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THE WELLAND INEQUALITY ON HYPERGROUPS

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Abstract. The Welland inequality for fractional integrals on hypergroups with quasi-metric and Haar measure is proved. This inequality gives pointwise estimates of fractional integrals by fractional maximal operators.

1. Introduction

Let $0 < \alpha < n$. The operator

$$\mathbb{I}_\alpha f(x) = \int_{R^n} |x-y|^{\alpha-n} f(y)\, dy$$

is known as the classical Riesz potential. We refer to the monographs [1], [19], [21] for various properties of Riesz potentials.

Also define fractional maximal operator on R^n as

$$\mathbb{M}_\alpha f(x) = \sup_{r>0} \frac{1}{|b(0,r)|^{1-\frac{\alpha}{n}}} \int_{b(0,r)} |f(x-y)| dy,$$

where $b(0,r)$ is the open bold in R^n with center zero and radius $r > 0$, and $|b(0,r)|$ is the Lebesgue measure of $b(0,r)$.

It is well known the following fact. Let $0 < \alpha < n$. For any ϵ, $0 < \epsilon < \min(\alpha, n-\alpha)$, there exists constant $C > 0$ such that for any nonnegative locally integrable function $f : R^n \to R_+$ and for any $x \in R^n$ the following inequality holds:

$$\mathbb{I}_\alpha f(x) \le C\sqrt{\mathbb{M}_{\alpha-\epsilon} f(x)\, \mathbb{M}_{\alpha+\epsilon} f(x)}\,.$$

This is known as the Welland inequality for classical Riesz potentials and was proved by G. Welland (see inequality (2.3) in [24]). In [17] and [8], the same results have been obtained for Riesz potentials on homogeneous and nonhomogeneous spaces, correspondingly. For multilinear fractional integrals on R^n this inequality was proved by G. Pradolini (see Theorem 2.26 in [20]). The Welland inequalities for potentials on hypergroups associated with the Laplace-Bessel differential operator and for potentials on Dunkl hypergroups were introduced in [9] and [22] respectively.

In this paper we prove the Welland inequality for fractional integrals on more general hypergroups.

2. Preliminaries

Let K be a set. A function $\rho : K \times K \to [0, \infty)$ is called quasi-metric if:

(1) $\rho(x,y) = 0 \Leftrightarrow x = y$;
(2) $\rho(x,y) = \rho(y,x)$;

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(3) there exists a constant $C_\rho \geq 1$ such that for every $x, y, z \in K$

$$\rho(x, y) \leq C_\rho (\rho(x, z) + \rho(z, y)).$$

In the theory of locally compact groups there arise certain spaces which, though not groups, have some of the structure of groups. Often, the structure can be expressed in terms of an abstract convolution of measures on the space.

A hypergroup $(K, *)$ consists of a locally compact Hausdorff space K together with a bilinear, associative, weakly continuous convolution on the Banach space of all bounded regular Borel measures on K with the following properties:

1. For all $x, y \in K$, the convolution of the point measures $\delta_x * \delta_y$ is a probability measure with compact support.
2. The mapping: $K \times K \to \mathcal{C}(K)$, $(x, y) \mapsto supp(\delta_x * \delta_y)$ is continuous with respect to the Michael topology on the space $\mathcal{C}(K)$ of all nonvoid compact subsets of K, where this topology is generated by the sets
$$U_{V,W} = \{L \in \mathcal{C}(K) : L \cap V \neq \varnothing, L \subset W\}$$
with V, W open in K.
3. There is an identity $e \in K$ with $\delta_e * \delta_x = \delta_x * \delta_e = \delta_x$ for all $x \in K$.
4. There is a continuous involution $\sim$ on K such that
$$(\delta_x * \delta_y)^\sim = \delta_{y^\sim} * \delta_{x^\sim}$$
and $e \in supp(\delta_x * \delta_y) \Leftrightarrow x = y^\sim$ for $x, y \in K$ (see [16], [23], [3], [2], [18]).

The measure λ on K is called Haar measure if for every Borel measurable function f on K,

$$\int_K f(\delta_x * \delta_y) d\lambda(y) = \int_K f(y) d\lambda(y) \quad (x \in K).$$

Hypergroup K is called commutative if $\delta_x * \delta_y = \delta_y * \delta_x$ for all $x, y \in K$.

It is well known that every commutative hypergroup K possesses a Haar measure (see [23]).

Define the generalized translation operators T^x, $x \in K$, by

$$T^x f(y) = \int_K f d(\delta_x * \delta_y)$$

for all $y \in K$. If K is a commutative hypergroup, then $T^x f(y) = T^y f(x)$.

Let K be hypergroup with Haar measure λ. The convolution of two functions is defined by

$$f * g(x) = \int_K T^x f(y^\sim) g(y) d\lambda(y).$$

Let $(K, *)$ be a hypergroup, with quasi-metric ρ, Haar measure λ and all balls $B(x, r) = \{y \in K : \rho(x, y) < r\}$ be λ-measurable. We will say Haar measure λ is a doubling on an identity, if there exists a constant $C_\lambda > 0$, not depending $r > 0$, such that

$$\lambda B(e, 2r) \leq C_\lambda \lambda B(e, r). \tag{1}$$

In this condition holds the triple $(K, *, \lambda)$ we will call a space of homogeneous type on an identity.

To avoid trivial measures we will always assume that $0 < \lambda B(e, r) < +\infty$, for all $r > 0$.

Given a space of homogeneous type $(K, *, \lambda)$ on an identity, we will call that it is a reverse doubling space on an identity if there exists a constant $0 < \gamma < 1$ such that for every $r > 0$ such that $B(e, r) \neq K$,

$$\lambda B\left(e, \frac{r}{2}\right) \leq \gamma \lambda B(e, r). \tag{2}$$

If this condition holds we also say that the measure λ is reverse doubling on an identity.

Lemma 2.1. *Let $(K, *)$ be a hypergroup, with quasi-metric ρ and Haar measure λ, (K, ρ, λ) reverse doubling space on an identity and $\lambda(K) = +\infty$ Then $\lambda\{e\} = 0$.*

Proof. By (2) we have

$$\lambda\{e\} = \lim_{n\to\infty} \lambda B(e, 2^{-n}) \leq \lim_{n\to\infty} \gamma^n \lambda B(e, 1) = 0.$$

□

Lemma 2.2. *Let $(K, *)$ be a hypergroup, with quasi-metric ρ and Haar measure λ, (K, ρ, λ) reverse doubling space on an identity and diam$(K) < +\infty$. Then $\lambda\{e\} = 0$.*

Proof. If $\lambda(K) = +\infty$ then we have the validity of lemma from Lemma 2.1.

Let $\lambda(K) < +\infty$. There exist two points $x, y \in K$ such that $\rho(x, y) > \dfrac{\operatorname{diam}(K)}{2}$. Choose

$$0 < r < \frac{\operatorname{diam}(K)}{8C_\rho}.$$

Then at least one of points x and y does not belong to $B(e, r)$. Otherwise, we have $x, y \in B(e, r)$ and

$$\frac{\operatorname{diam}(K)}{2} < \rho(x, y) \leq C_\rho\left(\rho(e, x) + \rho(e, y)\right) < 2C_\rho r < \frac{\operatorname{diam}(K)}{4}.$$

The obtained contradiction shows that the ball $B(e, r)$ is strictly contained in K. Hence the balls $B(e, 2r^{-n})$ are strictly contained in K, for integer n, and by reverse doubling property we have

$$\lambda\{e\} = \lim_{n\to\infty} \lambda B(e, 2^{-n} r) \leq \lim_{n\to\infty} \gamma^n \lambda B(e, r) = 0.$$

□

3. Main result

Let $(K, *)$ be a hypergroup, with quasi-metric ρ and Haar measure λ and $0 < \beta < 1$. For λ-locally integrable function f on hypergroup K, define fractional maximal operator

$$M_\beta f(x) = \sup_{r>0} \frac{1}{\lambda B(e, r)^{1-\beta}} (|f| * \chi_{B(e,r)})(x)$$

and fractional integral

$$I_\beta f(x) = (f * \lambda B(e, \rho(e, \cdot))^{\beta-1})(x)$$

on hypergroup K. Different problems of fractional integrals and fractional maximal operators were investigated in [5]–[7], [10]–[15].

Now formulate the main result.

Theorem 3.1. *Let $(K, *)$ be a hypergroup, with quasi-metric ρ and Haar measure λ, (K, ρ, λ) reverse doubling space on an identity, ε is a positive number satisfying $\varepsilon < min\{\beta, 1 - \beta\}$. Assume also $\lambda(K) = +\infty$ or diam $(K) < +\infty$. Then there exists a positive constant C such that for every $f \in L^1_{loc}(K)$ and for every $x \in K$,*

$$|\mathrm{I}_\beta f(x)| \leq C\sqrt{\mathrm{M}_{\beta-\varepsilon} f(x) \mathrm{M}_{\beta+\varepsilon} f(x)}. \tag{3}$$

Proof. Without loss of generality we can exclude the cases where $f = 0$ almost everywhere by λ or $|f| = +\infty$ on a set of positive λ-measure. Take any $x \in K$ and fix.

First assume $\lambda(K) = +\infty$. Define set

$$\left\{ \varrho > 0 : \lambda B(e, \varrho) \leq \left(\frac{M_{\beta+\varepsilon} f(x)}{M_{\beta-\varepsilon} f(x)} \right)^{\frac{1}{2\varepsilon}} \right\}. \tag{4}$$

Show that this set is non-empty. Assume the contrary. Then

$$\lambda B(e, \varrho) > \left(\frac{M_{\beta+\varepsilon} f(x)}{M_{\beta-\varepsilon} f(x)} \right)^{\frac{1}{2\varepsilon}}$$

for all $\varrho > 0$. In particular,

$$\lambda B\left(e, \frac{1}{n}\right) > \left(\frac{M_{\beta+\varepsilon} f(x)}{M_{\beta-\varepsilon} f(x)} \right)^{\frac{1}{2\varepsilon}},$$

for an integer n. Hence we have

$$\lambda\{e\} = \lim_{n \to +\infty} \lambda B\left(e, \frac{1}{n}\right) \geq \left(\frac{M_{\beta+\varepsilon} f(x)}{M_{\beta-\varepsilon} f(x)} \right)^{\frac{1}{2\varepsilon}} > 0.$$

But the last relation contradicts Lemma 2.1.

Therefore, the set (4) is non-empty. Also, this set is bounded above. Let

$$\xi = \sup \left\{ \varrho > 0 : \lambda B(e, \varrho) \leq \left(\frac{M_{\beta+\varepsilon} f(x)}{M_{\beta-\varepsilon} f(x)} \right)^{\frac{1}{2\varepsilon}} \right\}.$$

Take any r with $\frac{1}{2}\xi < r < \xi$ and fix. Then $\left(\frac{M_{\beta+\varepsilon} f(x)}{M_{\beta-\varepsilon} f(x)} \right)^{\frac{1}{2\varepsilon}} < \lambda B(e, 2r)$ and by doubling property

$$\frac{1}{C_\lambda} \left(\frac{M_{\beta+\varepsilon} f(x)}{M_{\beta-\varepsilon} f(x)} \right)^{\frac{1}{2\varepsilon}} < \frac{1}{C_\lambda} \lambda B(e, 2r)$$

$$\leq \lambda B(e, r) \leq \lambda B(e, \xi) \leq \left(\frac{M_{\beta+\varepsilon} f(x)}{M_{\beta-\varepsilon} f(x)} \right)^{\frac{1}{2\varepsilon}}.$$

Therefore, we have chose r such that

$$\lambda B(e, r) \sim \left(\frac{M_{\beta+\varepsilon} f(x)}{M_{\beta-\varepsilon} f(x)} \right)^{\frac{1}{2\varepsilon}}. \tag{5}$$

Fix this number r. It is clear that

$$|I_\beta f(x)| \leq |U_1(f(x), r)| + |U_2(f(x), r)|, \tag{6}$$

where

$$U_1(f(x), r) = \int\limits_{B(x,r)} T^x f(y^\sim) \lambda B(e, \rho(e, y))^{\beta - 1} d\lambda(y),$$

$$U_2(f(x), r) = \int\limits_{K \setminus B(x,r)} T^x f(y^\sim) \lambda B(e, \rho(e, y))^{\beta - 1} d\lambda(y).$$

Estimate $|U_1(f(x), r)|$.

$$\begin{aligned}
|U_1\left(f(x),r\right)| &\le \int\limits_{B(x,r)} T^x|f(y^\sim)|\lambda B(e,\rho(e,y))^{\beta-1}d\lambda(y)\\
&=\sum_{k=1}^{\infty}\int\limits_{2^{-k}r\le\rho(e,y)<2^{-k+1}r} T^x|f(y^\sim)|\lambda B(e,\rho(e,y))^{\beta-1}d\lambda(y)\\
&\le\sum_{k=1}^{\infty}\lambda B(e,2^{-k}r)^{\beta-1}\int\limits_{\rho(e,y)<2^{-k+1}r} T^x|f(y^\sim)|d\lambda(y)\\
&=\sum_{k=1}^{\infty}\left(\frac{\lambda B(e,2^{-k+1}r)}{\lambda B(e,2^{-k}r)}\right)^{1-\beta}\frac{\lambda B(e,2^{-k+1}r)^{\varepsilon}}{\lambda B(e,2^{-k+1}r)^{1-\beta+\varepsilon}}\\
&\quad\times\int\limits_{\rho(e,y)<2^{-k+1}r} T^x|f(y^\sim)|d\lambda(y)\\
&\le\sum_{k=1}^{\infty}C_\lambda^{1-\beta}(\gamma^\varepsilon)^k\lambda B(e,r)^\varepsilon M_{\beta-\varepsilon}f(x)\\
&\le\frac{C_\lambda^{1-\beta}}{1-\gamma^\varepsilon}\lambda B(e,r)^\varepsilon M_{\beta-\varepsilon}f(x)
\end{aligned}\tag{7}$$

Now estimate $|U_2\left(f(x),r\right)|$.

$$\begin{aligned}
|U_2\left(f(x),r\right)| &\le \int\limits_{K\setminus B(x,r)} |T^x f(y^\sim)|\lambda B(e,\rho(e,y))^{\beta-1}d\lambda(y)\\
&\le\sum_{k=0}^{\infty}\int\limits_{2^{k}r\le\rho(e,y)<2^{k+1}r} T^x|f(y^\sim)|\lambda B(e,\rho(e,y))^{\beta-1}d\lambda(y)\\
&\le\sum_{k=0}^{\infty}\left(\frac{\lambda B(e,2^{k+1}r)}{\lambda B(e,2^{k}r)}\right)^{1-\beta}\frac{\lambda B(e,2^{k+1}r)^{-\varepsilon}}{\lambda B(e,2^{k+1}r)^{1-\beta-\varepsilon}}\int\limits_{\rho(e,y)<2^{k+1}r} T^x|f(y^\sim)|d\lambda(y)\\
&\le\sum_{k=0}^{\infty}C_\lambda^{1-\beta}(\gamma^\varepsilon)^{k+1}\lambda B(e,r)^{-\varepsilon} M_{\beta+\varepsilon}f(x)\\
&\le\frac{C_\lambda^{1-\beta}}{1-\gamma^\varepsilon}\lambda B(e,r)^{-\varepsilon} M_{\beta+\varepsilon}f(x)
\end{aligned}\tag{8}$$

Substituting (5) into inequalities (7) and (8), we obtain

$$|I_\beta f(x)|\le C\sqrt{M_{\beta-\varepsilon}f(x)M_{\beta+\varepsilon}f(x)},$$

when $\lambda(K)=+\infty$.

Let now $\mathrm{diam}(K)<+\infty$. Without loss of generality we can assume $\lambda(K)<+\infty$. We have

$$\begin{aligned}
&\frac{1}{\lambda B(e,r)^{1-\beta-\varepsilon}}\int\limits_K T^x|f(y^\sim)|\chi_{B(e,r)}(y)d\lambda(y)\\
&\quad\le\lambda(K)^{2\varepsilon}\frac{1}{\lambda B(e,r)^{1-\beta+\varepsilon}}\int\limits_K T^x|f(y^\sim)|\chi_{B(e,r)}(y)d\lambda(y)\\
&\quad\le\lambda(K)^{2\varepsilon}M_{\beta-\varepsilon}f(x).
\end{aligned}$$

Hence

$$M_{\beta+\varepsilon}f(x)=\sup_{r>0}\frac{1}{\lambda B(e,r)^{1-\beta-\varepsilon}}\int\limits_K T^x|f(y^\sim)|\chi_{B(e,r)}(y)d\lambda(y)\leq\lambda(K)^{2\varepsilon}M_{\beta-\varepsilon}f(x)$$

and

$$\left(\frac{M_{\beta+\varepsilon}f(x)}{M_{\beta-\varepsilon}f(x)}\right)^{\frac{1}{2\varepsilon}}\leq\lambda(K)<+\infty. \tag{9}$$

Define set

$$\left\{\varrho>0:\lambda B(e,\varrho)\leq\frac{1}{2}\left(\frac{M_{\beta+\varepsilon}f(x)}{M_{\beta-\varepsilon}f(x)}\right)^{\frac{1}{2\varepsilon}}\right\}. \tag{10}$$

Show that this set is non-empty. Assume the contrary. Then

$$\lambda B(e,\varrho)>\frac{1}{2}\left(\frac{M_{\beta+\varepsilon}f(x)}{M_{\beta-\varepsilon}f(x)}\right)^{\frac{1}{2\varepsilon}}$$

for all $\varrho>0$. Let $\tau>0$ is chosen such that the ball $B(e,\tau)$ is strictly contained in K (Repeating the method in the proof of Lemma 2.2 one can be sure the existence of such ϱ). Then the balls $\lambda B\left(e,\frac{\tau}{n}\right)$ are strictly contained in K and

$$\lambda B\left(e,\frac{\tau}{n}\right)>\left(\frac{M_{\beta+\varepsilon}f(x)}{M_{\beta-\varepsilon}f(x)}\right)^{\frac{1}{2\varepsilon}},$$

for an integer n. Hence we have

$$\lambda\{e\}=\lim_{n\to+\infty}\lambda B\left(e,\frac{\tau}{n}\right)\geq\left(\frac{M_{\beta+\varepsilon}f(x)}{M_{\beta-\varepsilon}f(x)}\right)^{\frac{1}{2\varepsilon}}>0.$$

But the last relation contradicts Lemma 2.2.

Therefore, the set (10) is non-empty. By (9) we have this set is bounded above. Let

$$\xi=\sup\left\{\varrho>0:\lambda B(e,\varrho)\leq\frac{1}{2}\left(\frac{M_{\beta+\varepsilon}f(x)}{M_{\beta-\varepsilon}f(x)}\right)^{\frac{1}{2\varepsilon}}\right\}.$$

Take any r with $\frac{1}{2}\xi<r<\xi$ and fix. Then

$$\frac{1}{2}\left(\frac{M_{\beta+\varepsilon}f(x)}{M_{\beta-\varepsilon}f(x)}\right)^{\frac{1}{2\varepsilon}}<\lambda B(e,2r)$$

and by doubling property

$$\frac{1}{2C_\lambda}\left(\frac{M_{\beta+\varepsilon}f(x)}{M_{\beta-\varepsilon}f(x)}\right)^{\frac{1}{2\varepsilon}}<\frac{1}{C_\lambda}\lambda B(e,2r)$$

$$\leq\lambda B(e,r)\leq\lambda B(e,\xi)\leq\frac{1}{2}\left(\frac{M_{\beta+\varepsilon}f(x)}{M_{\beta-\varepsilon}f(x)}\right)^{\frac{1}{2\varepsilon}}.$$

We showed that if $\mathrm{diam}(K)<+\infty$ then one can choose r such that

$$\frac{1}{2C_\lambda}\left(\frac{M_{\beta+\varepsilon}f(x)}{M_{\beta-\varepsilon}f(x)}\right)^{\frac{1}{2\varepsilon}}<\lambda B(e,r)\leq\frac{1}{2}\left(\frac{M_{\beta+\varepsilon}f(x)}{M_{\beta-\varepsilon}f(x)}\right)^{\frac{1}{2\varepsilon}}. \tag{11}$$

Fix this number r. Split $|I_\beta f(x)|$ as in (6). Since

$$\lambda B(e,r)\leq\frac{1}{2}\left(\frac{M_{\beta+\varepsilon}f(x)}{M_{\beta-\varepsilon}f(x)}\right)^{\frac{1}{2\varepsilon}}\leq\frac{1}{2}\lambda(K)$$

we have that the ball $B(e,r)$ is strictly contained in K. Then the balls $B(e,2^{-k+1}r)$ are strictly contained in K for integer k and and we may apply the reverse doubling condition to the these balls. Therefore one can be sure that the estimate (7) for $|U_1\left(f(x),r\right)|$ is valid when $\operatorname{diam}(K)<+\infty$.

Let $d=\min\{k: k \text{ is an integer}, 2^k r>\operatorname{diam}(K)\}$ Then the sets $\{y: 2^k r\le \rho\left(e,y\right)<2^{k+1}r\}$ are empty, for all integer $k>d-1$. Estimating $|U_2\left(f(x),r\right)|$ we have

$$|U_2\left(f(x),r\right)| \le \sum_{k=0}^{d-1} \int\limits_{2^k r\le\rho(e,y)<2^{k+1}r} T^x|f(y^{\sim})|\lambda B(e,\rho(e,y))^{\beta-1}d\lambda(y)$$

$$\le \sum_{k=0}^{d-1}\left(\frac{\lambda B(e,2^{k+1}r)}{\lambda B(e,2^k r)}\right)^{1-\beta}\frac{\lambda B(e,2^{k+1}r)^{-\varepsilon}}{\lambda B(e,2^{k+1}r)^{1-\beta-\varepsilon}}\int\limits_{\rho(e,y)<2^{k+1}r} T^x|f(y^{\sim})|d\lambda(y).$$

Now let $j=\min\{i: i \text{ is an integer}, 2^i\ge C_\rho\}$. For $0\le k\le d-1$ we have

$$2^{k-j-3}r\le\frac{2^{d-1}r}{8C_\rho}\le\frac{\operatorname{diam}(K)}{8C_\rho}.$$

In the proof of Lemma 2.2 we showed that the ball $B(e,\delta)$, for any $0<\delta<\dfrac{\operatorname{diam}(K)}{8C_\rho}$, is strictly contained in K . Hence the balls $B(e,2^{k-j-3}r)$ are strictly contained in K. By conditions (1) and (2) we have

$$\lambda B(e,2^{k+1}r)\ge\lambda B(e,2^{k-j-3}r)\ge\frac{\lambda B(e,2^{-j-3}r)}{\gamma^k}\ge\frac{\lambda B(e,r)}{C_\lambda^{j+3}\gamma^k}.$$

Continue the estimate of $|U_2\left(f(x),r\right)|$.

$$|U_2\left(f(x),r\right)| \le \sum_{k=0}^{d-1}\left(\frac{\lambda B(e,2^{k+1}r)}{\lambda B(e,2^k r)}\right)^{1-\beta}\frac{\lambda B(e,2^{k+1}r)^{-\varepsilon}}{\lambda B(e,2^{k+1}r)^{1-\beta-\varepsilon}}\int\limits_{\rho(e,y)<2^{k+1}r} T^x|f(y^{\sim})|d\lambda(y)$$

$$\le \sum_{k=0}^{d-1} C_\lambda^{1-\beta}\left(\frac{\lambda B(e,r)}{C_\lambda^{j+3}\gamma^k}\right)^{-\varepsilon} M_{\beta+\varepsilon}f(x) \tag{12}$$

$$\le\frac{C_\lambda^{1-\beta+\varepsilon(j+3)}}{1-\gamma^\varepsilon}\lambda B(e,r)^{-\varepsilon}M_{\beta+\varepsilon}f(x).$$

Substituting (11) into inequalities (7) and (12), we obtain

$$|I_\beta f(x)|\le C\sqrt{M_{\beta-\varepsilon}f(x)M_{\beta+\varepsilon}f(x)},$$

for $\operatorname{diam}(K)<+\infty$. Theorem is proved. □

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