(t)=0\quad\text{or}\quad v_{n_j+1}^{(j)}(t)=0. \tag{7}$$

Substituting $v_k^{(j)}(t)=e^{i\lambda t}u_k^{(j)}$, $z=\lambda^2$ into (4)–(7) we obtain

$$\frac{u_k^{(j)}-u_{k+1}^{(j)}}{l_k^{(j)}}+\frac{u_k^{(j)}-u_{k-1}^{(j)}}{l_{k-1}^{(j)}}-m_k^{(j)}zu_k^{(j)}=0 \tag{8}$$
$$(k=1,2,\dots,n_j,\ j=1,2,\dots,g).$$

For each interior vertex with incoming edges e_j $(j\in W_i^-)$ and outgoing edges e_r $(r\in W_i^+)$ we have

$$u_0^{(r)}=u_{n_j+1}^{(j)}, \tag{9}$$

$$\sum_{r\in W_i^+}\frac{u_1^{(r)}-u_0^{(r)}}{l_0^{(r)}}=\sum_{j\in W_i^-}\frac{u_{n_j+1}^{(j)}-u_{n_j}^{(j)}}{l_{n_j}^{(j)}}. \tag{10}$$

If $W_i^-=\emptyset$ $(W_i^+=\emptyset)$ then instead of (9) we have $u_{n_j+1}^{(j)}=u_{n_s+1}^{(s)}$ $(u_0^{(j)}=u_0^{(s)})$ for all $j,s\in W_i^-$ $(j,s\in W_i^+)$. Here the the right-hand side (left-hand side) of (10) must be taken zero in case of $W_i^+=\emptyset$ $(W_i^-=\emptyset)$.

For each edge e_j incident with a pendant vertex

$$u_0^{(j)}=0 \tag{11}$$

or

$$u_{n_j+1}^{(j)}=0. \tag{12}$$

The fundamental system of two linearly independent solutions to (8) can be composed by the polynomials $R_{2k-2}^{(j)}(z,0)$ and $R_{2k-2}^{(j)}(z,1)$ which satisfy (see e.g. [18]) the initial conditions $R_0^{(j)}(z,0)=1$, $R_{-1}^{(j)}(z,0)=\frac{1}{l_0^{(j)}}$, $R_0^{(j)}(z,1)=1$, $R_{-1}^{(j)}(z,1)=0$ and the recurrence relations

$$R^{(j)}_{2k-1}(z,0)=-zm^{(j)}_k R^{(j)}_{2k-2}(z,0)+R^{(j)}_{2k-3}(z,0),$$

$$R^{(j)}_{2k-1}(z,1)=-zm^{(j)}_k R^{(j)}_{2k-2}(z,1)+R^{(j)}_{2k-3}(z,1), \tag{13}$$

$$R^{(j)}_{2k}(z,0)=l^{(j)}_k R^{(j)}_{2k-1}(z,0)+R^{(j)}_{2k-2}(z,0) \quad (k=1,2,\dots,n_j),$$

$$R^{(j)}_{2k}(z,1)=l^{(j)}_k R^{(j)}_{2k-1}(z,1)+R^{(j)}_{2k-2}(z,1) \quad (k=1,2,\dots,n_j), \tag{14}$$

We are looking for the solutions of (8) in the form

$$u^{(j)}_k=R^{(j)}_{2k-2}(z,0)q^{(j)}_1+R^{(j)}_{2k-2}(z,1)h^{(j)}_1$$

on the edge e_j with constants $q^{(j)}_1$ and $h^{(j)}_1$.

With this notations we obtain using (9)–(12):

For each interior vertex with incoming edges e_j and outgoing edge e_k

$$R^{(j)}_{2n_j}(z,0)q^{(j)}_1+R^{(j)}_{2n_j}(z,1)h^{(j)}_1=h^{(r)}_1, \tag{15}$$

$$\sum_{r\in W_i^+}\frac{q^{(r)}_1}{l^{(r)}_0}=\sum_{j\in W_i^-}(R^{(j)}_{2n_j-1}(z,0)g^{(j)}_1+R^{(j)}_{2n_j-1}(z,1)h^{(j)}_1). \tag{16}$$

For each edge e_j incident with a pendant vertex

$$h^{(j)}_1=0 \tag{17}$$

or

$$R^{(j)}_{2n_j-1}(z,0)q^{(j)}_1+R^{(j)}_{2n_j-1}(z,1)h^{(j)}_1=0. \tag{18}$$

Then the *characteristic polynomial* of problem (14)–(18), i.e. a polynomial whose zeros coincide with the spectrum of the problem can be expressed by $l^{(j)}_0R^{(j)}_{2n_j}(z,0)$, $l^{(j)}_0R^{(j)}_{2n_j-1}(z,0)$, $R^{(j)}_{2n_j}(z,1)$ and $R^{(j)}_{2n_j-1}(z,1)$. To do it we introduce the following system of vectors

$$\psi_j(z)$$
$$=\operatorname{col}\{\underbrace{0,0,\dots,0,l^{(j)}_0R^{(j)}_{-2}(z,0),l^{(j)}_0R^{(j)}_0(z,0),\dots,l^{(j)}_0R^{(j)}_{2n_j}(z,0),0,0,\dots,0}_{n+2g},\underbrace{0,0,\dots,0}_{n+2g}\},$$

$$\psi_{j+g}(z)$$
$$=\operatorname{col}\{\underbrace{0,0,\dots,0}_{n+2g},\underbrace{0,0,\dots,0,R^{(j)}_{-2}(z,1),R^{(j)}_0(z,1),\dots,R^{(j)}_{2n_j}(z,1),0,0,\dots,0}_{n+2g}\}$$

for $j=1,2,\dots,g$, where g is the number of edges in G, $n=\sum\limits_{j=1}^{g}n_j$. As in [21] we denote by L_j $(j=1,2,\dots,2g)$ the linear functionals $C^{2n+4g}\to C$ generated by (14)–(18). Then $\Phi(z)=\{L_j(\psi_p(z)\}^{2g}_{j,p}$ is the characteristic matrix which represents the system of linear equations describing the boundary conditions at pendant vertices and continuity and balance of forces conditions for the interior vertices. We call

$$\phi_N(z):=\det(\Phi(z))$$

the *characteristic polynomial* of problem (8)–(11). It is easy to see that the characteristic function satisfies

$$\phi_N(\overline{z})=\overline{\phi_N(z)}.$$

Remark 2.9. If W_i^+ or W_i^- is empty then the 0 must stand in the left- or the right-hand side of (16), correspondingly. Also condition (15) should look like

$$h_1^{(j)} = h_1^{(s)} \quad \text{for all} \quad j, s \in W_i^+$$

or

$$R^{(j)}_{2n_j-1}(z,0)q_1^{(j)} + R^{(j)}_{2n_j-1}(z,1)h_1^{(j)} = R^{(s)}_{2n_s-1}(z,0)q_1^{(s)} + R^{(s)}_{2n_s-1}(z,1)h_1^{(s)} \text{ for all } j, s \in W_i^-,$$

correspondingly.

Let G be the above described graph with local coordinates on its edges. Changing the masses $m_k^{(j)}$ ($k = 1, 2, \dots, n_j, j = 1, 2, \dots, g$) and the intervals $l_k^{(j)}$ ($k = 0, 1, \dots, n_j, j = 1, 2, \dots, g$) we change the operator $\mathcal{L}$ and therefore its spectrum and the multiplicities of its eigenvalues too. We denote the set of all obtained operators by $\mathcal{L}_G$. In this section we find the maximal possible value of an eigenvalue multiplicity of the operators $\mathcal{L} \in \mathcal{L}_G$ for a graph G of a given form. Namely, we show that the maximal multiplicity of an eigenvalue depends only on such parameters of the graph as the cyclomatic number, the number of noncyclic edges, the number of noncyclic interior vertices and the number of the so-called maximal cyclically connected subgraphs. This question can be reduced to the following one. Let arbitrary $z \geq 0$ be fixed. For the set of all $\mathcal{L} \in \mathcal{L}_G$ for which λ is an eigenvalue what the maximal multiplicity of it can be. It will be clear that the result does not depend on the lengths of the edges. It is required that $n_j \geq 1$ and for some of our results even $n_j \geq 3$ for all $j = 1, 2, \dots, g$.

3. Cyclically connected graphs

In this section the directions of edges are arbitrary. Let us notice that in a cyclically connected graph all the vertices are interior.

Since in this section graphs have no pendant vertices we consider problem (8)–(10). We consider also the auxiliary problem which we obtain from problem (8)–(10) imposing additionally any of the conditions

$$u_0^{(j)} = 0, \quad \text{for some } j \in W_i^+ \text{ and some } i \tag{19}$$

or

$$u_{n_j+1}^{(j)} = 0, \quad \text{for some } j \in W_i^- \text{ and some } i. \tag{20}$$

We will use the following lemma.

Lemma 3.1 (Lemma 3.1 in [12]). *Let G be a cyclically connected graph with the cyclomatic number $\mu \geq 2$. Then there exists an edge in G such that after deleting this edge we obtain a graph G' which is cyclically connected and its cyclomatic number is $\mu' = \mu - 1$.*

Theorem 3.2.

1. Let $n_j \geq 3$ for all $j = 1, 2, \dots, g$. The maximal multiplicity of an eigenvalue of problem (8)–(10) on a cyclically connected graph is $\mu + 1$.

2. If in addition to (8)–(10) we impose condition (19) or (20) at any of the vertices in G, then the maximal multiplicity of an eigenvalue of problem (8)–(10), (19) or problem (8)–(10), (20), is μ.

Proof. By assumption there are no vertices of degree 2 and, therefore, each vertex is of degree higher than 2.

First of all let us show that the multiplicity can reach $\mu + 1$. For a fixed positive z we can choose masses and subintervals such that $m_k^{(j)} = m_{n_j-k+1}^{(j)}$, $l_k^{(j)} = l_{n_j-k}^{(j)}$ for all k and j, and $R^{(j)}_{2n_j}(z,0) = 0$. By the Lagrange identity (see Lemma 3.5 in [19])

$$R^{(j)}_{2n_j}(z,1)R^{(j)}_{2n_j-1}(z,0) - R^{(j)}_{2n_j}(z,0)R^{(j)}_{2n_j-1}(z,1) = R^{(j)}_{2n_j}(z,1)R^{(j)}_{2n_j-1}(z,0) = \frac{1}{l_0^{(j)}}. \tag{21}$$

Since for a symmetric string $R^{(j)}_{2n_j-1}(z,0) = \pm l^{(j)}_0 R^{(j)}_{2n_j}(z,1)$, we obtain from (21) $R^{(j)}_{2n_j-1}(z,0) = l^{(j)}_0 R^{(j)}_{2n_j}(z,1) = \pm 1$. The masses $\{m^{(j)}_k\}_{k=1}^{n_j}$ and the intervals $\{l^{(j)}_k\}_{k=0}^{n_j}$ can be chosen such that not only $R^{(j)}_{2n_j}(z,0) = 0$ but also $R^{(j)}_{2n_j-1}(z,0) = 1$. This is possible because due to the condition $n_j \geq 3$ the number of eigenvalues of Dirichlet problem (8), (11), (12) on an edge is not less than 3 and z can be chosen equal to the second eigenvalue of problem (8), (11), (12).

Then we choose a cycle C_i with the edges $e_{j^i_k}$, $k = 1, 2, \ldots, r(i)$, $j^i_k < j^i_{k+1}$ and compose the corresponding eigenvector

$$Y_i = \{0, 0, \ldots, S_{j^i_1}(z,0), 0, 0, \ldots, 0, S_{j^i_2}(z,0), 0, 0, \ldots, 0, S_{j^i_{r(i)}}(z,0), 0, 0, \ldots, 0\},$$

where

$$S_{j^i_s}(z,0) = \{l^{(j^i_s)}_0 R^{(j^i_s)}_0(z,0), l^{(j^i_s)}_0 R^{(j^i_s)}_2(z,0), \ldots, l^{(j^i_s)}_0 R^{(j^i_s)}_{2n_{j^i_s}-2}(z,0)\}. \tag{22}$$

Thus, we can compose an eigenvector corresponding to each of μ fundamental cycles. One more eigenvector is

$$Y_{\mu+1} = \{1, Q_1(z,1), 1, Q_2(z,1), 1, \ldots, 1, Q_g(z,1)\},$$

where $Q_j(z,1) = \{R^{(j)}_0(z,1), R^{(j)}_2(z,1), \ldots, R^{(j)}_{2n_j-2}(z,1)\}$.

All in all we have $\mu+1$ linearly independent eigenvectors. Since the vectors Y_j ($j = 1, 2, \ldots, \mu$) satisfy condition (19) or (20) we conclude that the maximal multiplicity of an eigenvalue of problem (8)–(10), (19) or (8)–(10), (20) can reach μ.

Now we need to prove that the maximal multiplicity does not exceed $\mu+1$ for problem (8)–(10) and that it does not exceed μ for problem (8)–(10), (19) or problem (8)–(10), (20).

First consider the case of $\mu = 1$. Here we face the periodic problem

$$\frac{u_k - u_{k+1}}{l_k} + \frac{u_k - u_{k-1}}{l_{k-1}} - m_k z u_k = 0 \tag{23}$$

$$(k = 1, 2, \ldots, n),$$

$$u_0 = u_{n+1}, \tag{24}$$

$$\frac{u_1 - u_0}{l_0} = \frac{u_{n+1} - u_n}{l_n}. \tag{25}$$

The maximal multiplicity of an eigenvalue of this problem is 2 with the eigenvectors $\{0, S_1(\lambda,0)\}$ and $\{1, Q_1(\lambda,1)\}$. Thus, Statement 1 of Theorem 3.2 is valid in this case. Statement 2 for $\mu = 1$ follows from uniqueness of the solution of problem (23)–(24) under additional condition $u_0 = 0$.

Let us assume the statements of Theorem 3.5 to be valid for all cyclically connected graphs with the cyclomatic number $\mu = k$. Let us consider a cyclically connected graph G with $\mu = k+1$. We need to prove that the maximal multiplicity m does not exceed $k+2$. Let us recall that we have no vertices of degree 2. According to Lemma 3.1 it is possible to delete an edge e_{j_1} from G in such a way that the obtained graph G' will be cyclically connected with the cyclomatic number $\mu' = k$. If after deleting the edge a vertex of degree 2 appears then we can remove that vertex and contract the incident edges (without changing μ') so that achieve the assumption. Let v be the vertex from which e_{j_1} is going away.

By our assumption the maximal multiplicity of an eigenvalue of problem (8)–(10) on G' is $k+1$ and for the problem with additional condition (19) or (20) on G' the maximal multiplicity is k. For any eigenvalue of problem (8)–(10) on G the eigen-subspace can be chosen in such a way that

$$\begin{aligned}
Y_1 &= \{U_{1,1}, \ldots, U_{1,j_1-1}, Q_{j_1}(\lambda, 1), U_{1,j_1+1}, \ldots, U_{1,g}(x)\},\\
Y_2 &= \{U_{2,1}, \ldots, U_{2,j_1-1}, S_{j_1}(\lambda, 0), U_{2,j_1+1}, \ldots, U_{2,g}\},\\
Y_3 &= \{U_{3,1}, \ldots, U_{3,j_1-1}, 0, 0, \ldots, 0, U_{3,j_1+1}, \ldots, U_{3,g}\},\\
&\ldots,\\
Y_m &= \{U_{m,1}, \ldots, U_{m,j_1-1}, 0, 0, \ldots, 0, U_{m,j_1+1}, \ldots, U_{m,g}\},
\end{aligned}$$

where $U_{k,j}(\lambda)$ are some solution of (8) on the edge e_j.

Consider the $m-2$ projections $\tilde{Y}_3 = \{U_{3,1}, \ldots, U_{3,j_1-1}, U_{3,j_1+1}, \ldots, U_{3,g}\}, \ldots,$ $\tilde{Y}_m = \{U_{m,1}, \ldots, U_{m,j_1-1}, U_{m,j_1+1}, \ldots, U_{m,g}\}$ of vectors $Y_3, \ldots, Y_m$. These projections are eigenvectors of the problem on G' because the continuity conditions and the balance of forces condition are satisfied by $\tilde{Y}_j$, $j = 3, 4, \ldots, m$ as the vectors on G'. The additional condition $u_0^{(j_2)} = 0$ or $u_{n_{j_2}}^{(j_2)} = 0$ is satisfied, where e_{j_2} is an edge incident with the vertex for which e_{j_1} was an outgoing edge in G. By the induction assumption $m - 2 \leq k$, i.e. $m \leq k + 2$. Since Y_1 does not satisfy $u_0^{(j_2)} = 0$, we conclude that the maximal eigenvalue multiplicity does not exceed $k + 1$ for problem (8)–(10). Theorem is proved by induction. □

Remark 3.3. It should be mentioned that if $n = 1$ problem (23)–(24) has only one simple eigenvalue $\lambda = 0$ and in case of $n = 2$ it has two simple eigenvalues. This is in accordance with the known result (see [8], Theorem 3.2) for the eigenvalues of a periodic Jacobi matrices which in our notation is

$$z_1 < z_2 \leq z_3 < z_4 \leq z_5 < \cdots$$

4. Quasi-trees

Let us describe orientation of edges and the way of their enumeration we use for quasi-trees. First consider a tree rooted at a pendant vertex. We direct the edges away from the root and enumerate them successively such that the combinatorial distance from the root is a non-decreasing sequence of the number of the edge. Thus each vertex except for the root has one incoming edge and if $j \in W_i^+$ and $k \in W_i^-$ then $k < j$.

Now let G be a quasi-tree. Denote by T the tree obtained by contracting each maximal cyclically connected subgraph of G into a vertex. Having no cycles the obtained graph is a tree according to Definition 2.5. Since G is not cyclically connected, T has at least two pendant vertices. Then we direct the edges of T in the way described above. We enumerate the branches in G, i.e. the edges common for G and T in the way described above but when meeting a vertex in T obtained by contracting a maximal cyclically connected subgraph Γ we omit g^Γ successive numbers where g^Γ is the number of edges in Γ.

Let us direct and enumerate the edges of a maximal cyclically connected subgraph of our quasi-tree. Let Γ be such a subgraph. We call *entrance* the vertex $v_0 \in \Gamma$ incident with the incoming branch while the vertices incident with the outgoing branches v_i $(i = 1, 2, \ldots, n)$ we call *exits.*

The following two lemmas were proved in [12].

Lemma 4.1. *Let Γ be a maximal cyclically connected subgraph of a quasi-tree. Then the edges of Γ can be directed and enumerated such that*

1) $W_i^- = \emptyset$ only for the entrance (for v_0),

2) $W_i^+ = \emptyset$ can be only for exits (for v_i, $i = 1, 2, \ldots, n$),

3) if $j \in W_i^+$ and $k \in W_i^-$ then $k < j$.

Lemma 4.2. *The edges of any quasi-tree G can be oriented and enumerated such that*

1) $W_i^- = \emptyset$ only for the root,
2) $W_i^+ = \emptyset$ only for pendant vertices except for the root,
3) if $j \in W_i^+$ and $k \in W_i^-$ then $k < j$.

The maximum for eigenvalue multiplicities for quasi-trees is given by the following theorem.

Theorem 4.3. *Let $n_j \geq 1$ for all j. Then the maximal multiplicity of an eigenvalue of the operator $\mathcal{L}$ defined on a quasi-tree with $g \geq 1$ is $\omega := \mu + p_{pen} - 1$, where p_{pen} is the number of pendant vertices. Equivalently, $\omega = g - p_{in}$ where p_{in} is the number of interior vertices.*

Proof. First of all let us show that the multiplicity can be equal to ω.

As we have seen in the proof of Theorem 3.2 we can choose the masses $\{m_k^{(j)}\}_{k=1}^{n_j}$ and the intervals $\{l_k^{(j)}\}_{k=0}^{n_j}$ $(j = 1, 2, \dots, q)$ such that $R_{2n_j}^{(j)}(z, 0) = 0$ and $R_{2n_j-1}^{(j)}(z, 0) = -1$ for all $j = 1, 2, \dots, g$.

We choose ω linearly independent paths, i.e. paths which correspond to linearly independent rows of the adjacency matrix, $\{e_{i_k}\}_{k=1}^{r[i]}$ ($i = 1, 2, \dots, \omega$), connecting the root with the pendant vertices. This is possible because for a tree the number of such paths is $p_{pen} - 1$ and each linearly independent cycle contributes 1 into the number of such paths. Let us denote by

$$Y_i = \{0, 0, \dots, 0, S_{i_1}(z, 0), 0, 0, \dots, 0, -S_{i_2}(z, 0), 0, 0, \dots, 0, (-1)^{i_{r[i]}-1} S_{i_{r[i]}}(z, 0), 0, 0, \dots, 0\}.$$

The eigenvectors $\{Y_i\}_{i=1}^{\omega}$ are linearly independent. This shows that the multiplicity of an eigenvalue can reach ω.

Now let us prove that the multiplicity cannot exceed ω. The solution of equations (8) on an edge e_j is

$$U_j = B_j(1, Q_j(z, 1)) + A_j(0, S_j(z, 0)), \tag{26}$$

where A_j and B_j are constants.

The boundary condition at the root is $u_0^{(1)} = 0$ and therefore

$$B_1 = 0. \tag{27}$$

Let $W_i^- = \{i_1^-, i_2^-, \dots, i_{d^-(i)}^-\}$ and $W_i^+ = \{i_1^+, i_2^+, \dots, i_{d^+(i)}^+\}$, where we enumerate such that $i_k^- < i_{k'}^-$, $i_k^+ < i_{k'}^+$ if $k < k'$ and $i_k^- < i_p^+$ for all $i_k^- \in W_i^-$ and all $i_k^+ \in W_i^+$. The matching conditions at a vertex $v_i \in K$ are

$$\left.\begin{array}{ll} u_0^{(j)} = u_0^{(j')}, & \text{for all } j \in W_i^+ \text{ and } j' \in W_i^+, \\ u_{l_j}^{(j)} = u_{l_{j'}}^{(j')}, & \text{for all } j \in W_i^- \text{ and } j' \in W_i^-, \\ u_0^{(j')} = u_{l_j}^{(j)}, & \text{if } j' \in W_i^+ \text{ and } j \in W_i^- \end{array}\right\}, \tag{28}$$

$$\sum_{j \in W_i^+} \frac{u_1^{(j)} - u_0^{(j)}}{l_0^{(j)}} = \sum_{p \in W_i^-} \frac{u_{n_p+1}^{(p)} - u_{n_p}^{(p)}}{l_{n_p}}, \quad i \in K \quad \{\text{balance of forces}\}. \tag{29}$$

The boundary conditions at the pendant vertices (except for the root) are

$$u_{n_j+1}^{(j)} = 0, \quad j \in J. \tag{30}$$

Substituting (26) into (28), (29) and (30) we obtain

$$B_j - B_{i_1^-} R_{2n_{i_1^-}}^{(i_1^-)}(z, 1) - A_{i_1^-} R_{2n_{i_1^-}}^{(i_1^-)}(z, 0) = 0, \quad \text{for each} \quad j \in W_i^+, \tag{31}$$

$$\sum_{j \in W_i^+} A_j - \sum_{j \in W_i^-} \Big(B_j R_{2n_j-1}^{(j)}(z, 1) + A_j R_{2n_j-1}^{(j)}(z, 0) \Big) = 0. \tag{32}$$

If $d^-(v_i)>1$ we have additionally for each $j\in W_i^-$, $j\neq i_1^-$

$$B_{i_1^-}R^{(i_1^-)}_{2n_{i_1^-}}(z,1)+A_{i_1^-}R^{(i_1^-)}_{2n_{i_1^-}}(z,0)-B_jR^{(j)}_{2n_j}(z,1)-A_jR^{(j)}_{2n_j}(z,0)=0. \tag{33}$$

For each pendant vertex v_j except for the root the Dirichlet boundary conditions give

$$B_jR^{(j)}_{2n_j}(z,1))+A_jR^{(j)}_{2n_j}(z,0)=0. \tag{34}$$

Equations (27), (31)–(34) compose a system of homogeneous linear algebraic equations with unknowns $B_1,A_1,B_2,A_2,\dots,B_g,A_g$. We enumerate the unknowns successively, i.e. B_1 is the first and A_g is the unknown number $2g$. Equations (27), (31), (32) we call *leading* and equip each of the leading equations with a numerical *rating* equal to the maximal number among its unknowns with non-zero coefficients. This coefficient before the unknown of the maximal number we call *leading coefficient* for the corresponding leading equation. According to our way of enumeration $j>j'$ for all $j\in W_i^+$ and $j'\in W_i^-$ and, therefore, all the ratings of the leading equation are different. Let us place the leading equations in the order of growing rating. The rest of the equations we place after the leading equations in arbitrary order. The multiplicity of an eigenvalue λ is equal to the number of linearly independent solutions of the obtained system of equations.

Since the rating of each succeeding leading equation is higher than that of the previous leading equation, each row in the matrix of the system corresponding to a leading equation can not be presented as a linear combination of the previous rows and, consequently, the rows corresponding to the leading equations are linearly independent. Thus, the rank of the systems matrix is not less than the number of leading equations. Let us count this number.

Equation (27) is leading and corresponds to the edge e_1. Each of the other edges of G is outgoing away from a vertex and, therefore, the number of equations of the form (31) is equal to $g-1$. The number of an equation of the form (32) is equal to the number of interior vertices p_{in}. Thus, the number of leading equation is $1+g-1+p_{in}=g+p_{in}$ and the rank of the matrix is not less than $g+p_{in}$.

Since the number of linearly independent solutions of a system of R homogeneous linear equations is equal to $R-r$ where r is the rank of the systems matrix, we conclude that in our case the number of linearly independent solutions does not exceed $2g-(g+p_{in})=g-p_{in}$.

Let us show that $g-p_{in}=\mu+g^T-p^T_{in}$.

Denote by g^{Γ_k} the number of edges and by p^{Γ_k} the number of vertices in a maximal cyclically connected subgraph Γ_k. We denote by $\tilde p^T_{in}$ the number of interior vertices in G which does not belong to $\Gamma=\bigcup\limits_{k=1}^{t}\Gamma_k$, where t is the number of maximal cyclically connected subgraphs in G. Then

$$\begin{aligned} g-p_{in}&=g^T+\sum_{k=1}^{t}g^{\Gamma_k}-\sum_{k=1}^{t}p^{\Gamma_k}-\tilde p^T_{in}\\ &=g^T+\sum_{k=1}^{t}(g^{\Gamma_k}-p^{\Gamma_k}+1)-t-\tilde p^T_{in}=g^T+\mu-p^T_{in}.\end{aligned} \tag{35}$$

On the other hand, $\mu=g-p+1=g-p_{in}-p_{pen}+1$ and, therefore, $g-p_{in}=\mu+p_{pen}-1$. □

Theorem 4.4. *The statement of Theorem 4.3 remains true if the Dirichlet conditions $u^{(j_1)}_{n_{j_1}+1}=0$ at some of the pendant vertices v_{j_1} of the quasi-tree are replaced with the corresponding Neumann conditions $u^{(j_1)}_{n_{j_1}+1}=u^{(j_1)}_{n_{j_1}}$.*

Proof. Let us show that the eigenvalue multiplicity can reach ω. It is possible to change the masses and subintervals on the edge incident with the pendant vertex v_j such that instead of $R_{2n_{j_1}}(z,0)=0$ we obtain $R_{2n_{j_1}-1}(z,0)=0$. The rest of the proof is the same as in the proof of Theorem 4.3. □

5. Connected Graphs

Now we consider a connected graph G_0 which is not cyclically connected (the case of cyclically connected graphs has been considered in Sec. 3). We exclude the case $g=0$, i.e. our graph has at least one edge. It is clear from Definition 2.6 that any such graph can be obtained by attaching cyclically connected graphs Γ_k $(k=1,2,\ldots,t)$ to a quasi-tree G in such a way that each Γ_k has only one common vertex with G. If we contract each maximal cyclically connected subgraph of G_0 into a vertex then we obtain a connected graph with at least one edge which has no cycles, i.e. a tree T (see Definition 2.5). For the quasi-tree G obtained from G_0 by contracting each maximal cyclically connected subgraph which has only one common vertex with its complement in G_0 we denote by μ the cyclomatic number of G. We denote by μ_0 the cyclomatic number of G_0 and by g_0 the number of edges in G_0. An example see at Fig. 2.

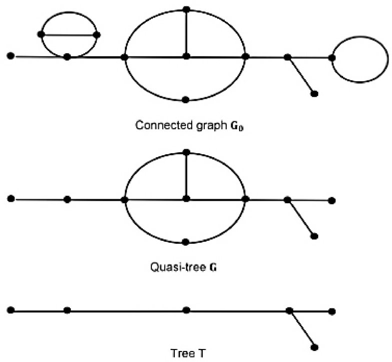

Figure 2

Theorem 5.1. *Let G_0 be a connected but not cyclically connected graph. Let $n_j\geq 3$ for all $j=1,2,\ldots,g$. Then the maximal possible multiplicity of an eigenvalue of the operator $\mathcal{L}$ is $\mu_0+g^T-p_{in}^T$.*

Proof. If our graph G_0 is a quasi-tree then Theorem 5.1 is nothing but Theorem 4.3. Now let us show that the maximal possible multiplicity of an eigenvalue of the operator $\mathcal{L}$ on G_0 is $\mu_0+g^T-p_{in}^T$ if we attach a maximal cyclically connected subgraph Γ of cyclomatic number $\mu^\Gamma=\mu_0-\mu$ to a quasi-tree G. First let us show that the multiplicity of an eigenvalue of $\mathcal{L}$ on G_0 can reach $\mu_0+g^T-p_{in}^T$. Consider the following two cases.

1. We attach the maximal cyclically connected subgraph Γ of cyclomatic number μ^Γ to an interior vertex of the quasi-tree G.

Let us start enumerating the edges of G_0 with the edges $e_j\in G$ $(j=1,2,\ldots,g)$. In the proof of Theorem 4.4 it was shown that for some value of z we can choose the sets $\{m_k^{(j)}\}_{k=1}^{n_j}$ and $\{l_k^{(j)}\}_{k=0}^{n_j}$ $(j=1,2,\ldots,g)$ such that there will be $\mu+g^T-p_{in}^T$ linearly independent eigenvectors of $\mathcal{L}$ on G satisfying $R_{2n_j}^{(j)}(z,0)=0$ and $l_0^{(j)}R_{2n_j-1}^{(j)}(z,0)=R_{2n_j}^{(j)}(z,1)=1$ for $j=1,2,\ldots,g$. Being prolonged by 0 onto Γ they compose $\mu+g^T-p_{in}^T$ linearly independent eigenvectors $\{Y_i\}_{i=1}^{\mu+g^T-p_{in}^T}$ of $\mathcal{L}$ on $G_0=G\cup\Gamma$. We can choose the set $\{m_k^{(j)}\}_{k=1}^{n_j}$ and $\{l_k^{(j)}\}_{k=0}^{n_j}$ $(j=g+1,g+2,\dots,g_0)$ where $e_j\in\Gamma$ such that $R_{2n_j}^{(j)}(z,0)=0$ and $l_0^{(j)}R_{2n_j-1}^{(j)}(z,0)=R_{2n_j}^{(j)}(z,1)=1$ for each edge of the attached maximal cyclically connected subgraph Γ. Let $\{C_i^\Gamma\}_{i=1}^{\mu_0-\mu}$ be some fundamental set of cycles in the maximal cyclically connected subgraph Γ. Like in Section 3 for each cycle C_i^Γ in Γ with the edges $e_{i[k]}^\Gamma$, $i=1,2,\dots,r(i)$, $i[k]<i[k+1]$ we compose the corresponding eigenvector $Y_i^\Gamma=\{0,0,\dots,0,S_{i[1]}^\Gamma(z,0),0,0,\dots,0,S_{i[2]}^\Gamma(z,0),0,0,\dots,0,S_{i[r(i)]}^\Gamma(z,0),0,0,\dots,0\}$. Thus we obtain the set $\{Y_i^\Gamma\}_{i=\mu+g^T-p_{in}^T+1}^{\mu_0+g^T-p_{in}^T}$ of linearly independent eigenvectors for $\mathcal{L}$ on Γ which being prolonged by 0 onto G are also eigenvectors for $\mathcal{L}$ on $G_0=G\cup\Gamma$.

Therefore, all in all we have $\mu_0-\mu+\mu+g^T-p_{in}^T=\mu_0+g^T-p_{in}^T$ linearly independent eigenvectors.

2. We attach the maximal cyclically connected subgraph of cyclomatic number $\mu^\Gamma=\mu_0-\mu$ to a pendant vertex v_i of the quasi-tree G. According to Theorem 4.4 the set $\{q_j(x)\}_{j=1}^g$ can be chosen such that there exist $\mu+g^T-p_{in}^T$ linearly independent eigenvectors of $\mathcal{L}$ on G with Neumann condition at v_i and Dirichlet conditions at the rest of pendant vertices. Each of these eigenvectors being prolonged onto Γ by $c_j(\lambda,x)$ for each $e_j\in\Gamma$ form an eigenvector on G_0. It is clear that all these eigenvectors are linearly independent and their number is $\mu+g^T-p_{in}^T$.

Then for each cycle C_i^Γ in Γ with the edges $e_{i[k]}^\Gamma$, $k=1,2,\dots,r(i)$, $i[k]<i[k+1]$ we compose the corresponding eigenvector
$Y_i^\Gamma=\{0,0,\dots,0,S_{i[1]}^\Gamma(z,0),0,0,\dots,0,S_{i[2]}^\Gamma(z,0),0,0,\dots,0,S_{i[r(i)]}^\Gamma(z,0),0,0,\dots,0\}$ for the operator $\mathcal{L}$ on Γ. Being prolonged onto G by 0 these vectors are linearly independent eigenvectors for the operator $\mathcal{L}$ on $G_0=G\cup\Gamma$. It is clear that all these eigenvectors are linearly independent. Thus, again all in all we have $\mu_0-\mu+\mu_0+g^T-p_{in}^T=\mu_0+g^T-p_{in}^T$ linearly independent eigenvectors. Attaching cyclically connected subgraphs successively we see that the multiplicity of an eigenvalue of $\mathcal{L}$ on G_0 can reach $\mu_0+g^T-p_{in}^T$ in case of finite number of cyclically connected subgraphs attached.

Now let us prove that the maximal multiplicity does not exceed $\mu_0+g^T-p_{in}^T$.

Let us attach a cyclically connected graph Γ_1 to the quasi-tree G in such a way that Γ_1 has only one common vertex with G. We denote the obtained graph by $G^{(1)}=\Gamma_1\cup G$.

It follows from Theorem 3.2 that there exist not more than μ^{Γ_1} linearly independent eigenvectors of $\mathcal{L}$ on $G^{(1)}$ which are identically zero on the edges of G.

The number of linearly independent eigenvectors of $\mathcal{L}$ on G does not exceed $\mu+g^T-p_{in}^T$ according to Theorem 4.4. Therefore, the number of linearly independent eigenvectors of the problem on $G^{(1)}$ which are not identically zero on G does not exceed $\mu+g^T-p_{in}^T$. This means that the number of eigenvectors for the operator $\mathcal{L}$ on $G^{(1)}$ does not exceed $\mu^{\Gamma_1}+\mu+g^T-p_{in}^T$. Thus, Theorem 5.1 is proved for the case of a quasi-tree with one attachment. Now we attach a cyclically connected graph Γ_2 to $G^{(1)}$ such that Γ_2 and $G^{(1)}$ have only one vertex in common and denote $G^{(2)}=\Gamma_2\cup G^{(1)}$. Again Theorem 3.2 implies that there exist not more than μ^{Γ_2} linearly independent eigenvectors of $\mathcal{L}$ on $G^{(2)}$ which are identically zero on the edges of $G^{(1)}$.

The number of linearly independent eigenvectors of $\mathcal{L}$ on $G^{(1)}$ does not exceed $\mu^{\Gamma_1}+\mu+g^T-p_{in}^T$ according to what has been proved above. Therefore, the number of linearly independent eigenvectors of the problem on $G^{(2)}$ which are not identically zero on $G^{(1)}$ does not exceed $\mu^{G_1}+\mu+g^T-p_{in}^T$. This means that the number of eigenvectors for the operator $\mathcal{L}$ on $G^{(2)}$ does not exceed $\mu^{\Gamma_2}+\mu^{\Gamma_1}+\mu+g^T-p_{in}^T$. By this procedure we can include all the attachments and finally obtain that the maximal multiplicity of an eigenvalue of $\mathcal{L}$ on $G_0=G\bigcup\limits_{k=1}^{t}\Gamma_k$ does not exceed $\sum\limits_{k=1}^{t}\mu^{\Gamma_k}+\mu+g^T-p_{in}^T=\mu_0+g^T-p_{in}^T$. $\square$

References

1. O. Boyko and V. Pivovarchik, *Inverse spectral problem for a star graph of Stieltjes strings*, Methods Funct. Anal. Topology **14** (2008), 159–167.
2. F. R. Gantmakher and M. G. Krein, *Oscillating Matrices and Kernels and Vibrations of Mechanical Systems*, GITTL, Moscow-Leningrad, 1950 (Russian). German transl. Akademie Verlag, Berlin, 1960. English transl. AMS Chelsea Publishing, Providence, RI, 2002.
3. J. Genin and J. S. Maybee, *Mechanical vibrations trees*, J. Math. Anal. Appl. **45** (1974), 746–763.
4. G. Gladwell, *Inverse Problems in Vibration*, Kluwer Academic Publishers, Dordrecht, 2004.
5. G. Gladwell, *Matrix inverse eigenvalue problems*. In: G. Gladwell, A. Morassi, eds., Dynamical Inverse Problems: Theory and Applications, CISM Courses and Lectures, vol. 529, 2011, pp. 1–29.
6. P. Glendinning, *Shaking and whirling: dynamics of spiders and their webs*, MIMS E Print 2018.7, http://eprints.maths.manchester.ac.uk.
7. E. A. Guillemin, *Synthesis of Passive Networks. Theory and Methods Appropriate to the Realization and Approximation Problems*, John Wiley and Sons, Inc., New York, 1958.
8. W. E. Ferguson, Jr., *The construction of Jacobi and periodic Jacobi matrices with perturbed spectra*, Math. Comp., **35** (1980), no. 152, 1203–1220.
9. C. R. Johnson and A. Leal Duarte, *On the possible multiplicities of the eigenvalues of a Hermitian matrix whose graph is a tree*, Linear Alg. Appl. **348** (2002), 7–21.
10. C. R. Johnson, A. Leal Duarte, and C. M. Saiago, *Inverse eigenvalue problems and list of multiplicities of eigenvalues for matrices whose graph is a tree: the case of generalized stars and double generalized stars*, Linear Alg. Appl. **373** (2003), 311–330.
11. C. R. Johnson, B. Kroschel, and M. Omladic, *Eigenvalue multiplicities in principal submatrices*, Linear Alg. Appl. **390** (2004), 111–120.
12. I. S. Kac and V. Pivovarchik, *On multiplicity of a quantum graph spectrum*, J. Phys. A: Math. Theor. **44** (2011), 10530, 14 p.
13. M. G. Krein, *On a generalization of Stieltjes investigations*, Doklady Acad. Nauk SSSR **87** (1952), 881–884. (Russian)
14. A. Leal Duarte, *Construction of acyclic matrices from spectral data*, Linear Alg. Appl. **113** (1989), 173–182.
15. J. A. Lubary, *Multiplicity of solutions of second-order linear differential equations on networks*, Linear Alg. Appl. **274** (1998), 301–315.
16. V. A. Marchenko, *Introduction to the Theory of Inverse Problems of Spectral Analysis*, Acta, Kharkov, 2005. (Russian)
17. O. Martynyuk, V. Pivovarchik, C. Tretter, *Inverse problem for a damped Stieltjes string from parts of spectra*, Applicable Analysis **95** (2015), no 12, 2605–2619.
18. V. Pivovarchik, *Existence of a tree of Stieltjes strings corresponding to two given spectra*, J. Phys. A **42** (2009), 375213, 16 p.
19. V. Pivovarchik, N. Rozhenko, C. Tretter, *Dirichlet-Neumann inverse spectral problem for a star graph of Stieltjes strings*, Linear Alg. Appl. **439** (2013), no. 8, 2263–2292.
20. V. Pivovarchik, C. Tretter, *Location and multiplicities of eigenvalues for a star graph of Stieltjes strings*, J. Diffenence Equ. Appl. **21** (2015), 383–402.
21. Yu. V. Pokorny, and V. L. Pryadiev, *The qualitative Sturm-Liouville theory on spatial networks*, J. Mathematical Sciences, **119** (2004), 788-835.
22. S. Seshu, M. B. Reed, *Linear Graphs and Electrical Networks*, Addison-Wesley Pub. Co., 1961.

23. K. Veselić, *On linear vibrational systems with one dimensional damping*, Applicable Analysis **29** (1988), 1–18.

South-Ukrainian National Pedagogical University, 26 Staroportofrankovskaya, Odesa, Ukraine

South-Ukrainian National Pedagogical University, 26 Staroportofrankovskaya, Odesa, Ukraine

South-Ukrainian National Pedagogical University, 26 Staroportofrankovskaya, Odesa, Ukraine
E-mail address: vpivovarchik@gmail.com

Received 03/06/2018