Methods of Functional Analysis and Topology
Vol. 25 (2019), no. 2, pp. 161–176

KÖTHE-ORLICZ VECTOR-VALUED WEAKLY SEQUENCE SPACES OF DIFFERENCE OPERATORS

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Abstract. In the present article, we propose vector-valued weakly null, weakly convergent and weakly bounded sequences over n-normed spaces associated with infinite matrix, Musielak-Orlicz function and difference operator. We make an effort to study some algebraic and topological properties of these sequence spaces. Further, we shall investigate some inclusion relations between newly formed sequence spaces.

1. Introduction and Preliminaries

In 1972, E. Pietsch [14] introduced some classes of vector-valued weakly summable and absolutely summable sequences. Thereafter, many authors have worked on these sequence spaces. Recently, the study of vector-valued sequence spaces has been explored by Mohuiddine and Raj [11], Mursaleen and Raj [12], Oğur and Sağir [13] and many others.

Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of natural, real and complex numbers respectively. We write

$$\omega = \{x = (x_k) : x_k \in \mathbb{R} \text{ or } \mathbb{C}\}$$

the space of all real or complex sequences.

An increasing non-negative integer sequence $\theta = (k_r)$ with $k_0 = 0$ and $k_r - k_{r-1} \to \infty$ as $r \to \infty$ is known as lacunary sequence. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$. We write $h_r = k_r - k_{r-1}$ and q_r denotes the ratio $\frac{k_r}{k_{r-1}}$. Freedman et al. [4] defined the sequence space

$$N_\theta = \Big\{x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |x_k - L| = 0 \text{ for some } L\Big\}$$

known as lacunary strongly convergent sequence space. Many authors have explored the concept of lacunary convergence. Some of them are Colak et al. [1], Esi et al. [3], Kilicman and Borgohain [7], Raj and Sharma [15], Tripathy and Baruah [18], Tripathy and Dutta [20], Tripathy and Dutta [21], Tripathy and Et [24], Tripathy et al. [25] and Tripathy and Mahanta [26].

Let $\mathcal{A} = (a_{jk})$ be an infinite matrix of real or complex numbers a_{jk}, where $j,\ k \in \mathbb{N}$. The $\mathcal{A}$ transform of $x = (x_k)$ is written as $\mathcal{A}x$ and $\mathcal{A}x = (\mathcal{A}_k(x))$ if $\mathcal{A}_k(x) = \sum_{k=1}^{\infty} a_{jk}x_k$ converges for each $k \in \mathbb{N}$.

An Orlicz function $M : [0,\infty) \to [0,\infty)$ is convex, continuous and non-decreasing function which also satisfy $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. Lindenstrauss and Tzafriri [9] used the idea of Orlicz function to define the following

2010 *Mathematics Subject Classification.* 40A05, 40A30.

Key words and phrases. Weakly sequence, lacunary sequence, Musielak-Orlicz function, difference operator, n-normed space, infinite matrix.

sequence space:

$$\ell_M = \left\{ x \in \omega : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\},$$

which is called as an Orlicz sequence space. An Orlicz function is said to satisfy Δ_2-condition if for a constant K, $M(Px) \leq KPM(x)$ for all values of $x \geq 0$ and for $P > 1$. A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called as Musielak-Orlicz function.

Kızmaz [8] proposed the concept of difference sequence spaces $Z(\Delta)$, where $Z = c, c_0$ and ℓ_∞. These sequence spaces were further generalized by Et and Çolak in [2] where they introduced sequence spaces $Z(\Delta^m)$. Subsequently, the difference operator Δ_n for a non-negative integer n was introduced by Tripathy and Esi [22] and later on generalized by Tripathy et al. [23]. The difference sequence spaces $Z(\Delta_n^m)$, where $m, n \in \mathbb{N}$ are defined as follows:

$$Z(\Delta_n^m) = \{x = (x_k) \in \omega : (\Delta_n^m x_k) \in Z\},$$

where $\Delta_n^0 x_k = x_k, \Delta_n^m x_k = (\Delta_n^{m-1} x_k - \Delta_n^{m-1} x_{k+n})$ and binomial representation is

$$\Delta_n^m x_k = \sum_{i=0}^{m} (-1)^m \begin{pmatrix} m \\ i \end{pmatrix} x_{k+ni}.$$

Misiak [10] developed the concept of $n-$normed spaces. Many Mathematicians have studied the concept of $n-$normed spaces. In [19] Tripathy and Borgogain discussed some interesting results on $n-$normed sequences related to ℓ_p space. Moreover, Tripathy et al. [27] introduced $I-$convergence in probabilistic $n-$normed space. For definition and results on $n-$normed spaces (see [5], [6], [12]). A sequence (x_k) in a n-normed space $(W, \|\cdot, \cdots, \cdot\|)$ is said to converge to some $L \in W$ if

$$\lim_{k \to \infty} \|(x_k - L, z_1, \cdots, z_{n-1})\| = 0 \quad \text{for every} \quad z_1, \cdots, z_{n-1} \in W.$$

A sequence (x_k) in a n-normed space $(W, \|\cdot, \cdots, \cdot\|)$ is said to be Cauchy with respect to the n-norm if

$$\lim_{k,p \to \infty} \|(x_k - x_p, z_1, \cdots, z_{n-1})\| = 0 \quad \text{for every} \quad z_1, \cdots, z_{n-1} \in W.$$

A sequence (x_k) in a $n-$normed space W is said to be bounded if for a positive constant Q, $\|(x_k, z_1, \cdots, z_{n-1})\| \leq Q$ for all $z_1, \cdots, z_{n-1} \in W$ and is written as $x_k = O(1)$. To know more about sequence spaces see ([12], [16], [17]) and references therein.

Let W be a linear metric space. A function $\vartheta : W \to \mathbb{R}$ is called paranorm, if

(1) $\vartheta(x) \geq 0$, for all $x \in W$,
(2) $\vartheta(-x) = \vartheta(x)$, for all $x \in W$,
(3) $\vartheta(x + y) \leq \vartheta(x) + \vartheta(y)$, for all $x, y \in W$,
(4) if (μ_n) is a sequence of scalars with $\mu_n \to \mu$ as $n \to \infty$ and (x_n) is a sequence of vectors with $\vartheta(x_n - x) \to 0$ as $n \to \infty$, then $\vartheta(\mu_n x_n - \mu x) \to 0$ as $n \to \infty$.

If Y is any sequence space then the $\alpha-$dual of Y is denoted by Y^* and is defined as

$$Y^* = \left\{ (\gamma_k) \in \omega : \sum_{k=1}^{\infty} |\alpha_k \gamma_k| \text{ coverges for all } (\alpha_k)_k \in Y \right\}.$$

Let W be a $n-$Banach space and $R(n - W)$ denotes the space of $W-$valued sequences. Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers, $u = (u_k)$ be any sequence of positive real numbers, $\mathcal{A} = (a_{jk})$ be an infinite matrix and θ be a lacunary sequence. In the present paper we define weakly convergent, weakly null and weakly bounded sequence spaces as follows:

$$\Big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\Big]$$
$$= \Big\{x = (x_k) \in R(n-W) :$$
$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} a_{jk} \Big[M_k\Big(\Big\|\Big(\frac{u_k \Delta_n^m (f_k g(x_k)) - L}{\rho}, z_1, \cdots, z_{n-1}\Big)\Big\|\Big)\Big]^{p_k} = 0,$$
$$\text{for all } g \in W^*, f_k \in Y^* \text{ and for some } \rho > 0\Big\},$$

$$\Big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\Big]_0$$
$$= \Big\{x = (x_k) \in R(n-W) :$$
$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} a_{jk} \Big[M_k\Big(\Big\|\Big(\frac{u_k \Delta_n^m (f_k g(x_k))}{\rho}, z_1, \cdots, z_{n-1}\Big)\Big\|\Big)\Big]^{p_k} = 0,$$
$$\text{for all } g \in W^*, f_k \in Y^* \text{ and for some } \rho > 0\Big\}$$

and

$$\Big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\Big]_\infty$$
$$= \Big\{x = (x_k) \in R(n-W) :$$
$$\sup_j \sup_r \frac{1}{h_r} \sum_{k \in I_r} a_{jk} \Big[M_k\Big(\Big\|\Big(\frac{u_k \Delta_n^m (f_k g(x_k))}{\rho}, z_1, \cdots, z_{n-1}\Big)\Big\|\Big)\Big]^{p_k} < \infty,$$
$$\text{for all } g \in W^*, f_k \in Y^* \text{ and for some } \rho > 0\Big\}.$$

If $0 < p_k \le \sup p_k = H, D = \max(1, 2^{H-1})$. Then

$$|c_k + d_k|^{p_k} \le D(|c_k|^{p_k} + |d_k|^{p_k}), \tag{1.1}$$

for all $c_k, d_k \in \mathbb{R}$ for all $k \in \mathbb{N}$.

The main purpose of this paper is to study some new classes of lacunary weakly sequences over $n-$normed spaces by means of sequence of Orlicz functions and infinite matrix. We shall study some algebraic properties, topological properties and interesting inclusion relations between these sequence spaces.

2. Main Results

Theorem 2.1. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ be a bounded sequence of positive real numbers. Then the sequence spaces $\big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]_0$, $\big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]$ and $\big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]_\infty$ are linear spaces over the field $\mathbb{C}$ of complex numbers.*

Proof. Let $(x_k), (y_k) \in \big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]_0$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} a_{jk} \Big[M_k\Big(\Big\|\Big(\frac{u_k \Delta_n^m (f_k g(x_k))}{\rho_1}, z_1, \cdots, z_{n-1}\Big)\Big\|\Big)\Big]^{p_k} < \infty$$

and

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}a_{jk}\Big[M_k\Big(\Big\|\Big(\frac{u_k\Delta_n^m(f_kg(y_k))}{\rho_2},z_1,\cdots,z_{n-1}\Big)\Big\|\Big)\Big]^{p_k}<\infty.$$

Define $\rho_3=\max(2|\alpha|\rho_1,2|\beta|\rho_2)$. Since $\mathcal{M}$ is non-decreasing and convex. Hence, by using (1.1), we have

$$\begin{aligned}\lim_{r\to\infty}&\frac{1}{h_r}\sum_{k\in I_r}a_{jk}\Big[M_k\Big(\Big\|\Big(\frac{u_k\Delta_n^m(\alpha(f_kg(x_k))+\beta(f_kg(y_k)))}{\rho_3},z_1,\cdots,z_{n-1}\Big)\Big\|\Big)\Big]^{p_k}\\
&\le\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}a_{jk}\Big[M_k\Big(\Big\|\Big(\frac{u_k\Delta_n^m(\alpha(f_kg(x_k)))}{\rho_3}+\frac{u_k\Delta_n^m(\beta(f_kg(y_k)))}{\rho_3},\\
&\qquad z_1,\cdots,z_{n-1}\Big)\Big\|\Big)\Big]^{p_k}\\
&\le D\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\frac{1}{2^{p_k}}a_{jk}\Big[M_k\Big(\Big\|\Big(\frac{u_k\Delta_n^m(f_kg(x_k))}{\rho_1},z_1,\cdots,z_{n-1}\Big)\Big\|\Big)\Big]^{p_k}\\
&+D\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\frac{1}{2^{p_k}}a_{jk}\Big[M_k\Big(\Big\|\Big(\frac{u_k\Delta_n^m(f_kg(y_k))}{\rho_2},z_1,\cdots,z_{n-1}\Big)\Big\|\Big)\Big]^{p_k}<\infty.\end{aligned}$$

Thus, the sequence space $\big[W,Y,\mathcal{M},u,\mathcal{A},\theta,p,\Delta_n^m,\|\cdot,\cdots,\cdot\|\big]_0$ is a linear space over the field $\mathbb{C}$ of complex numbers. Similarly, $\big[W,Y,\mathcal{M},u,\mathcal{A},\theta,p,\Delta_n^m,\|\cdot,\cdots,\cdot\|\big]$ and $\big[W,Y,\mathcal{M},u,\mathcal{A},\theta,p,\Delta_n^m,\|\cdot,\cdots,\cdot\|\big]_\infty$ are also linear spaces over the field $\mathbb{C}$ of complex numbers. □

Theorem 2.2. *Let $\mathcal{M}=(M_k)$ be a Musielak-Orlicz function and $p=(p_k)$ be a bounded sequence of positive real numbers. Then the sequence space $\big[W,Y,\mathcal{M},u,\mathcal{A},\theta,p,\Delta_n^m,\|\cdot,\cdots,\cdot\|\big]_0$ is a paranormed space with respect to a paranorm*

$$\vartheta(x)=\inf\Big\{(\rho)^{p_r/H}:\Big(\frac{1}{h_r}\sum_{k\in I_r}a_{jk}\Big[M_k\Big(\Big\|\Big(\frac{u_k\Delta_n^m(f_kg(x_k))}{\rho},z_1,\cdots,z_{n-1}\Big)\Big\|\Big)\Big]^{p_k}\Big)^{1/H}\le1\Big\},$$

where $H=\max(1,\sup_k p_k<\infty)$.

Proof. (i) It is obvious that $\vartheta(x)\ge0$, for $x=(x_k)\in\big[W,Y,\mathcal{M},u,\mathcal{A},\theta,p,\Delta_n^m,\|\cdot,\cdots,\cdot\|\big]_0$. Since $M_k(0)=0$, we get $\vartheta(0)=0$.
(ii) $\vartheta(-x)=\vartheta(x)$.
(iii) Let $(x_k),(y_k)\in\big[W,Y,\mathcal{M},u,\mathcal{A},\theta,p,\Delta_n^m,\|\cdot,\cdots,\cdot\|\big]_0$, then there exist positive numbers ρ_1,ρ_2 such that

$$\frac{1}{h_r}\sum_{k\in I_r}a_{jk}\Big[M_k\Big(\Big\|\Big(\frac{u_k\Delta_n^m(f_kg(x_k))}{\rho_1},z_1,\cdots,z_{n-1}\Big)\Big\|\Big)\Big]^{p_k}\le1$$

and

$$\frac{1}{h_r}\sum_{k\in I_r}a_{jk}\Big[M_k\Big(\Big\|\Big(\frac{u_k\Delta_n^m(f_kg(y_k))}{\rho_2},z_1,\cdots,z_{n-1}\Big)\Big\|\Big)\Big]^{p_k}\le1.$$

Now, by using Minkowski's inequality, for $\rho = \rho_1 + \rho_2$, we have

$$\Big(\frac{1}{h_r}\sum_{k\in I_r} a_{jk}\Big[M_k\Big(\Big\|\Big(\frac{u_k\Delta_n^m((f_kg(x_k)) + (f_kg(y_k)))}{\rho}, z_1, \cdots, z_{n-1}\Big)\Big\|\Big)\Big]^{p_k}\Big)^{1/H}$$
$$= \Big(\frac{1}{h_r}\sum_{k\in I_r} a_{jk}\Big[M_k\Big(\Big\|\Big(\frac{u_k\Delta_n^m((f_kg(x_k))) + u_k\Delta_n^m((f_kg(y_k)))}{\rho_1+\rho_2}, z_1, \cdots, z_{n-1}\Big)\Big\|\Big)\Big]^{p_k}\Big)^{1/H}$$
$$\leq \Big(\frac{1}{h_r}\sum_{k\in I_r} a_{jk}\Big[\Big(\frac{\rho_1}{\rho_1+\rho_2}\Big)M_k\Big(\Big\|\Big(\frac{u_k\Delta_n^m((f_kg(x_k)))}{\rho_1}, z_1, \cdots, z_{n-1}\Big)\Big\|$$
$$+ \Big(\frac{\rho_2}{\rho_1+\rho_2}\Big)M_k\Big(\Big\|\Big(\frac{u_k\Delta_n^m((f_kg(y_k)))}{\rho_2}, z_1, \cdots, z_{n-1}\Big)\Big\|\Big)\Big]^{p_k}\Big)^{1/H}$$
$$\leq \Big(\frac{\rho_1}{\rho_1+\rho_2}\Big)\Big(\frac{1}{h_r}\sum_{k\in I_r} a_{jk}\Big[M_k\Big(\Big\|\Big(\frac{u_k\Delta_n^m((f_kg(x_k)))}{\rho_1}, z_1, \cdots, z_{n-1}\Big)\Big\|\Big)\Big]^{p_k}\Big)^{1/H}$$
$$+ \Big(\frac{\rho_2}{\rho_1+\rho_2}\Big)\Big(\frac{1}{h_r}\sum_{k\in I_r} a_{jk}\Big[M_k\Big(\Big\|\Big(\frac{u_k\Delta_n^m((f_kg(y_k)))}{\rho_2}, z_1, \cdots, z_{n-1}\Big)\Big\|\Big)\Big]^{p_k}\Big)^{1/H}$$
$$\leq 1.$$

Since, $\rho \geq 0$, so

$$\vartheta(x+y)$$
$$= \inf\Big\{(\rho)^{p_r/H} : \Big(\frac{1}{h_r}\sum_{k\in I_r} a_{jk}\Big[M_k\Big(\Big\|\Big(\frac{u_k\Delta_n^m(f_kg(x_k)) + u_k\Delta_n^m(f_kg(y_k))}{\rho}, z_1, \cdots, z_{n-1}\Big)\Big\|\Big)\Big]^{p_k}\Big)^{1/H} \leq 1\Big\}$$
$$\leq \inf\Big\{(\rho_1)^{p_r/H} : \Big(\frac{1}{h_r}\sum_{k\in I_r} a_{jk}\Big[M_k\Big(\Big\|\Big(\frac{u_k\Delta_n^m(f_kg(x_k))}{\rho_1}, z_1, \cdots, z_{n-1}\Big)\Big\|\Big)\Big]^{p_k}\Big)^{1/H} \leq 1\Big\}$$
$$+ \inf\Big\{(\rho_2)^{p_r/H} : \Big(\frac{1}{h_r}\sum_{k\in I_r} a_{jk}\Big[M_k\Big(\Big\|\Big(\frac{u_k\Delta_n^m(f_kg(y_k))}{\rho_2}, z_1, \cdots, z_{n-1}\Big)\Big\|\Big)\Big]^{p_k}\Big)^{1/H} \leq 1\Big\}.$$

Therefore, $\vartheta(x+y) \leq \vartheta(x) + \vartheta(y)$.
Next, we show that the scalar multiplication is continuous. Let μ be any complex number. Then

$$\vartheta(\mu x)$$
$$= \inf\Big\{(\rho)^{p_r/H} : \Big(\frac{1}{h_r}\sum_{k\in I_r} a_{jk}\Big[M_k\Big(\Big\|\Big(\frac{u_k\Delta_n^m\mu(f_kg(x_k))}{\rho}, z_1, \cdots, z_{n-1}\Big)\Big\|\Big)\Big]^{p_k}\Big)^{1/H} \leq 1\Big\}$$
$$= \inf\Big\{(|\mu|b)^{p_r/H} : \Big(\frac{1}{h_r}\sum_{k\in I_r} a_{jk}\Big[M_k\Big(\Big\|\Big(\frac{u_k\Delta_n^m(f_kg(x_k))}{b}, z_1, \cdots, z_{n-1}\Big)\Big\|\Big)\Big]^{p_k}\Big)^{1/H} \leq 1\Big\},$$

where $b = \frac{\rho}{|\mu|} > 0$. Now,

$$\vartheta(\mu x) \leq \max(1, |\mu|^{\sup p_r}) \inf \Big\{ (g)^{p_r/H} : \Big(\frac{1}{h_r} \sum_{k \in I_r} a_{jk} \Big[M_k \Big(\Big\| \Big(\frac{u_k \Delta_n^m (f_k g(x_k))}{g}, z_1, \cdots, z_{n-1} \Big) \Big\| \Big) \Big]^{p_k} \Big)^{1/H} \leq 1 \Big\},$$

since $|\mu|^{p_r} \leq \max(1, |\mu|^{\sup p_r})$. Hence, the sequence space $\big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]_0$ is a paranormed space with respect to a paranorm $\vartheta(x)$. □

Theorem 2.3. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ and (s_k) be two bounded sequences of positive real numbers with $0 < p_k \leq s_k$ for each k and $(\frac{s_k}{p_k})$ be bounded. Then*
(i) $\big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, s, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]_0 \subseteq \big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]_0$,
(ii) $\big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, s, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big] \subseteq \big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]$,
(iii) $\big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, s, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]_\infty \subseteq \big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]_\infty$.

Proof. Let $(x_k) \in \big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, s, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]_\infty$. Then

$$\sup_j \sup_r \Big(\frac{1}{h_r} \sum_{k \in I_r} a_{jk} \Big[M_k \Big(\Big\| \Big(\frac{u_k \Delta_n^m (f_k g(x_k))}{\rho}, z_1, \cdots, z_{n-1} \Big) \Big\| \Big) \Big]^{s_k} < \infty.$$

Consider $\nu_k = a_{jk} \Big[M_k \Big(\Big\| \Big(\frac{u_k \Delta_n^m (f_k g(x_k))}{\rho}, z_1, \cdots, z_{n-1} \Big) \Big\| \Big) \Big]^{s_k}$ and $\eta_k = p_k / s_k$. Since $p_k \leq s_k$, hence $0 < \eta < \eta_k \leq 1$. Write $z_k = \nu_k, y_k = 0$ if $\nu_k \geq 1$ and $y_k = \nu_k, z_k = 0$ if $\nu_k < 1$. Thus, $\nu_k = z_k + y_k$ and $\nu_k^{\eta_k} = z_k^{\eta_k} + y_k^{\eta_k}$. It follows that $z_k^{\eta_k} \leq z_k \leq \nu_k$ and $y_k^{\eta_k} \leq y_k^{\eta}$.
So,

$$\frac{1}{h_r} \sum_{k \in I_r} \nu_k^{\eta_k} = \frac{1}{h_r} \sum_{k \in I_r} (z_k^{\eta_k} + y_k^{\eta_k}) \leq \frac{1}{h_r} \sum_{k \in I_r} z_k + \frac{1}{h_r} \sum_{k \in I_r} y_k^{\eta}.$$

Now $\frac{1}{\eta} > 1$, as $\eta < 1$. By applying Holder's inequality, we have

$$\begin{aligned} \frac{1}{h_r} \sum_{k \in I_r} y_k^{\eta} &= \sum_{k \in I_r} \Big(\frac{1}{h_r} y_k \Big)^{\eta} \Big(\frac{1}{h_r} \Big)^{1-\eta} \\ &\leq \Big(\sum_{k \in I_r} \Big[\Big(\frac{1}{h_r} y_k \Big)^{\eta} \Big]^{1/\eta} \Big)^{\eta} \Big(\sum_{k \in I_r} \Big[\Big(\frac{1}{h_r} \Big)^{1-\eta} \Big]^{\frac{1}{1-\eta}} \Big)^{1-\eta} = \Big(\frac{1}{h_r} \sum_{k \in I_r} y_k \Big)^{\eta}. \end{aligned}$$

Hence,

$$\frac{1}{h_r} \sum_{k \in I_r} y_k^{\eta} \leq \frac{1}{h_r} \sum_{k \in I_r} \nu_k + \Big(\frac{1}{h_r} \sum_{k \in I_r} y_k \Big)^{\eta}.$$

Therefore, $(x_k) \in \big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]_\infty$. One can easily prove (i) and (ii). □

Theorem 2.4. *Let $\mathcal{M} = (M_k)$ and $\mathcal{M}' = (M_k')$ be two Musielak-Orlicz functions. Then the following inclusions hold:*

(i) $\big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]_0 \bigcap \big[W, Y, \mathcal{M}', u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]_0 \subset \big[W, Y, \mathcal{M} + \mathcal{M}', u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]_0$,

(ii) $\big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big] \bigcap \big[W, Y, \mathcal{M}', u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big] \subset \big[W, Y, \mathcal{M} + \mathcal{M}', u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]$,

(iii) $\big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]_\infty \bigcap \big[W, Y, \mathcal{M}', u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]_\infty \subset \big[W, Y, \mathcal{M} + \mathcal{M}', u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]_\infty$.

Proof. Let $(x_k) \in \big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]_\infty \bigcap \big[W, Y, \mathcal{M}', u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]_\infty$.

Then for all $g \in W^*$, $f_k \in Y^*$ and for some $\rho > 0$, we have

$$\sup_j \sup_r \frac{1}{h_r} \sum_{k \in I_r} a_{jk} \Big[M_k\Big(\Big\|\Big(\frac{u_k \Delta_n^m (f_k g(x_k))}{\rho}, z_1, \cdots, z_{n-1}\Big)\Big\|\Big)\Big]^{p_k} < \infty$$

and

$$\sup_j \sup_r \frac{1}{h_r} \sum_{k \in I_r} a_{jk} \Big[M'_k\Big(\Big\|\Big(\frac{u_k \Delta_n^m (f_k g(x_k))}{\rho}, z_1, \cdots, z_{n-1}\Big)\Big\|\Big)\Big]^{p_k} < \infty.$$

Now, by using inequality (1.1), we have

$$\begin{aligned} a_{jk} \Big[(M_k + M'_k)\Big(\Big\|\Big(\frac{u_k \Delta_n^m (f_k g(x_k))}{\rho}, z_1, \cdots, z_{n-1}\Big)\Big\|\Big)\Big]^{p_k} \\ \leq \; & D\Big[a_{jk}\Big[M_k\Big(\Big\|\Big(\frac{u_k \Delta_n^m (f_k g(x_k))}{\rho}, z_1, \cdots, z_{n-1}\Big)\Big\|\Big)\Big]^{p_k}\Big] \\ & + D\Big[a_{jk}\Big[M'_k\Big(\Big\|\Big(\frac{u_k \Delta_n^m (f_k g(x_k))}{\rho}, z_1, \cdots, z_{n-1}\Big)\Big\|\Big)\Big]^{p_k}\Big]. \end{aligned}$$

This implies

$$\begin{aligned} \frac{1}{h_r} \sum_{k \in I_r} a_{jk} \Big[(M_k + M'_k)\Big(\Big\|\Big(\frac{u_k \Delta_n^m (f_k g(x_k))}{\rho}, z_1, \cdots, z_{n-1}\Big)\Big\|\Big)\Big]^{p_k} \\ \leq \; & \frac{D}{h_r} \sum_{k \in I_r} a_{jk} \Big[M_k\Big(\Big\|\Big(\frac{u_k \Delta_n^m (f_k g(x_k))}{\rho}, z_1, \cdots, z_{n-1}\Big)\Big\|\Big)\Big]^{p_k} \\ & + \frac{D}{h_r} \sum_{k \in I_r} a_{jk} \Big[M'_k\Big(\Big\|\Big(\frac{u_k \Delta_n^m (f_k g(x_k))}{\rho}, z_1, \cdots, z_{n-1}\Big)\Big\|\Big)\Big]^{p_k} < \infty. \end{aligned}$$

Hence, $(x_k) \in \big[W, Y, \mathcal{M} + \mathcal{M}', u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]_\infty$. Similarly we can prove for (i) and (ii). □

Theorem 2.5. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ be a bounded sequence of positive real numbers. Then* $\big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]_0 \subset \big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big] \subset \big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]_\infty$.

Proof. We establish the inclusion $\big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big] \subset \big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]_\infty$, since the first inclusion is obvious. Let $(x_k) \in \big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]$. Then there exists some positive number ρ_1, such that

$$\frac{1}{h_r} \sum_{k \in I_r} a_{jk} \Big[M_k\Big(\Big\|\Big(\frac{u_k \Delta_n^m (f_k g(x_k)) - L}{\rho_1}, z_1, \cdots, z_{n-1}\Big)\Big\|\Big)\Big]^{p_k} \to 0 \quad \text{as} \quad r \to \infty.$$

Since $\mathcal{M} = (M_k)$ is non-decreasing, convex and so by using inequality (1.1), for $\rho = 2\rho_1$, we have

$$\frac{1}{h_r}\sum_{k\in I_r} a_{jk}\Big[M_k\Big(\Big\|\Big(\frac{u_k\Delta_n^m(f_kg(x_k))}{\rho}, z_1,\cdots,z_{n-1}\Big)\Big\|\Big)\Big]^{p_k}$$
$$\leq \frac{D}{h_r}\sum_{k\in I_r} a_{jk}\Big[M_k\Big(\Big\|\Big(\frac{u_k\Delta_n^m(f_kg(x_k))-L}{\rho_1}, z_1,\cdots,z_{n-1}\Big)\Big\|\Big)\Big]^{p_k}$$
$$+\frac{D}{h_r}\sum_{k\in I_r} a_{jk}\Big[M_k\Big(\Big\|\Big(\frac{L}{\rho_1}, z_1,\cdots,z_{n-1}\Big)\Big\|\Big)\Big]^{p_k}$$
$$\leq \frac{D}{h_r}\sum_{k\in I_r} a_{jk}\Big[M_k\Big(\Big\|\Big(\frac{u_k\Delta_n^m(f_kg(x_k))-L}{\rho_1}, z_1,\cdots,z_{n-1}\Big)\Big\|\Big)\Big]^{p_k}$$
$$+D\max\Big\{1, a_{jk}\Big[M_k\Big(\Big\|\Big(\frac{L}{\rho_1}, z_1,\cdots,z_{n-1}\Big)\Big\|\Big)\Big]^{H}\Big\} < \infty.$$

This implies $(x_k) \in \big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot,\cdots,\cdot\|\big]_\infty$. Hence, the proof. □

Theorem 2.6. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function which satisfies Δ_2-condition for all k and $p = (p_k)$ be a bounded sequence of positive real numbers. Then*

$$\big[W, Y, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot,\cdots,\cdot\|\big] \subset \big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot,\cdots,\cdot\|\big].$$

Proof. Let $(x_k) \in \big[W, Y, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot,\cdots,\cdot\|\big]$. Then, we have

$$\mathcal{S}_r = \frac{1}{h_r}\sum_{k\in I_r} a_{jk}[(\|(u_k\Delta_n^m(f_kg(x_k)) - L, z_1,\cdots,z_{n-1})\|)]^{p_k} \to \infty \quad \text{as} \quad r\to\infty,$$

for all $g \in W^*$, $f_k \in Y^*$ and for some L. Let $\epsilon > 0$ and choose φ with $0 < \varphi < 1$ such that $M_k(s) < \epsilon$ for $0 \leq s \leq \varphi$ for all k. Thus

$$\frac{1}{h_r}\sum_{k\in I_r} a_{jk}\Big[M_k\Big(\Big\|\Big(\frac{u_k\Delta_n^m(f_kg(x_k))-L}{\rho}, z_1,\cdots,z_{n-1}\Big)\Big\|\Big)\Big]^{p_k}$$
$$= \frac{1}{h_r}\sum_{1} a_{jk}\Big[M_k\Big(\Big\|\Big(\frac{u_k\Delta_n^m(f_kg(x_k))-L}{\rho}, z_1,\cdots,z_{n-1}\Big)\Big\|\Big)\Big]^{p_k}$$
$$+ \frac{1}{h_r}\sum_{2} a_{jk}\Big[M_k\Big(\Big\|\Big(\frac{u_k\Delta_n^m(f_kg(x_k))-L}{\rho}, z_1,\cdots,z_{n-1}\Big)\Big\|\Big)\Big]^{p_k},$$

where $\sum_1$ is taken over $k \in I_r$, $\|(u_k\Delta_n^m(f_kg(x_k))-L, z_1,\cdots,z_{n-1})\| \leq \varphi$ and $\sum_2$ is taken over $k \in I_r$, $\|(u_k\Delta_n^m(f_kg(x_k)) - L, z_1,\cdots,z_{n-1})\| > \varphi$. Also, by using the properties of Musielak-Orlicz function we have $\sum_1 \leq \varphi^H$ and for $\sum_2$ we have

$$\Big\|\Big(u_k\Delta_n^m(f_kg(x_k)) - L, z_1,\cdots,z_{n-1}\Big)\Big\| \leq 1 + \Big\|\Big(\frac{u_k\Delta_n^m(f_kg(x_k))-L}{\varphi}, z_1,\cdots,z_{n-1}\Big)\Big\|.$$

Since (M_k) is non-decreasing and convex for all k,

$$a_{jk}\Big[M_k\Big(\Big\|\Big(u_k\Delta_n^m(f_kg(x_k)) - L, z_1,\cdots,z_{n-1}\Big)\Big\|\Big)\Big]$$
$$< a_{jk}\Big[M_k\Big(1 + \Big\|\Big(\frac{u_k\Delta_n^m(f_kg(x_k))-L}{\varphi}, z_1,\cdots,z_{n-1}\Big)\Big\|\Big)\Big]$$
$$\leq \frac{1}{2}a_{jk}(M_k(2)) + \frac{1}{2}a_{jk}\Big[M_k\Big((2)\Big\|\Big(\frac{u_k\Delta_n^m(f_kg(x_k))-L}{\varphi}, z_1,\cdots,z_{n-1}\Big)\Big\|\Big)\Big].$$

Since (M_k) satisfies Δ_2-condition for all k, we have
$a_{jk}[M_k(\|u_k\Delta_n^m(f_kg(x_k)) - L, z_1, \cdots, z_{n-1}\|)]$

$$\begin{aligned} &\leq \frac{1}{2}K\left\|\left(\frac{u_k\Delta_n^m(f_kg(x_k)) - L}{\varphi}, z_1, \cdots, z_{n-1}\right)\right\|M_k(2) \\ &+ \frac{1}{2}K\left\|\left(\frac{u_k\Delta_n^m(f_kg(x_k)) - L}{\varphi}, z_1, \cdots, z_{n-1}\right)\right\|M_k(2) \\ &= K\left\|\left(\frac{u_k\Delta_n^m(f_kg(x_k)) - L}{\varphi}, z_1, \cdots, z_{n-1}\right)\right\|M_k(2). \end{aligned}$$

Thus,

$$\frac{1}{h_r}\sum_{k\in I_r} a_{jk}[M_k(\|(\frac{u_k\Delta_n^m(f_kg(x_k)) - L}{\rho}, z_1, \cdots, z_{n-1})\|)]^{p_k} \leq \varepsilon^H + [\max(1, KM_k(2))\varphi]^H \mathcal{S}_r.$$

Hence, $(x_k) \in \left[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\right]$ as $r \to \infty$. □

Theorem 2.7. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function which satisfies Δ_2-condition for all k and $p = (p_k)$ be a bounded sequence of positive real numbers. Then the following statements are equivalent:*
(i) $\left[W, Y, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\right]_\infty \subset \left[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\right]_\infty$,
(ii) $\left[W, Y, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\right]_0 \subset \left[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\right]_\infty$,
(iii) $\sup_r \frac{1}{h_r}\sum_{k\in I_r} a_{jk}[M_k(s)]^{p_k} < \infty$, *for all* $s > 0$.

Proof. $(i) \Rightarrow (ii)$: It is enough to prove that $\left[W, Y, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\right]_0 \subset \left[W, Y, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\right]_\infty$. Consider $(x_k) \in \left[W, Y, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\right]_0$. Then there exists $r > 0$, for $\varepsilon > 0$, such that

$$\frac{1}{h_r}\sum_{k\in I_r} a_{jk}[(\|(u_k\Delta_n^m(f_kg(x_k)), z_1, \cdots, z_{n-1})\|)]^{p_k} < \varepsilon.$$

Hence, there exists $N > 0$, such that

$$\sup_j \sup_r \frac{1}{h_r}\sum_{k\in I_r} a_{jk}[(\|(u_k\Delta_n^m(f_kg(x_k)), z_1, \cdots, z_{n-1})\|)]^{p_k} < N,$$

for all j and r. Thus, we have $(x_k) \in \left[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\right]_\infty$.
$(ii) \Rightarrow (iii)$: $-$ Assume that

$$\sup_r \frac{1}{h_r}\sum_{k\in I_r} a_{jk}[M_k(s)]^{p_k} = \infty$$

and hence, we can find a subinterval $I_{r(n)}$ of the set of interval I_r, such that

$$\frac{1}{h_{r(n)}}\sum_{k\in I_{r(n)}} a_{jk}\left[M_k\left(\frac{1}{n}\right)\right]^{p_k} > n, \quad n = 1, 2, \cdots. \tag{2.1}$$

Define $x_k = \frac{1}{n}$, if $k \in I_{r(n)}$ and $x_k = 0$ if $k \notin I_{r(n)}$. Then $(x_k) \in \left[W, Y, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\right]_0$, but by equation (2.1), $(x_k) \notin \left[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\right]_\infty$, which contradicts (ii). Hence $\sup_r \frac{1}{h_r}\sum_{k\in I_r} a_{jk}[M_k(s)]^{p_k} < \infty$, for all $s > 0$.
$(iii) \to (i)$: Assume that $(x_k) \in \left[W, Y, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\right]_\infty$ and $(x_k) \notin \left[W, Y, \mathcal{M},\right.$

$u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]_\infty$, then we have

$$\sup_j \sup_r \frac{1}{h_r} \sum_{k \in I_r} a_{jk} [M_k(\|(\frac{u_k \Delta_n^m (f_k g(x_k)) - L}{\rho}, z_1, \cdots, z_{n-1})\|)]^{p_k} = \infty. \tag{2.2}$$

Let $s = \|\frac{u_k \Delta_n^m (f_k g(x_k)) - L}{\rho}, z_1, \cdots, z_{n-1}\|$, for each k. Then equation (2.2) becomes

$$\sup_r \frac{1}{h_r} \sum_{k \in I_r} a_{jk} [M_k(s)]^{p_k} = \infty,$$

which contradicts (iii). Therefore, (i) must holds. □

Theorem 2.8. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function which satisfies Δ_2-condition for all k and $p = (p_k)$ be a bounded sequence of positive real numbers. Then the following statements are equivalent:*
(i) $\big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]_0 \subset \big[W, Y, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]_0$,
(ii) $\big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]_0 \subset \big[W, Y, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]_\infty$,
(iii) $\inf_r \frac{1}{h_r} \sum_{k \in I_r} a_{jk} [M_k(s)]^{p_k} > 0$, *for all* $s > 0$.

Proof. $(i) \Rightarrow (ii) :-$ One can easily show this .
$(ii) \Rightarrow (iii) :-$ Assume that

$$\inf_r \frac{1}{h_r} \sum_{k \in I_r} a_{jk} [M_k(s)]^{p_k} = 0, \quad \text{for some} \quad s > 0.$$

Hence, we can find a subinterval $I_{r(n)}$ of the set of interval I_r such that

$$\frac{1}{h_{r(n)}} \sum_{k \in I_{r(n)}} a_{jk} [M_k(n)]^{p_k} < \frac{1}{n}, \quad n = 1, 2, \cdots . \tag{2.3}$$

Define $(x_k) = n$, if $k \in I_{r(n)}$ and $x_k = 0$, if $k \notin I_{r(n)}$. Hence, by equation (2.3), $(x_k) \in \big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]_0$ but $(x_k) \notin \big[W, Y, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]_\infty$, which contradicts (ii). Therefore, (iii) must holds.
$(iii) \Rightarrow (i) :-$ Assume that $x = (x_k) \in \big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]_0$. Then

$$\frac{1}{h_r} \sum_{k \in I_r} a_{jk} \left[\left(M_k \left\|\left(\frac{u_k \Delta_n^m (f_k g(x_k))}{\rho}, z_1, \cdots, z_{n-1}\right)\right\|\right)\right]^{p_k} \to 0 \quad \text{as} \quad r \to \infty. \tag{2.4}$$

Now suppose that $(x_k) \notin \big[W, Y, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]_0$, for some number $\varepsilon > 0$ and a subinterval $I_{r(n)}$, of the set of interval I_r, we have

$$\left\|\left(\frac{u_k \Delta_n^m (f_k g(x_k))}{\rho}, z_1, \cdots, z_{n-1}\right)\right\| \geq \varepsilon, \quad \forall k.$$

Then by using the properties of the Orlicz function, we have

$$\left[M_k\left(\left\|\left(\frac{u_k \Delta_n^m (f_k g(x_k))}{\rho}, z_1, \cdots, z_{n-1}\right)\right\|\right)\right]^{p_k} \geq (M_k(\varepsilon))^{p_k}, \quad \forall k.$$

Hence, by using (2.4), we have

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} a_{jk} [M_k(s)]^{p_k} = 0,$$

which contradicts to $\inf_r \frac{1}{h_r} \sum_{k \in I_r} a_{jk} [M_k(s)]^{p_k} > 0$, for all $s > 0$. Therefore, (i) must holds. □

Theorem 2.9. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ be a bounded sequence of positive real numbers. If $\sup_k [M_k(x)]^{p_k} < \infty$ for all fixed $x > 0$, then*

$$\big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big] \subseteq \big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]_\infty.$$

Proof. Suppose $(x_k) \in \big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]$, then there exist positive number ρ, such that

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \left[M_k\left(\left\|\left(\frac{u_k\Delta_n^m(f_k g(x_k)) - L}{\rho}, z_1, \cdots, z_{n-1}\right)\right\|\right)\right]^{p_k} = 0.$$

Since M_k is non-decreasing and convex, then by using inequality (1.1), for $\rho = 2\rho_1$, we have

$$\begin{aligned}
&\sup_j \sup_r \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \left[M_k\left(\left\|\left(\frac{u_k\Delta_n^m(f_k g(x_k))}{\rho}, z_1, \cdots, z_{n-1}\right)\right\|\right)\right]^{p_k} \\
= \;& \sup_j \sup_r \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \left[M_k\left(\left\|\left(\frac{u_k\Delta_n^m(f_k g(x_k) + L - L)}{\rho}, z_1, \cdots, z_{n-1}\right)\right\|\right)\right]^{p_k} \\
\leq \;& D \sup_j \sup_r \frac{1}{h_r} \sum_{k\in I_r} \frac{1}{2^{p_k}} a_{jk} \left[M_k\left(\left\|\left(\frac{u_k\Delta_n^m(f_k g(x_k)) - L}{\rho_1}, z_1, \cdots, z_{n-1}\right)\right\|\right)\right]^{p_k} \\
& + D \sup_j \sup_r \frac{1}{h_r} \sum_{k\in I_r} \frac{1}{2^{p_k}} a_{jk} \left[M_k\left(\left\|\left(\frac{L}{\rho_1}, z_1, \cdots, z_{n-1}\right)\right\|\right)\right]^{p_k} \\
\leq \;& D \sup_j \sup_r \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \left[M_k\left(\left\|\left(\frac{u_k\Delta_n^m(f_k g(x_k)) - L}{\rho_1}, z_1, \cdots, z_{n-1}\right)\right\|\right)\right]^{p_k} \\
& + D \sup_j \sup_r \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \left[M_k\left(\left\|\left(\frac{L}{\rho_1}, z_1, \cdots, z_{n-1}\right)\right\|\right)\right]^{p_k} < \infty.
\end{aligned}$$

Hence, $(x_k) \in \big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]_\infty$. □

Theorem 2.10. *Let $\mathcal{M} = (M_k)$, $\mathcal{M}' = (M'_k)$ be two Musielak-Orlicz functions satisfying Δ_2-condition and $0 < \inf p_k = h \leq p_k \leq \sup p_k = H < \infty$. Then the following inclusion relations hold:*
(i) $\big[W, Y, \mathcal{M}', u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]_0 \subset \big[W, Y, \mathcal{M} \circ \mathcal{M}', u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]_0$,
(ii) $\big[W, Y, \mathcal{M}', u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big] \subset \big[W, Y, \mathcal{M} \circ \mathcal{M}', u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]$,
(iii) $\big[W, Y, \mathcal{M}', u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]_\infty \subset \big[W, Y, \mathcal{M} \circ \mathcal{M}', u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]_\infty$.

Proof. Let $(x_k) \in \big[W, Y, \mathcal{M}', u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]_0$, then

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \left[M'_k\left(\left\|\left(\frac{u_k\Delta_n^m(f_k g(x_k))}{\rho}, z_1, \cdots, z_{n-1}\right)\right\|\right)\right]^{p_k} = 0.$$

Let $\varepsilon > 0$ and choose φ with $0 < \varphi < 1$ such that $M_k(s) < \varepsilon$, for $0 \leq s \leq \varphi$.

Also let $(y_k) = M'_k\left(\left\|\left(\frac{u_k\Delta_n^m(f_k g(x_k))}{\rho}, z_1, \cdots, z_{n-1}\right)\right\|\right)$, for all $k \in \mathbb{N}$. We have

$$\frac{1}{h_r} \sum_{k\in I_r} a_{jk} (M_k[y_k])^{p_k} = \frac{1}{h_r} \sum_{\substack{k\in I_r \\ y_k \leq \varphi}} a_{jk} (M_k[y_k])^{p_k} + \frac{1}{h_r} \sum_{\substack{k\in I_r \\ y_k \geq \varphi}} a_{jk} (M_k[y_k])^{p_k}.$$

Thus, we have

$$\begin{aligned}\frac{1}{h_r}\sum_{\substack{k\in I_r\\ y_k\leq\varphi}} a_{jk}(M_k[y_k])^{p_k} &\leq [M_k(1)]^H \frac{1}{h_r}\sum_{\substack{k\in I_r\\ y_k\leq\varphi}} a_{jk}(M_k[y_k])^{p_k},\\ \frac{1}{h_r}\sum_{\substack{k\in I_r\\ y_k\leq\varphi}} a_{jk}(M_k[y_k])^{p_k} &\leq [M_k(2)]^H \frac{1}{h_r}\sum_{\substack{k\in I_r\\ y_k\leq\varphi}} a_{jk}(M_k[y_k])^{p_k}.\end{aligned} \tag{2.5}$$

Since $(M_k)'s$ are non-decreasing and convex, so for $y_k > \varphi, y_k < y_k/\varphi < 1 + y_k/\varphi$,

$$M_k(y_k) < M_k(1 + y_k/\varphi) < \frac{1}{2}M_k(2) + \frac{1}{2}M_k\left(\frac{2y_k}{\varphi}\right).$$

Since, $\mathcal{M} = (M_k)$ satisfies Δ_2-condition,

$$M_k(y_k) < \frac{1}{2}\frac{Ry_k}{\varphi}M_k(2) + \frac{1}{2}\frac{Ry_k}{\varphi}M_k(2) = \frac{Ry_k}{\varphi}M_k(2).$$

Therefore,

$$\frac{1}{h_r}\sum_{\substack{k\in I_r\\ y_k\geq\varphi}} a_{jk}(M_k[y_k])^{p_k} \leq \max\left(1, \left(R\frac{M_k(2)}{\varphi}\right)^H\right)\frac{1}{h_r}\sum_{\substack{k\in I_r\\ y_k\leq\varphi}} a_{jk}[y_k]^{p_k}. \tag{2.6}$$

Now by (2.5) and (2.6), we have $(x_k) \in \left[W, Y, \mathcal{M}\circ\mathcal{M}', u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot,\cdots,\cdot\|\right]_0$. Hence the proof. In the same manner, (ii) and (iii) can be proved. ☐

Theorem 2.11. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $0 < h = \inf p_k$. Then $\left[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot,\cdots,\cdot\|\right]_\infty \subset \left[W, Y, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot,\cdots,\cdot\|\right]_0$ if and only if*

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} a_{jk}(M_k(s))^{p_k} = \infty, \tag{2.7}$$

for some $s > 0$.

Proof. Let $\left[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot,\cdots,\cdot\|\right]_\infty \subset \left[W, Y, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot,\cdots,\cdot\|\right]_0$ and $\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} a_{jk}(M_k(s))^{p_k} = \infty$ does not holds. Thus, there is subinterval $I_{r(n)}$ of the set of intervals I_r and a number $s_0 > 0$, where

$$s_0 = \left\|\left(\frac{u_k\Delta_n^m(f_kg(x_k))}{\rho}, z_1, \cdots, z_{n-1}\right)\right\|, \quad \forall k$$

such that

$$\frac{1}{h_{r(n)}}\sum_{k\in I_{r(n)}} a_{jk}(M_k(s_0))^{p_k} \leq C < \infty, \quad n = 1, 2, 3, \cdots. \tag{2.8}$$

Define $(x_k) = s_0$, if $k \in I_{r(n)}$ and $(x_k) = 0$, if $k \notin I_{r(n)}$. Therefore, by using (2.8), $(x_k) \in \left[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot,\cdots,\cdot\|\right]_\infty$ and $(x_k) \notin \left[W, Y, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot,\cdots,\cdot\|\right]_0$. Hence, (2.7) must holds.

Conversely, assume that (2.7) holds and let $(x_k) \in \left[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot,\cdots,\cdot\|\right]_\infty$. Then for each r,

$$\frac{1}{h_r}\sum_{k\in I_r} a_{jk}\left[M_k\left(\left\|\left(\frac{u_k\Delta_n^m(f_kg(x_k))}{\rho}, z_1, \cdots, z_{n-1}\right)\right\|\right)\right]^{p_k} \leq C < \infty. \tag{2.9}$$

Let $(x_k) \notin \left[W, Y, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\right]_0$. Then for some number $\varepsilon > 0$, there is a number t_0 such that for a subinterval $I_{r(n)}$ of the set of intervals I_r,

$$\left\|\left(\frac{u_k \Delta_n^m (f_k g(x_k))}{\rho}, z_1, \cdots, z_{n-1}\right)\right\| > \varepsilon, \quad \text{for} \quad t > t_0.$$

Now, by using the properties of sequence of Orlicz functions, we have

$$a_{jk}\left[M_k\left(\left\|\left(\frac{u_k \Delta_n^m (f_k g(x_k))}{\rho}, z_1, \cdots, z_{n-1}\right)\right\|\right)\right]^{p_k} \geq a_{jk}[M_k(\varepsilon)]^{p_k},$$

which contradicts (2.7). Hence, by using (2.9), we have

$$\left[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\right]_\infty \subset \left[W, Y, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\right]_0$$

□

Theorem 2.12. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and Δ_n^m be a difference operator for $m, n \geq 1$. Then*
(i) $\left[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^{m-1}, \|\cdot, \cdots, \cdot\|\right]_0 \subseteq \left[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\right]_0$,
(ii) $\left[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^{m-1}, \|\cdot, \cdots, \cdot\|\right] \subseteq \left[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\right]$,
(iii) $\left[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^{m-1}, \|\cdot, \cdots, \cdot\|\right]_\infty \subseteq \left[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\right]_\infty$.

Proof. Let $(x_k) \in \left[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^{m-1}, \|\cdot, \cdots, \cdot\|\right]_0$ and $\varepsilon > 0$ be given. Then there exists $\rho > 0$ such that

$$\frac{1}{h_r} \sum_{k \in I_r} a_{jk}\left[M_k\left(\left\|\left(\frac{u_k \Delta_n^{m-1} (f_k g(x_k))}{\rho}, z_1, \cdots, z_{n-1}\right)\right\|\right)\right]^{p_k} \to 0, \quad \text{as} \quad r \to \infty. \tag{2.10}$$

By using the properties of Musielak-Orlicz function, we have

$$\begin{aligned}
&\frac{1}{h_r} \sum_{k \in I_r} a_{jk}\left[M_k\left(\left\|\left(\frac{u_k \Delta_n^m (f_k g(x_k))}{2\rho}, z_1, \cdots, z_{n-1}\right)\right\|\right)\right] \\
&= \frac{1}{h_r} \sum_{k \in I_r} a_{jk}\left[M_k\left(\left\|\left(\frac{u_k(\Delta_n^{m-1}(f_{k+1} g(x_{k+1})) - \Delta_n^{m-1}(f_k g(x_k)))}{2\rho}, z_1, \cdots, z_{n-1}\right)\right\|\right)\right] \\
&\leq \frac{1}{2} \frac{1}{h_r} \sum_{k \in I_r} a_{jk}\left[M_k\left(\left\|\left(\frac{u_k \Delta_n^{m-1} (f_{k+1} g(x_{k+1}))}{\rho}, z_1, \cdots, z_{n-1}\right)\right\|\right)\right] \\
&+ \frac{1}{2} \frac{1}{h_r} \sum_{k \in I_r} a_{jk}\left[M_k\left(\left\|\left(\frac{u_k \Delta_n^{m-1} (f_k g(x_k))}{\rho}, z_1, \cdots, z_{n-1}\right)\right\|\right)\right].
\end{aligned}$$

Thus, by using inequality (1.1), we have

$$\begin{aligned}
&\frac{1}{h_r} \sum_{k \in I_r} a_{jk}\left[M_k\left(\left\|\left(\frac{u_k \Delta_n^m (f_k g(x_k))}{2\rho}, z_1, \cdots, z_{n-1}\right)\right\|\right)\right]^{p_k} \\
&\leq D\left\{\frac{1}{2} \frac{1}{h_r} \sum_{k \in I_r} a_{jk}\left[M_k\left(\left\|\left(\frac{u_k \Delta_n^{m-1} (f_{k+1} g(x_{k+1}))}{\rho}, z_1, \cdots, z_{n-1}\right)\right\|\right)\right]^{p_k}\right. \\
&\left.+ \frac{1}{2} \frac{1}{h_r} \sum_{k \in I_r} a_{jk}\left[M_k\left(\left\|\left(\frac{u_k \Delta_n^{m-1} (f_k g(x_k))}{\rho}, z_1, \cdots, z_{n-1}\right)\right\|\right)\right]^{p_k}\right\}.
\end{aligned}$$

Then, by using (2.10), we have

$$\lim_{r \to 0} \frac{1}{h_r} \sum_{k \in I_r} a_{jk}\left[M_k\left(\left\|\left(\frac{u_k \Delta_n^m (f_k g(x_k))}{\rho}, z_1, \cdots, z_{n-1}\right)\right\|\right)\right]^{p_k} = 0.$$

Therefore, $(x_k) \in \left[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\right]_0$. □

Theorem 2.13. *Let $p = (p_k)$ be a bounded sequence of positive real numbers.*
(i) If $0 < \inf p_k \le p_k \le 1$ for all k, then

$$\big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big] \subseteq \big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big].$$

(ii) If $1 \le p_k \le \sup p_k = H < \infty$ for all k, then

$$\big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big] \subseteq \big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big].$$

Proof. (i) Let $(x_k) \in \big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]$. Then

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \left[M_k \left(\left\| \left(\frac{u_k \Delta_n^m (f_k g(x_k)) - L}{\rho}, z_1, \cdots, z_{n-1} \right) \right\| \right) \right]^{p_k} = 0.$$

Since $0 < \inf p_k \le 1$, we have

$$\begin{aligned} &\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \left[M_k \left(\left\| \left(\frac{u_k \Delta_n^m (f_k g(x_k)) - L}{\rho}, z_1, \cdots, z_{n-1} \right) \right\| \right) \right] \\ &\quad \le \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \left[M_k \left(\left\| \left(\frac{u_k \Delta_n^m (f_k g(x_k)) - L}{\rho}, z_1, \cdots, z_{n-1} \right) \right\| \right) \right]^{p_k} = 0. \end{aligned}$$

Thus, $(x_k) \in \big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]$.
(ii) Let $(x_k) \in \big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]$.

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \left[M_k \left(\left\| \left(\frac{u_k \Delta_n^m (f_k g(x_k)) - L}{\rho}, z_1, \cdots, z_{n-1} \right) \right\| \right) \right] = 0.$$

Also, $1 \le p_k \le \sup p_k = H < \infty$ for all k. This implies that

$$\begin{aligned} &\frac{1}{h_r} \sum_{k\in I_r} a_{jk} \left[M_k \left(\left\| \left(\frac{u_k \Delta_n^m (f_k g(x_k)) - L}{\rho}, z_1, \cdots, z_{n-1} \right) \right\| \right) \right]^{p_k} \\ &\quad \le \frac{1}{h_r} \sum_{k\in I_r} a_{jk} \left[M_k \left(\left\| \left(\frac{u_k \Delta_n^m (f_k g(x_k)) - L}{\rho}, z_1, \cdots, z_{n-1} \right) \right\| \right) \right] = 0. \end{aligned}$$

Thus, $(x_k) \in \big[W, Y, \mathcal{M}, u, \mathcal{A}, \theta, p, \Delta_n^m, \|\cdot, \cdots, \cdot\|\big]$. Hence, the proof. □

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Received 15/10/2018; Revised 30/11/2018