Methods of Functional Analysis and Topology
Vol. 25 (2019), no. 2, pp. 152–160

# ON ISOMETRIES SATISFYING DEFORMED COMMUTATION RELATIONS

OLHA OSTROVSKA AND ROMAN YAKYMIV

Abstract. We consider an $C^*$-algebra $\mathcal{E}_{1,n}^q$, $q \le 1$, generated by isometries satisfying $q$-deformed commutation relations. For the case $|q| < 1$, we prove that $\mathcal{E}_{1,n}^q \simeq \mathcal{E}_{1,n}^0 = \mathcal{O}_{n+1}^0$. For $|q| = 1$ we show that $\mathcal{E}_{1,n}^q$ is nuclear and prove that its Fock representation is faithful. In this case we also discuss the representation theory, in particular construct a commutative model for representations.

## 1. Introduction

In this paper we consider a certain perturbation of a family of pairwise orthogonal isometries. Namely, we study properties and representation theory for the $C^*$-algebra $\mathcal{E}_{1,n}^q$ generated by isometries $t$, $s_j$, $j = \overline{1,n}$, subject to the relations

$$s_i^* s_j = 0, \quad i \ne j, \quad t^* s_j = q s_j t^*.$$

In a recent paper [9] the authors study the $C^*$-algebra $\mathcal{E}_{n,m}^q$ with $n, m \ge 2$, generated by families $\{t_j\}_{j=1}^m$ and $\{s_i\}_{i=1}^n$. In particular, it was shown that for $|q| < 1$ one has $\mathcal{E}_{n,m}^q \simeq \mathcal{E}_{n,m}^0$ and for $|q| = 1$ the $C^*$-isomorphism class of the quotient of $\mathcal{E}_{n,m}^q$ by the unique maximal ideal is independent of $q$ and isomorphic to the tensor product of Cuntz algebras $\mathcal{O}_n \otimes \mathcal{O}_m$.

We show that the result for $|q| < 1$ remains true for $\mathcal{E}_{1,n}^q$ and show that $\mathcal{E}_{1,n}^q$ for $|q| = 1$ is nuclear and its Fock representation is faithful. We also prove an analog of the Wold decomposition Theorem for our family of isometries and present the uniform form (commutative model) of its representations.

## 2. The case $|q| < 1$

Here we discuss briefly the case $|q| < 1$. Namely we explain the way to show the isomorphism $\mathcal{E}_{1,n}^q \simeq \mathcal{E}_{1,n}^0$. Since the ideas we use are similar to the ones presented in [9], we omit the proofs and restrict ourselves to giving basic statements.

Set $\tilde{s}_j = (\mathbf{1} - tt^*) s_j$, $j = \overline{1,n}$.

**Lemma 1.** *The following commutation relations hold:*

$$t^* \tilde{s}_j = 0, \quad j = \overline{1,n}.$$

**Proposition 1.** *For any $j = \overline{1,n}$, one has*

$$s_j = \sum_{k=0}^{\infty} q^k \, t^k \, \tilde{s}_j \, t_k^*.$$

*In particular, the family $\{t, \tilde{s}_j, \ j = \overline{1,n}\}$ generates $\mathcal{E}_{1,n}^q$.*

---

2010 *Mathematics Subject Classification.* 46L05, 46L35, 46L80, 46L65, 47A67, 81R10.

*Key words and phrases.* Cuntz-Toeplitz algebra, $q$-deformation, Fock representation, commutative model.

Suppose that $\mathcal{E}^q_{1,n}$ is realized by Hilbert space operators. Consider the left polar decomposition $\tilde{s}_j = \hat{s}_j \cdot c_j$, where $c_j^2 = \tilde{s}_j^* \tilde{s}_j = \mathbf{1} - |q|^2 tt^* > 0$, implying that $\hat{s}_j$ is an isometry and

$$\hat{s}_j = \tilde{s}_j c_j^{-1} \in \mathcal{E}^q_{1,n}, \quad 1 = \overline{1, n}.$$

**Lemma 2.** *The following commutation relations hold:*

$$t^* \hat{s}_j = 0, \quad j = \overline{1, n},$$
$$\hat{s}_j^* \hat{s}_i = \delta_{ji} \mathbf{1}, \quad j, i = \overline{1, n}.$$

Summing up the results stated above, we get the following.

**Theorem 1.** *Let $\hat{s}_j = (\mathbf{1} - tt^*) s_j (\mathbf{1} - |q|^2 tt^*)^{-1/2}$, $j = \overline{1, n}$. Then the family $\{t, \hat{s}_j\}_{j=1}^n$ generates $\mathcal{E}^q_{1,n}$, and*

$$t^* t = \mathbf{1}, \quad t^* \hat{s}_j = 0, \quad \hat{s}_j^* \hat{s}_i = \delta_{ij} \mathbf{1}, \quad i, j = \overline{1, n}.$$

**Corollary 1.** *Denote by $v_i$, $i = \overline{1, n+1}$, the isometries generating $\mathcal{E}^0_{1,n} = \mathcal{O}^0_{n+1}$. Then Theorem 1 implies that the correspondence*

$$v_1 \mapsto s_1, \quad v_{1+j} \mapsto \hat{s}_j, \quad j = \overline{1, n},$$

*extends uniquely to a surjective homomorphism $\varphi \colon \mathcal{E}^0_{1,n} \to \mathcal{E}^q_{1,n}$.*

Let us present an inverse homomorphism, $\psi \colon \mathcal{E}^q_{n,m} \to \mathcal{E}^0_{n,m}$. To do this, put

$$\tilde{w}_j = v_{1+j} (1 - |q|^2 v_1 v_1^*)^{1/2}, \quad j = \overline{1, n}.$$

Then $\tilde{w}_j^* \tilde{w}_j = 1 - |q|^2 v_1 v_1^*$, and $\tilde{w}_j^* \tilde{w}_i = 0$ if $j \neq i$, $j, i = \overline{1, n}$. Construct

$$w_j = \sum_{k=0}^{\infty} q^k v_1^k \tilde{w}_j (v_1^k)^*, \quad j = \overline{1, n},$$

Note that the series above converges with respect to the norm in $\mathcal{E}^0_{1,n}$.

**Lemma 3.** *The following commutation relations hold:*

$$w_j^* w_i = \delta_{ji} \mathbf{1}, \quad v_1^* w_j = q w_j v_1^*, \quad i, j = \overline{1, n}.$$

**Lemma 4.** *For any $r = \overline{1, m}$, one has $\tilde{w}_j = (\mathbf{1} - v_1 v_1^*) w_j$.*

**Theorem 2.** *Let $v_i$, $i = \overline{1, n+1}$, be the isometries generating $\mathcal{E}^0_{1,n}$. Put*

$$\tilde{w}_j = v_{1+j} (\mathbf{1} - |q|^2 v_1 v_1^*)^{1/2} \quad \textit{and} \quad w_j = \sum_{k=0} q^k v_1^k \tilde{w}_r (v_1^k)^*.$$

*Then*

$$w_j^* w_i = \delta_{ji} \mathbf{1}, \quad v_1^* w_j = q w_j v_1^*, \quad i, j = \overline{1, n}.$$

*Moreover, the family $\{v_1, w_j\}_{j=1}^n$ generates $\mathcal{E}^0_{1,n}$.*

**Corollary 2.** *The statement of Theorem 2 and the universal property of $\mathcal{E}^q_{1,n}$ imply the existence of a surjective homomorphism $\psi \colon \mathcal{E}^q_{1,n} \to \mathcal{E}^0_{1,n}$ defined by*

$$\psi(t) = v_1, \quad \psi(s_j) = w_j, \quad i = \overline{1, n}.$$

**Theorem 3.** *For any $q \in \mathbb{C}$, $|q| < 1$, one has an isomorphism $\mathcal{E}^q_{1,n} \simeq \mathcal{E}^0_{1,n}$.*

## 3. The case $|q| = 1$

In this section, we study $\mathcal{E}^q_{1,n}$ with $|q| = 1$. In this case the generating relations imply a relation of the form

$$s_j t = q t s_j, \quad j = 1, \ldots, n.$$

3.1. **The Fock representation of $\mathcal{E}^q_{1,n}$, $|q| = 1$.** In this part we construct a Fock representation of $\mathcal{E}^q_{1,n}$, $|q| = 1$, and show its faithfulness.

**Definition 1.** The Fock representation, $\pi^q_F$, of $\mathcal{E}^q_{1,n}$, is a unique, up to unitary equivalence, irreducible $*$-representation having the vacuum vector $\Omega$, $||\Omega|| = 1$, such that

$$\pi^q_F(s^*_j)\Omega = 0, \quad \pi^q_F(t^*)\Omega = 0, \quad j = \overline{1, n}.$$

Let $S$ denote the unilateral shift on $l_2(\mathbb{Z}_+)$, and $d_1(q)\colon l_2(\mathbb{Z}_+) \to l_2(\mathbb{Z}_+)$ be a diagonal unitary operator defined by

$$d_1(q)e_k = q^k h_k, \quad k \in \mathbb{Z}_+,$$

where $h_n$, $n \in \mathbb{Z}_+$, are vectors of standard orthonormal basis.

Denote by $\pi_{F,n}$ the Fock representation of $\mathcal{O}^{(0)}_n \subset \mathcal{E}^q_{1,n}$ acting on the space

$$\mathcal{F}_n = \mathcal{T}(\mathcal{H}_n) = \mathbb{C}\Omega_n \oplus \bigoplus_{d=1}^{\infty} \mathcal{H}_n^{\otimes d}, \quad \mathcal{H}_n = \mathbb{C}^n,$$

by the formulas

$$\pi_{F,n}(s_j)\Omega_n = e_j, \quad \pi_{F,n}(s_j)e_{i_1} \otimes e_{i_2} \cdots \otimes e_{i_d} = e_j \otimes e_{i_1} \otimes e_{i_2} \cdots \otimes e_{i_d},$$
$$\pi_{F,n}(s^*_j)\Omega = 0, \quad \pi_{F,n}(s^*_j)e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_d} = \delta_{ji_1} e_{i_2} \otimes \cdots \otimes e_{i_d}, \quad d \in \mathbb{N},$$

where $e_1, \ldots, e_n$ is the standard orthonormal basis of $\mathcal{H}_n$. We also denote by $d_n(q)\colon \mathcal{F}_n \to \mathcal{F}_n$ the unitary operator such that

$$d_n(q)\Omega_n = \Omega_n, \quad d_n(q)X = q^d X, \quad X \in \mathcal{H}_n^{\otimes d}.$$

**Theorem 4.** *The Fock representation of $\mathcal{E}^q_{1,n}$ exists. Up to a unitary equivalence, the Fock space is $\mathcal{F}^q = l_2(\mathbb{Z}_+) \otimes \mathcal{F}_n$ and*

$$\pi^q_F(t) = S \otimes \mathbf{1}_{\mathcal{F}_n},$$
$$\pi^q_F(s_j) = d_1(q) \otimes \pi_{F,n}(s_j), \quad j = 1, \ldots, d.$$

*Proof.* Let $\Omega = h_0 \otimes \Omega_n \in \mathcal{F}^q$. It is easy to see that

$$\pi_F(t^*)\Omega = 0, \quad \pi^q_F(s^*_j)\Omega = 0, \quad j = 1, \ldots, n.$$

Since the pair $(S, S^*)$ acts irreducibly on $l_2(\mathbb{Z}_+)$, and $\pi_{F,n}$ is irreducible, we get that the family $\{\pi^q_F(t), \pi^q_F(t^*), \pi^q_F(s_j), \pi^q_F(s^*_j)\}_{j=1}^n$ is irreducible on $\mathcal{F}_q$. □

*Remark* 1. It follows from the main result of [7] that $\pi^q_F$ is faithful on $*$-subalgebra $E^q_{1,n} \subset \mathcal{E}^q_{1,n}$ generated by $t$ and $s_j$, $j = \overline{1, n}$.

*Remark* 2. Below we present two another forms of generators of $\mathcal{E}^q_{1,n}$ in the Fock representation,

$$\pi^q_F(s_j) = \mathbf{1}_{l_2(\mathbb{Z}_+)} \otimes \pi_{F,n}(s_j), \quad j = \overline{1, n},$$
$$\pi^q_F(t) = S \otimes d_n(q^{-1}),$$

or

$$\pi^q_F(s_j) = d(q^{1/2}) \otimes \pi_{F,n}(s_j), \quad j = \overline{1, n},$$
$$\pi^q_F(t) = S \otimes d_n(q^{-1/2}).$$

Next we show that $\pi^q_F$ is a faithful representation of $\mathcal{E}^q_{1,n}$. To this end we consider the action $\alpha$ of $\mathbb{T}^2$ on $\mathcal{E}^q_{n,m}$,

$$\alpha_{\varphi_1,\varphi_2}(s_j) = e^{2\pi i\varphi_1} s_j, \quad \alpha_{\varphi_1,\varphi_2}(t) = e^{2\pi i\varphi_2} t.$$

In the following we denote by $\Lambda_n$ the set of all words in the alphabet $\{1, 2, \ldots, n\}$ including the empty word.

**Proposition 2.** *The fixed point $C^*$-subalgebra $(\mathcal{E}^q_{1,n})^\alpha \subset \mathcal{E}^q_{1,n}$ with respect to $\alpha$ is an AF-algebra and the restriction of $\pi^q_F$ to $(\mathcal{E}^q_{1,n})^\alpha$ is faithful.*

*Proof.* Indeed, it is easy to see that $(\mathcal{E}^q_{1,n})^\alpha$ is generated by the family

$$\{t^k(t^k)^* s_{\mu_1} s^*_{\nu_1},\ |\mu_1| = |\nu_1|,\ \mu_1, \nu_1 \in \Lambda_n,\ k \in \mathbb{Z}_+\}. \tag{1}$$

Furthermore, the Fock representation of the $*$-algebra, generated by family (1) is faithful. To finish the proof we recall that a representation of AF-$C^*$-algebra is faithful iff it is faithful on finite-dimensional subalgebras. □

**Corollary 3.** *The Fock representation of $\mathcal{E}^q_{1,n}$ is faithful.*

*Proof.* It is easy to see that $\pi^q_F$ is an equivariant homomorphism between the $C^*$-algebras $\mathcal{E}^q_{1,n}$ and $\pi^q_F(\mathcal{E}^q_{1,n})$. It remains to notice that equivariant homomorphism between $C^*$-algebras with group action is faithful iff it is faithful on fixed point subalgebras, see [2]. □

**Theorem 5.** *The $C^*$-algebra $\mathcal{E}^q_{1,n}$ is nuclear.*

*Proof.* For $q = e^{2\pi i \varphi_0}$, consider the action $\alpha_q$ of $\mathbb{Z}$ on $\mathcal{E}^q_{1,n}$ defined on the generators by

$$\alpha^k_q(s_j) = e^{\pi i k \varphi_0} s_j, \quad \alpha^k_q(t) = e^{-\pi i k \varphi_0} t, \quad j = 1, \dots, n, \quad k \in \mathbb{Z}.$$

Denote by the same symbol a similar action on $\mathcal{E}^1_{1,n} \simeq \mathcal{T}(C(\mathbb{T})) \otimes \mathcal{O}^{(0)}_n$. Here we denote by $\tilde{s}_j$ and $\tilde{t}$ the generators of $\mathcal{E}^1_{1,n}$.

We claim that for any $\varphi_0 \in [0, 1)$, one has an isomorphism $\mathcal{E}^q_{1,n} \rtimes_{\alpha_q} \mathbb{Z} \simeq \mathcal{E}^1_{1,n} \rtimes_{\alpha_q} \mathbb{Z}$. Recall that $\mathcal{E}^1_{1,n} \rtimes_{\alpha_q} \mathbb{Z}$ is generated as a $C^*$-algebra by the elements $\tilde{s}_j$, $\tilde{t}$ and a unitary $u$, such that the following relations are satisfied

$$u \tilde{s}_j u^* = e^{i\pi\varphi_0}\, \tilde{s}_j, \quad u \tilde{t} u^* = e^{-i\pi\varphi_0}\, \tilde{t}, \quad j = \overline{1, n}.$$

Put $\hat{s}_j = \tilde{s}_j u$ and $\hat{t} = \tilde{t} u$. Obviously, $\hat{s}_j$, $\hat{t}_r$ and $u$ generate $\mathcal{E}^1_{1,n} \rtimes_{\alpha_q} \mathbb{Z}$. Further,

$$\hat{s}^*_j \hat{s}_k = \delta_{jk} \mathbf{1}, \quad \hat{t}^* \hat{t} = \mathbf{1},$$

and

$$\hat{s}_j \hat{t} = \tilde{s}_j u \tilde{t} u = e^{-i\pi\varphi_0} \tilde{s}_j \tilde{t} u^2 = e^{-i\pi\varphi_0} \tilde{t} \tilde{s}_j u^2 = e^{-2\pi i \varphi_0} \tilde{t} u \tilde{s}_j u = \bar{q}\, \hat{s}_j \hat{t}.$$

In a similar way we get $\hat{s}^*_j \hat{t} = q \hat{t} \hat{s}^*_j$, $j = \overline{1, n}$. Finally,

$$u \hat{s}_j u^* = e^{i\pi\varphi_0} \hat{s}_j, \quad u \hat{t} u^* = e^{-i\pi\varphi_0} \hat{t}.$$

Hence the correspondence

$$s_j \mapsto \hat{s}_j, \quad t \mapsto \hat{t}, \quad u \mapsto u,$$

determines a homomorphism $\Phi_q \colon \mathcal{E}^q_{1,n} \rtimes_{\alpha_q} \mathbb{Z} \to \mathcal{E}^1_{1,n} \rtimes_{\alpha_q} \mathbb{Z}$. The inverse is constructed evidently.

Recall that $C^*(t, t^* \mid t^* t = \mathbf{1}) \simeq \mathcal{T}(C(\mathbb{T}))$, the algebra of Toeplitz operator with continuous symbol. Let us show nuclearity of $\mathcal{E}^q_{1,n}$. Indeed, $\mathcal{E}^1_{1,n} = \mathcal{T}(C(\mathbb{T})) \otimes O^{(0)}_n$ is nuclear. Then so is the crossed product $\mathcal{E}^1_{1,n} \rtimes_{\alpha_q} \mathbb{Z}$. Then due to the above isomorphism, $\mathcal{E}^q_{1,n} \rtimes_{\alpha_q} \mathbb{Z}$ is nuclear, implying nuclearity of $\mathcal{E}^q_{1,n}$, see [1]. □

Now we prove an analogue of the Wold decomposition theorem. Denote

$$Q = \sum_{j=1}^n s_j s^*_j, \quad P = t t^*.$$

**Theorem 6** (Generalised Wold decomposition). *Let $\pi\colon \mathcal{E}_{1,n}^q \to \mathbb{B}(\mathcal{H})$ be a $*$-representation. Then*

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4,$$

*where each $\mathcal{H}_j$, $j = 1,2,3,4$, is invariant with respect to $\pi$, and for $\pi_j = \pi\restriction_{\mathcal{H}_j}$ one has*

- *$\mathcal{H}_1 = \mathcal{F} \otimes \mathcal{K}$ for some Hilbert space $\mathcal{K}$, and $\pi_1 = \pi_F^q \otimes \mathbf{1}_{\mathcal{K}}$;*
- *$\mathcal{H}_2 = l_2(\mathbb{Z}_+) \otimes \mathcal{G}_2$, $\pi_2(\mathbf{1} - Q) = 0$, $\pi_2(\mathbf{1} - P) \neq 0$,*

$$\pi_1(t) = S \otimes \mathbf{1},\ \pi_1(s_j) = d_1(q) \otimes \tilde{\pi}_1(s_j), \quad j = \overline{1,n},$$

*where the operators $\tilde{\pi}_1(s_j)$ determine representation of $\mathcal{O}_n$ on $\mathcal{G}_2$;*
- *$\mathcal{H}_3 = \mathcal{F}_n \otimes \mathcal{G}_3$, $\pi_3(\mathbf{1} - P) = 0$, $\pi_3(\mathbf{1} - Q) \neq 0$,*

$$\pi(s_j) = \pi_{F,n}(s_j) \otimes 1, \quad \pi(t) = d_n(q^{-1}) \otimes U,$$

*where $U$ is unitary on $\mathcal{G}_3$;*
- *$\pi_4(\mathbf{1} - Q) = 0$, $\pi_4(\mathbf{1} - P) = 0$,*

*where any of $\mathcal{H}_j$, $j = 1,2,3,4$, could be zero.*

*Proof.* Let $\pi$ be a representation of $\mathcal{E}_{1,n}^q$ acting on $\mathcal{H}$. Suppose that $\pi(\mathbf{1} - Q) \neq 0$ and $\pi(\mathbf{1} - P) \neq 0$. Put $\mathcal{K} = \ker P \cap \ker Q$. Then the minimal invariant subspace $\mathcal{H}_1 \subset \mathcal{H}$ is isomorphic to $\mathcal{F} \otimes \mathcal{K}$ and the restriction of $\pi$ to $\mathcal{H}_1$ is unitary equivalent to $\pi_F \otimes \mathbf{1}_{\mathcal{K}}$.

Consider the restriction of $\pi$ to $\mathcal{H}_1^{\perp}$. We keep for it the same notations, and put $\pi(t) = T$, $\pi(s_j) = S_j$. If $\pi(\mathbf{1} - P) \neq 0$, i.e. $\ker T^* \neq \{0\}$, put $\mathcal{G}_2 = \ker T^*$. It is easy to see that

$$\mathcal{H}_2 = \bigoplus_{n \in \mathbb{Z}_+} T^n(\mathcal{G}_2) \subset \mathcal{H}_1^{\perp}$$

is invariant with respect to $T$, $T^*$, $S_j$, $S_j^*$, $j = \overline{1,n}$, $\mathcal{H}_2 \simeq l_2(\mathbb{Z}_+) \otimes \mathcal{G}_2$. Let $\pi_2$ be a restriction of $\pi$ to $\mathcal{H}_2$. Then

$$\pi_2(t) = S \otimes \mathbf{1}_{\mathcal{G}_2}, \quad \pi_2(s_j) = d_1(q) \otimes \tilde{\pi}_2(s_j), \quad j = \overline{1,n},$$

where $\sum_{j=1}^n \tilde{\pi}_2(s_j s_j^*) = \mathbf{1}_{\mathcal{G}_2}$. Further, decompose $\mathcal{H}_1^{\perp} = \mathcal{H}_2 \oplus \mathcal{H}_2^{\perp}$. Notice that

$$\ker \pi(\mathbf{1} - T) \cap \mathcal{H}_2^{\perp} = \{0\}.$$

Suppose that $\ker \pi(\mathbf{1} - Q) \cap \mathcal{H}_2^{\perp} \neq \{0\}$. Put $\mathcal{G}_3 = \ker \pi(\mathbf{1} - Q) \cap \mathcal{H}_2^{\perp} \neq \{0\}$. Construct

$$\mathcal{H}_3 = \bigoplus_{\lambda \in \Lambda_n} S_\lambda(\mathcal{G}_3).$$

It is invariant with respect to $T$, $S_j$, $S_j^*$, $j = \overline{1,n}$. Denote by $\pi_3$ the restriction of $\pi$ to $\mathcal{H}_3$. Then

$$\pi_3(s_j) = \pi_{F,n}(s_j) \otimes \mathbf{1}_{\mathcal{G}_3},\ \pi_3(t) = d_n(q^{-1}) \otimes U, \quad j = \overline{1,n},$$

where $U$ is unitary on $\mathcal{G}_3$.

Finally, put $\mathcal{H}_4$ to be the orthogonal complement of $\mathcal{H}_3$ in $\mathcal{H}_2^{\perp}$. Evidently $\mathbf{1} - P = 0$ and $\mathbf{1} - Q = 0$ on $\mathcal{H}_4$. □

3.2. **Representations of $\mathcal{E}_{1,n}^q$.** In this part we describe classes of unitary equivalence of representations of $\mathcal{E}_{1,n}^q$ such that the unitary part in the Wold decomposition of isometry corresponding to some of $s_j$, $j = \overline{1,n}$, is non-zero.

In the following for a fixed representation $\pi$ of $\mathcal{E}_{1,n}^q$ we denote $\pi(t)$ by $T$ and $\pi(s_j)$ by $S_j$, $j = \overline{1,n}$. Below we denote by

$$\Lambda_n^j = \{\lambda = (\lambda_1, \dots, \lambda_k) \mid 1 \leq \lambda_j \leq n,\ \lambda_k \neq j,\ k \in \mathbb{N}\}.$$

We also denote by $S_\lambda := S_{\lambda_1} \cdots S_{\lambda_k}$.

**Proposition 3.** *Let $\pi\colon \mathcal{E}_{1,n}^q \to B(\mathcal{H})$ be a representation such that for some $j$ the unitary part of $S_j = \pi(s_j)$ is non-zero. Denote by $\mathcal{H}_{u,j} \subset \mathcal{H}$ the largest subspace invariant with respect to $S_j$, $S_j^*$, such that the restriction of $S_j$ is unitary. Then $\mathcal{H}_{u,j}$ is invariant with respect to $T$, $T^*$,*

$$S_k^*(\mathcal{H}_{u,j}) = \{0\}, \quad S_\lambda(\mathcal{H}_{u,j}) \perp S_\mu(\mathcal{H}_{u,j}), \quad k \neq j, \quad \lambda \neq \mu, \quad \lambda, \mu \in \Lambda_n^j.$$

*Proof.* To show that $\mathcal{H}_{u,j}$ is invariant with respect to $T$ we recall that

$$\mathcal{H}_{u,j} = \bigcap_{k=1}^{\infty} S_j^k(\mathcal{H}).$$

Then the result follows from the commutation relations $T^* S_j = q S_j T^*$ , $S_j T = q T S_j$.

Further for $k \neq j$ and $x \in \mathcal{H}_{u,j}$ one has

$$S_k^* x = S_k^*(S_j S_j^* x) = 0.$$

Let us show that for $x, y \in \mathcal{H}_{u,j}$ and $\lambda \in \Lambda_n^j$ one has $(S_\lambda x, y) = 0$. Indeed let $m$ be the first number such that $\lambda_m \neq j$, i.e., $\lambda = (j, \ldots, j, \lambda_m, \ldots, \lambda_k)$. Since $\mathcal{H}_{u,j}$ is invariant with respect to $S_j$, $S_j^*$, one has

$$\big(x, S_\lambda y\big) = \big((S_j^*)^n x, S_{\lambda_m} \cdots S_{\lambda_k} y\big) = 0,$$

where in the last equality we use the fact that $S_{\lambda_m}^*(\mathcal{H}_{u,j}) = \{0\}$. Let $|\mu| < |\lambda|$, $\lambda, \mu \in \Lambda_n^j$. Then $\lambda = \tilde{\lambda}\hat{\lambda}$, $|\tilde{\lambda}| = |\mu|$, $\tilde{\lambda} \in \Lambda_n^j$, and

$$\big(S_\lambda x, S_\mu y\big) = \delta_{\tilde{\lambda}\mu}\big(S_{\hat{\lambda}} x, y\big) = 0.$$

Finally, recall that if $|\lambda| = |\mu|$, then $S_\lambda^* S_\mu = \delta_{\lambda\mu}$ implying

$$\big(S_\lambda x, S_\mu y\big) = 0, \quad \lambda \neq \mu, \quad x, y \in \mathcal{H}_{u,j}. \qquad \square$$

*Remark* 3. 1. By a similar arguments one can show that if $\big(x, y\big) = 0$, $x, y \in \mathcal{H}_{u,j}$, then $\big(S_\lambda x, S_\mu y\big) = 0$ for any $\lambda, \mu \in \Lambda_n^j$.

2. The subspace

$$\tilde{H} = \mathcal{H}_{u,j} \oplus \bigoplus_{\lambda \in \Lambda_n^j} S_\lambda(\mathcal{H}_{u,j})$$

is invariant with respect to the operators of $\pi$.

Suppose that the conditions of Proposition 3 are satisfied, and denote by $\tilde{T}$ and $V_j$ the restrictions of $T$ and $S_j$ respectively to $\mathcal{H}_{u,j}$.

**Proposition 4.** *Let $\pi\colon \mathcal{E}_{1,n}^q \to B(\mathcal{H})$ and $1 \leq j \leq n$ be such that $\pi(s_j)$ has a non-trivial unitary part, and $\mathcal{H} = \tilde{\mathcal{H}}$ introduced above. Then $\pi$ is irreducible iff the family $\{\tilde{T}, \tilde{T}^*, V_j\}$ is irreducible on $\mathcal{H}_{u,j}$.*

*Proof.* Suppose that $\mathcal{G} \subset \mathcal{H}_{u,j}$ is invariant with respect to $T$, $T^*$ and $V_j$. Then Proposition 3 and Remark 3 imply that the subspace

$$\mathcal{G} \oplus \bigoplus_{\lambda \in \Lambda_n^j} S_\lambda(\mathcal{G})$$

is non-trivial and invariant with respect to the operators of $\pi$. $\square$

**Proposition 5.** *Let $\pi_k\colon \mathcal{E}_{1,n}^q \to \mathcal{H}_k$, $k = 1, 2$, be irreducible representations such that $\pi_k(s_j)$ have non-trivial unitary parts. Then $\pi_1$ is unitary equivalent to $\pi_2$ iff the corresponding families $\mathfrak{F}_k = \{\tilde{T}_k, \tilde{T}_k^*, V_{j,k}\}$, $k = 1, 2$, acting on $\mathcal{H}_{u,j}^{(k)}$, are unitary equivalent.*

*Proof.* Suppose that $\pi_1$ is unitary equivalent to $\pi_2$. Let $U\colon \mathcal{H}_1 \to \mathcal{H}_2$ be a unitary operator intertwining $\pi_1$ and $\pi_2$. It is easy to deduce that $U\colon \mathcal{H}_{u,j}^{(1)} \to \mathcal{H}_{u,j}^{(2)}$. Indeed, for any $m \in \mathbb{N}$ one has

$$U((S_j^{(1)})^m(\mathcal{H}_1)) \subset (S_j^{(2)})^m(U(\mathcal{H}_1)) = (S_j^{(2)})^m(\mathcal{H}_2).$$

Hence $U$ intertwines the families $\mathfrak{F}_1$ and $\mathfrak{F}_2$.

Conversely, let the families $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be unitarily equivalent and $\tilde{U}\colon \mathcal{H}_{u,j}^{(1)} \to \mathcal{H}_{u,j}^{2}$ be a unitary operator intertwining them. Construct an operator $U\colon \mathcal{H}_1 \to \mathcal{H}_2$ by the rule

$$U(x) = \tilde{U}(x), \quad U(S_\lambda^{(1)}x) = S_\lambda^{(2)}(\tilde{U}x), \quad x \in \mathcal{H}_{u,j}^{(1)}.$$

Then Proposition 3 and Remark 3 imply that $U$ is a unitary operator intertwining the representations $\pi_1$ and $\pi_2$. $\square$

Combining the results of Propositions 4, 5, we get the following statement.

**Theorem 7.** *Let $\pi$ be an irreducible representation of $\mathcal{E}_{1,n}^q$ on the space $\mathcal{H}$ such that for a fixed $1 \le j \le n$ the isometry $S_j = \pi(s_j)$ has a non-trivial unitary part in its Wold decomposition. Then $\pi$ is determined uniquely, up to a unitary equivalence, by the irreducible family of operators $\{\tilde{T}, \tilde{T}^*, V_j\}$ on the Hilbert space $\mathcal{G}$, satisfying the relations*

$$\tilde{T}^*\tilde{T} = \mathbf{1}, \quad V_j^*V_j = V_jV_j^* = \mathbf{1}, \quad \tilde{T}^*V_j = qV_j\tilde{T}^*, \quad V_j\tilde{T} = q\tilde{T}V_j.$$

*Namely, let $f^{(i)}$, $i \in \mathcal{I}$ be an orthonormal basis of $S_k\mathcal{G}$. Then the orthonormal basis of $\mathcal{H}$ has the form*

$$f^{(i)} \otimes f_\emptyset, \quad f^{(i)} \otimes f_\lambda, \quad i \in \mathcal{I}, \quad \lambda \in \Lambda_n^j,$$

*and $i \in \mathcal{I}$,*

$$\begin{aligned}
&S_jf^{(i)} \otimes f_\emptyset = (V_jf^{(i)}) \otimes f_\emptyset, \quad Tf^{(i)} \otimes f_\emptyset = (\tilde{T}f^{(i)}) \otimes f_\emptyset,\\
&S_kf^{(i)} \otimes f_\emptyset = f^{(i)} \otimes f_{(k)}, \quad k \ne j, \quad S_kf^{(i)} \otimes f_\lambda = f^{(i)} \otimes f_{\sigma_k(\lambda)}, \quad k = \overline{1,n}, \quad i \in \mathcal{I},\\
&S_k^*f^{(i)} \otimes f_\emptyset = 0, \quad k \ne j, \quad S_k^*f^{(i)} \otimes f_\lambda = \delta_{k\lambda_1}f^{(i)} \otimes f_{\sigma(\lambda)}, \quad k = \overline{1,n}, \quad i \in \mathcal{I},\\
&Tf^{(i)} \otimes f_\lambda = q^{-|\lambda|}(\tilde{T}f^{(i)}) \otimes f_\lambda, \quad T^*f^{(i)} \otimes f_\lambda = q^{|\lambda|}(\tilde{T}^*f^{(i)}) \otimes f_\lambda, \quad \lambda \in \Lambda_n^j,
\end{aligned}$$

*where $\sigma_k(\lambda) = (k\lambda)$, $\sigma((\lambda_1)) = \emptyset$, $\sigma((\lambda_1, \dots, \lambda_m)) = (\lambda_2, \dots, \lambda_m)$.*

3.3. **Commutative model for representations.** The generalized Wold decomposition for representations of $\mathcal{E}_{1,n}^q$, Theorem 6, implies in particular, that any irreducible representation of $\mathcal{E}_{1,n}^q$ contains only one component $\mathcal{H}_i$, $j = 1, 2, 3, 4$. In $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$ all representations are described either explicitly or in terms of representations of $\mathcal{O}_n$. Here, we give a general form of representations in $\mathcal{H}_4$ (commutative model with respect to a commutative subalgebra).

The defining relations in $\mathcal{E}_{1,n}^q$ imply that in $\mathcal{H} = \mathcal{H}_4$, the operator $T$ is unitary, and the operators $P_\lambda = S_\lambda S_\lambda^*$ form a commutative family of projections, all of them commute with $T$, and

$$\sum_{|\lambda|=k} P_\lambda = I, \quad k = 1, 2, \dots.$$

Following [11], we write the joint spectral decomposition of the commuting family $(T, P_\lambda \mid \lambda \in \{1, \dots, n\}_0^\infty)$ as

$$\begin{aligned}
T &= \int_{\mathbb{T}\times\{1,\dots,n\}^\infty} t\, dE(t,\lambda), \quad \mathbb{T}\times\{1,\dots,n\}^\infty \ni (t,\lambda) = (t, \lambda_1, \lambda_2, \dots),\\
P_\lambda &= \int_{\mathbb{T}\times\{1,\dots,n\}^\infty} \chi_{\lambda\times\{1,\dots,n\}^\infty}(\lambda)\, d\mu(t,\lambda) = E(\mathbb{T}\times\mu\times\{1,\dots,n\}^\infty).
\end{aligned}$$

For each $j = 1, \dots, n$, we have

$$TS_j = \bar{q} S_j T, \quad P_\lambda S_j = \delta_{\lambda_1, j} S_j P_{\sigma(\lambda)},$$

and applying general commutative model formalism of [11] we obtain the following realization of representations of $\mathcal{E}^q_{1,n}$.

**Theorem 8.** *For any representation of* $\mathcal{E}^q_{1,n}$ *in the component* $\mathcal{H}_4$ *of the generalized Wold decomposition, the following holds. The space* $\mathcal{H} = \mathcal{H}_4$ *decomposes into a direct integral*

$$\mathcal{H} = \int^{\oplus}_{\mathbb{T}\times\{1,\dots,n\}^\infty} \mathcal{H}_{t,\lambda}\, d\mu(t,\lambda) \quad (t,\lambda) = (t, \lambda_1, \lambda_2, \dots),$$

*and the operators act by the following formula:*

$$\begin{aligned}
&(Uf)(t,\lambda) = tf(t,\lambda),\\
&(S_j f)(t,\lambda) = \delta_{j,\lambda_1} U_j(\bar{q}t, \sigma(\lambda)) \Big(\frac{d\delta_j(\lambda_1) \otimes \mu(\bar{q}t, \sigma(\lambda))}{d\mu(t,\lambda)}\Big)^{1/2} f(\bar{q}t, \sigma(\lambda)),\\
&(S_j^* f)(t,\lambda) = U_j^*(qt,\lambda) \Big(\frac{d\mu(qt, \sigma_j(\lambda))}{d\mu(t,\lambda)}\Big)^{1/2} f(qt, \sigma_j(\lambda)).
\end{aligned}$$

*Here,* $\mu$ *is a probability measure defined on the cylinder* $\sigma$*-algebra, quasi-invariant with respect to the transformations* $\mu(t,\lambda) \mapsto (qt, \sigma_j(\lambda))$, $j = 1, \dots, n$; $\mathcal{H}_{t,\lambda}$ *is a measurable field of Hilbert spaces such that* $\dim H_{t,\lambda} = \dim H_{qt,\sigma_j(\lambda)}$ $\mu$*-a.e.;* $U_j(t,\lambda)$, $j = 1, \dots, n$, *are measurable unitary operator-valued functions.*

*Conversely, any quasi-invariant measure* $\mu$*, measurable field* $\mathcal{H}_{t,\lambda}$ *with* $\dim H_{t,\lambda} = \dim H_{qt,\sigma_j(\lambda)}$ $\mu$*-a.e., and a collection of measurable unitary* $U_j(t,\lambda)$, $j = 1, \dots, n$, *give rise to a representation of* $\mathcal{E}^q_{1,n}$ *in the component* $\mathcal{H}_4$ *of the generalized Wold decomposition.*

The simplest class of representations arise as follows. Take any point $(t_0, \lambda_0) \in \mathbb{T} \times \{1, \dots, n\}^\infty$ and consider its orbit $O_{t_0,\lambda_0}$ under the mappings $(t,\lambda) \mapsto (\bar{q}t, \sigma(\lambda))$, $(t,\lambda) \mapsto (qt, \sigma_j(\lambda))$, $j = 1, \dots, n$. Let $\mu$ be the atomic measure uniformly distributed over the points of $O_{t_0,\lambda_0}$, $H_{t,\lambda} = \mathbb{C}$, $U_j(t,\lambda) = 1$, $j = 1, \dots, n$. Then the basis of $\mathcal{H}$ is $e_{t,\lambda}$, $(t,\lambda) \in O_{t_0,\lambda_0}$, and

$$Te_{t,\lambda} = te_{t,\lambda}, \quad S_j e_{t,\lambda} = e_{qt,\sigma_j(\lambda)}, \quad S_j^* e_{t,\lambda} = \delta_{j,\lambda_1} e_{\bar{q}t,\sigma(\lambda)},$$

$j = 1, \dots, n$. All such representations are irreducible, and the representations corresponding to $(t_0, \lambda_0)$ and $(t_0', \lambda_0')$ are unitarily equivalent if and only if these points belong to the same orbit. Notice that the representations corresponding to the points $(t, (j, j, \dots))$, $j = 1, \dots, n$, fall in, but does not cover the class of representations studied in the previous section.

## References


1. B. Blackadar, *K-Theory for Operator Algebras*, Mathematical Sciences Research Institute Publications, vol. 5, Springer-Verlag, New York, 1986. DOI 10.1007/978-1-4613-9572-0. URL https://doi.org/10.1007/978-1-4613-9572-0
2. N. P. Brown, N. Ozawa, *$C^*$-Algebras and Finite-Dimensional Approximations*, Graduate Studies in Mathematics, vol. 88, American Mathematical Society, Providence, RI, 2008. DOI 10.1090/gsm/088. URL https://doi.org/10.1090/gsm/088
3. L. A. Coburn, *The $C^*$-algebra generated by an isometry*, Bull. Amer. Math. Soc. **73** (1967), 722–726. DOI 10.1090/S0002-9904-1967-11845-7. URL https://doi.org/10.1090/S0002-9904-1967-11845-7
4. J. Cuntz, *Simple $C^*$-algebras generated by isometries*, Comm. Math. Phys. **57** (1977), no. 2, 173–185. URL http://projecteuclid.org/euclid.cmp/1103901288

5. K. R. Davidson, *$C^*$-Algebras by Example*, Fields Institute Monographs, vol. 6, American Mathematical Society, Providence, RI, 1996. DOI 10.1090/fim/006. URL https://doi.org/10.1090/fim/006
6. M. de Jeu, P. R. Pinto, *The structure of doubly non-commuting isometries*, 2018. URL https://arxiv.org/abs/1801.09716
7. P. E. T. Jørgensen, D. P. Proskurin, Y. S. Samoĭlenko, *The kernel of Fock representations of Wick algebras with braided operator of coefficients*, Pacific J. Math. **198** (2001), no. 1, 109–122. DOI 10.2140/pjm.2001.198.109. URL https://doi.org/10.2140/pjm.2001.198.109
8. P. E. T. Jorgensen, L. M. Schmitt, R. F. Werner, *Positive representations of general commutation relations allowing Wick ordering*, J. Funct. Anal. **134** (1995), no. 1, 33–99. DOI 10.1006/jfan.1995.1139. URL https://doi.org/10.1006/jfan.1995.1139
9. A. Kuzmin, V. Ostrovskyi, D. Proskurin, and R. Yakymiv, *On q-tensor product of Cuntz algebras*, Preprint, 2019. https://arxiv.org/abs/1812.08530
10. A. Kuzmin, N. Pochekai, *Faithfulness of the Fock representation of the $C^*$-algebra generated by $q_{ij}$-commuting isometries*, J. Operator Theory **80** (2018), no. 1, 77–93.
11. V. Ostrovskyi, Yu. Samoilenko, *Introduction to the Theory of Representations of Finitely Presented $*$-Algebras.* I. *Representations by Bounded Operators*, Rev. Math. and Math. Phys. **11**, Gordon & Breach, London, 1999.
12. D. Proskurin, *Stability of a special class of $q_{ij}$-CCR and extensions of higher-dimensional noncommutative tori*, Lett. Math. Phys. **52** (2000), no. 2, 165–175. DOI 10.1023/A:1007668304707. URL https://doi.org/10.1023/A:1007668304707
13. M. A. Rieffel, *$C^*$-algebras associated with irrational rotations*, Pacific J. Math. **93** (1981), no. 2, 415–429. URL http://projecteuclid.org/euclid.pjm/1102736269
14. M. Weber, *On $C^*$-algebras generated by isometries with twisted commutation relations.* J. Funct. Anal. **264** (2013), no. 8, 1975–2004. DOI 10.1016/j.jfa.2013.02.001. URL https://doi.org/10.1016/j.jfa.2013.02.001
15. D. P. Williams, *Crossed Products of $C^*$-Algebras*, Mathematical Surveys and Monographs, vol 134, American Mathematical Society, Providence, RI, 2007. DOI 10.1090/surv/134. URL https://doi.org/10.1090/surv/134

National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute"
*E-mail address*: `olyushka.ostrovska@gmail.com`

Faculty of Computer Sciences and Cybernetics, Kiev National Taras Shevchenko University
*E-mail address*: `yakymiv@univ.kiev.ua`

Received 02/04/2019