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SUBSCALARITY OF k-QUASI-CLASS A OPERATORS

M. H. M. RASHID

Abstract. In this paper, we show that every k-quasi-class A operator has a scalar extension and give some spectral properties of the scalar extensions of k-quasi-class A operators. As a corollary, we get that such an operator with rich spectrum has a nontrivial invariant subspace.

1. Introduction

Let $\mathscr{H}$ and $\mathscr{K}$ be separable complex Hilbert spaces, and let $\mathscr{B}(\mathscr{H},\mathscr{K})$ denote the algebra of all bounded linear operators from $\mathscr{H}$ to $\mathscr{K}$. When $\mathscr{H}=\mathscr{K}$, we write $\mathscr{B}(\mathscr{H})$ for $\mathscr{B}(\mathscr{H},\mathscr{H})$. Throughout this paper, the *range* and the *null* space of an operator T will be denoted by $\mathscr{R}(T)$ and $\ker(T)$, we write $\sigma(T)$, $\sigma_a(T)$, and $\sigma_e(T)$ for the *spectrum*, the *approximate point spectrum*, and the *essential spectrum*, respectively. An operator $T\in\mathscr{B}(\mathscr{H})$ is said to be *p-hyponormal* if $(T^*T)^p\geq(TT^*)^p$, where $0<p<\infty$. In particular, 1-*hyponormal* operator and $\frac{1}{2}$-*hyponormal* operators are called *hyponormal* operator and *semi-hyponormal* operators, respectively.

An arbitrary operator $T\in\mathscr{B}(\mathscr{H})$ has a unique polar decomposition $T=U|T|$, where $|T|=(T^*T)^{1/2}$ and U is a *partial isometry* satisfying $\ker(U)=\ker(|T|)=\ker(T)$ and $\ker(U^*=\ker(T^*)$. Associated with T is the operator $|T|^{1/2}U|T|^{1/2}$ called the *Aluthge transform* of T, and denoted throughout this paper by $\widehat{T}$. For every $T\in\mathscr{B}(\mathscr{H})$, the sequence $\{\widehat{T}^{(n)}\}$ of Aluthge iterates of T is defined by $\widehat{T}^{(0)}=T$ and $\widehat{T}^{(n+1)}=\widehat{\widehat{T}^{(n)}}$ for every positive integer n (see [2] and [11]).

An operator $T\in\mathscr{B}(\mathscr{H})$ is said to be *w-hyponormal* if $|\widehat{T}|\geq|T|\geq|\widehat{T}^*|$ (see [3]), and *paranormal* if $\|Tx\|^2\leq\|T^2x\|\,\|x\|$ for all $x\in\mathscr{H}$. We say that $T\in\mathscr{B}(\mathscr{H})$ is *normaloid* if $\|T\|=r(T)$, where $r(T)$ is the *spectral radius* of T. It is well-known that every p-hyponormal operator is w-hyponormal and that every w-hyponormal operator is normaloid. Furuta et al. [7] introduced the following interesting class of Hilbert space operators: We say that $T\in\mathscr{B}(\mathscr{H})$ belongs to *class A* if $|T^2|\geq|T|^2$. It is known that

$$\{\text{Hyponormal}\}\subset\{w\text{-hyponormal}\}\subset\{\text{Class } A\}\subset\{\text{Paranormal}\}.$$

More recently, the authors of [9] have extended class A operators to *quasi-class A* operators. An operator $T\in\mathscr{B}(\mathscr{H})$ is said to be *quasi-class A* if $T^*|T^2|T\geq T^*|T|^2T$, and *quasi-paranormal* if $\|TTx\|^2\leq\|T^3x\|\,\|Tx\|$ for all $x\in\mathscr{H}$. Hence we have the following inclusion:

$$\{\text{class } A\}\subset\{\text{quasi-class } A\}\subset\{\text{quasi-paranormal}\}.$$

As a further generalization, Tanahashi et al. [20] introduced the class of *k-quasi-class A* operators. An operator T is said to be a *k-quasi-class A* operator if

$$T^{*k}(|T^2|-|T|^2)T^k\geq 0,$$

where k is a positive integer number.

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An operator $T \in \mathscr{B}(\mathscr{H})$ is called *scalar* of order m if it possesses a spectral distribution of order m, i.e. a continuous unital morphism of topological algebras

$$\psi : C_0^m(\mathbb{C}) \to \mathscr{B}(\mathscr{H})$$

such that $\psi(z) = T$, where as usual z stands for the identity function on C_0^m, the complex-valued continuously differentiable functions of order m, $0 \le m \le \infty$. An operator is said to be *subscalar* of order m if it is similar to the restriction of a scalar operator of order m to an invariant subspace.

In 1984, M. Putinar [16] showed that every hyponormal operator has a scalar extension. In 1987, his theorem was used to show that hyponormal operators with thick spectra have nontrivial invariant subspaces, a result due to S. Brown (see [5]). Recently, S. Jung et al. [10] showed that every class A operator has a scalar extension. In particular, such operators with rich spectra have nontrivial invariant subspaces. Also they give some spectral properties of the scalar extension of a class A operator. In 2018, his theorem was used to show that w-hyponormal operators with thick spectra have nontrivial invariant subspaces, a result due to M. H. M. Rashid (see [18]). In this paper, we show that every k-quasi-class A operator has a scalar extension and give some spectral properties of the scalar extensions of k-quasi-class A operators. As a corollary, we get that such an operator with rich spectrum has a nontrivial invariant subspace.

2. Preliminaries

Let z be the coordinate in $\mathbb{C}$, and let $d\mu(z)$, or simply $d\mu$, denote the planar Lebesgue measure. Let U be a bounded open subset of $\mathbb{C}$. We shall denote by $L^2(U, \mathscr{H})$ the Hilbert space of measurable functions $f : U \to \mathscr{H}$ such that

$$\|f\|_{2,U} = \left(\int_U \|f(z)\|^2 \, d\mu \right)^{\frac{1}{2}} < \infty.$$

We denote the space $L^2(U, \mathscr{H}) \cap \mathcal{H}(U, \mathscr{H})$ by $A^2(U, \mathscr{H})$ where $\mathcal{H}(U, \mathscr{H})$ is the Fréechet space of analytic (holomorphic) $\mathscr{H}$-valued functions on U. Then $A^2(U, \mathscr{H})$ is a closed subspace of the $L^2(U, \mathscr{H})$, and the orthogonal projection of $L^2(U, \mathscr{H})$ onto this space will be denoted by P.

Now, we introduce a special Sobolev type space. Let U be a bounded open subset of $\mathbb{C}$ and m be a fixed nonnegative integer. Then the Sobolev space $W^m(U, \mathscr{H})$ is the space of functions $f \in L^2(U, \mathscr{H})$ whose derivatives $\overline{\partial} f, \overline{\partial}^2 f, \ldots, \overline{\partial}^m f$ in the sense of distributions still belong to $L^2(U, \mathscr{H})$. Endowed with the norm

$$\|f\|_{W^m}^2 = \sum_{i=0}^{m} \left\| \overline{\partial}^i f \right\|_{2,U}^2 ,$$

$W^m(U, \mathscr{H})$ becomes a Hilbert space contained continuously in $L^2(U, \mathscr{H})$. The linear operator M of multiplication by z on $W^m(U, \mathscr{H})$ is continuous and it has a spectral distribution ψ of order m defined by the following relation: for $\phi \in C_0^m(\mathbb{C})$ and $f \in W^m(U, \mathscr{H})$, $\psi(\phi) f = \phi f$. Hence M is a scalar operator of order m.

An operator $T \in \mathscr{B}(\mathscr{H})$ is said to have the single-valued extension property at z_0 if for every neighborhood D of z_0 and any analytic function $f : D \to \mathscr{H}$ with $(T - z) f(z) = 0$, we have $f(z) \equiv 0$. An operator $T \in \mathscr{B}(\mathscr{H})$ is said to have the single-valued extension property (or SVEP) if it has the single-valued extension property at every z in $\mathbb{C}$. For an operator $T \in \mathscr{B}(\mathscr{H})$ with SVEP and for $x \in \mathscr{H}$ we can consider the set $\rho_T(x)$ of elements z_0 in $\mathbb{C}$ such that there exists an analytic function $f(z)$ defined in a neighborhood of z_0, with values in $\mathscr{H}$, which satisfies $(T - z) f(z) \equiv x$. We denote $\sigma_T(x) = \mathbb{C} \setminus \rho_T(x)$ and $\mathscr{H}_T(F) = \{x \in \mathscr{H} : \sigma_T(x) \subset F\}$, where F is a subset of $\mathbb{C}$. An operator $T \in \mathscr{B}(\mathscr{H})$ is said to have Dunford's property (C) if $\mathscr{H}_T(F)$ is closed for each closed subset F of $\mathbb{C}$.

An operator $T \in \mathscr{B}(\mathscr{H})$ is said to have property (δ) if for every open covering (U, V) of $\mathbb{C}$, $\mathscr{H} = \mathscr{H}_T(\overline{U}) + \mathscr{H}_T(\overline{V})$. It is well known that that the adjoint of a bounded linear operator on a Hilbert space with the property (β) has the property (δ) (see [1]). An operator $T \in \mathscr{B}(\mathscr{H})$ is said to have the property (β) if for every open subset G of $\mathbb{C}$ and every sequence $f_n : G \to \mathscr{H}$ of $\mathscr{H}$-valued analytic functions such that $(T - z)f_n(z)$ converges uniformly to 0 in norm on compact subsets of G, $f_n(z)$ converges uniformly to 0 in norm on compact subsets of G. It is well-known that

$$\text{Property } (\beta) \Rightarrow \text{Dunford's property } (C) \Rightarrow \text{SVEP}.$$

The SVEP of operators was first introduced by N. Dunford to investigate a class of spectral operators; this is another important generalization of normal operators (see [15]). In local spectral theory, for a given operator T on a complex Banach space $\mathcal{X}$ and a vector $x \in \mathcal{X}$, one is often interested in the existence and uniqueness of an analytic solution $f(.) : U \to \mathcal{X}$ of the local resolvent equation

$$(T - z)f(z) = x$$

on a suitable open subset U of $\mathbb{C}$. Clearly, if T has the SVEP, then the existence of an analytic solution to any local resolvent equation (related to T) implies the uniqueness of its analytic solution. The SVEP is possessed by many important classes of operators such as hyponormal operators and decomposable operators.

The most satisfactory generalization of normal operators on a Hilbert space to a general Banach space is the concept of decomposable operators. These operators possess a spectral theorem and rich lattice structure for which it is possible to develop what is called local spectral theory, i.e., a local spectral analysis. Decomposability can be defined in several ways, for instance, by the concept of a local spectral subspace.

Definition 2.1. An operator $T \in \mathscr{B}(\mathscr{H})$ is said to be decomposable if T has both the Dunford property (C) and property (δ).

Standard examples of decomposable operators are normal operators on Hilbert spaces and operators that have totally disconnected spectra. Recall that an operator $X \in \mathscr{B}(\mathscr{H}, \mathscr{K})$ is quasiaffinity if it has a trivial kernel and dense range. An operator $S \in \mathscr{B}(\mathscr{H})$ is said to be a quasiaffine transform of $T \in \mathscr{B}(\mathscr{K})$ if there is a quasiaffinity $X \in \mathscr{B}(\mathscr{H}, \mathscr{K})$ such that $XS = TX$. Furthermore, S and T are quasisimilar if there are quasiaffinities X and Y such that $XS = TX$ and $SY = YT$.

Two important subspaces in local spectral theory and Fredholm theory are $H_T(\{\lambda\})$, which is associated with the singleton $\{\lambda\}$, and $H_T(\mathbb{C}\backslash\{\lambda\})$. Note that $H_T(\{\lambda\})$ coincides with the quasi-nilpotent part of an operator $\lambda - T$, defined by

$$H_0(\lambda - T) = \{x \in \mathscr{H} : \lim_{n \longrightarrow \infty} \|(\lambda - T)^n x\|^{\frac{1}{n}} = 0\}.$$

Clearly, $\ker(\lambda - T)^n \subseteq H_0(\lambda - T)$ for every $n \in \mathbb{N}$. Also, we note that $H_T(\mathbb{C} \setminus \{\lambda\})$ coincides with the analytic core $K(\lambda - T)$ which is defined by the set of all $x \in \mathscr{H}$ such that there exists a constant $c > 0$ and a sequence of elements $x_n \in \mathscr{H}$ such that $x_0 = x$, $Tx_n = x_{n-1}$, and $\|x_n\| \leq c^n \|x\|$ for all $n \in \mathbb{N}$. In general, $H_0(\lambda - T)$ and $K(\lambda - T)$ are not closed. Furthermore,

$$\text{the closdness of } H_0(\lambda - T) \Rightarrow T \text{ has the SVEP at } \lambda.$$

Definition 2.2. An operator $T \in \mathscr{B}(\mathscr{H})$ is said to have property (Q) if $H_0(\lambda - T)$ is closed for all $\lambda \in \mathbb{C}$.

It follows that

$$\text{Property } (\beta) \Rightarrow \text{Dunford's property } (C) \Rightarrow \text{Property } (Q) \Rightarrow \text{SVEP}.$$

3. Main Results

In this section, we will show that every k-quasi-class A operator has a scalar extension. For this, we begin with the following lemmas.

Lemma 3.1. [16, Proposition 2.1] *For a bounded open disk D in the complex plane $\mathbb{C}$, there is a constant C_D such that for an arbitrary operator $f \in \mathscr{B}(\mathscr{H})$ and $f \in W^k(D, \mathscr{H})$ we have*

$$\left\|(I-P)\overline{\partial}^i f\right\|_{2,D} \leq C_D \left(\left\|(T-z)^*\overline{\partial}^{i+1} f\right\|_{2,D} + \left\|(T-z)^*\overline{\partial}^{i+2} f\right\|_{2,D}\right)$$

for $i = 0, 1, \ldots, k-2$, where P denotes the orthogonal projection of $L^2(D, \mathscr{H})$ onto the Bergman space $A^2(D, \mathscr{H})$.

Lemma 3.2. *Let T be a class A operator and let D be a bounded disc in $\mathbb{C}$. If $\{f_n\}$ is a sequence in $W^m(D, \mathscr{H})$ $(m \geq 3)$ such that*

$$\lim_{n\to\infty} \left\|(T-z)\overline{\partial}^i f_n\right\|_{2,D} = 0$$

for $i = 1, 2, \ldots, m$, then $\lim_{n\to\infty} \left\|\overline{\partial}^i f_n\right\|_{2,D_0} = 0$ for $i = 1, 2, \ldots, m-2$, where D_0 is a disc strictly contained in D and P denotes the orthogonal projection of $L^2(D, \mathscr{H})$ onto the Bergman space $A^2(D, \mathscr{H})$.

Proof. Since T belongs to class A, by Lemma 3.1, there exists a constant C_D such that

$$\left\|(I-P)\overline{\partial}^i f_n\right\|_{2,D} \leq C_D \left(\left\|(T-z)^*\overline{\partial}^{i+1} f_n\right\|_{2,D} + \left\|(T-z)^*\overline{\partial}^{i+2} f_n\right\|_{2,D}\right) \tag{3.1}$$

for $i = 0, 1, 2, \ldots, m-2$. From (3.1), we have

$$\lim_{n\to\infty} \left\|(I-P)\overline{\partial}^i f_n\right\|_{2,D} = 0 \tag{3.2}$$

for $i = 0, 1, 2, \ldots, m-2$. So, it holds that

$$\lim_{n\to\infty} \left\|(T-z)\overline{\partial}^i f_n\right\|_{2,D} = 0$$

for $i = 0, 1, 2, \ldots, m-2$. Since T has the property (β), we have

$$\lim_{n\to\infty} \left\|P\overline{\partial}^i f_n\right\|_{2,D_0} = 0 \tag{3.3}$$

for $i = 0, 1, 2, \ldots, m-2$, where D_0 denotes a disc with $\overline{D_0} \subset D$. From (3.2) and (3.3), we have

$$\lim_{n\to\infty} \left\|\overline{\partial}^i f_n\right\|_{2,D_0} = 0$$

for $i = 0, 1, 2, \ldots, m-2$. □

Next lemma is the important result for the proof of our main theorem.

Lemma 3.3. *Let $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ be an operator defined on $\mathscr{H} \oplus \mathscr{K}$, where T_1 is class A and T_3 is nilpotent of order k and let D be a bounded disc in $\mathbb{C}$ containing $\sigma(T)$. Define the map $F : \mathscr{H} \oplus \mathscr{K} \to H(D)$ by*

$$Fh = \widetilde{1 \otimes h} (\equiv 1 \otimes h + \overline{(T-z)W^{12+2k}(D, \mathscr{H}) \oplus W^{12+2k}(D, \mathscr{K})},$$

where

$$H(D) := W^{12+2k}(D, \mathscr{H}) \oplus W^{12+2k}(D, \mathscr{K}) / \overline{(T-z)W^{12+2k}(D, \mathscr{H}) \oplus W^{12+2k}(D, \mathscr{K})}$$

and $1 \otimes h$ denotes the constant function sending any $z \in D$ to h. Then F is injective and has closed range.

Proof. Let $f_n = (f_n^1, f_n^2)^t \in W^{12+2k}(D, \mathscr{H}) \oplus W^{12+2k}(D, \mathscr{K})$ and $h_n = (h_n^1, h_n^2)^t \in \mathscr{H} \oplus \mathscr{K}$ be sequences such that

$$\lim_{n\to\infty} \|(T-z)f_n + 1\otimes h_n\|_{W^{12+2k}(D,\mathscr{H})\oplus W^{12+2k}(D,\mathscr{K})} = 0. \tag{3.4}$$

Then from (3.4) we have the following equations:

$$\begin{aligned} \lim_{n\to\infty} \left\|(T_1-z)f_n^1 + T_2 f_n^2 + 1\otimes h_n^1\right\|_{W^{12+2k}} &= 0, \\ \lim_{n\to\infty} \left\|(T_3-z)f_n^2 + 1\otimes h_n^2\right\|_{W^{12+2k}} &= 0. \end{aligned} \tag{3.5}$$

The definition of the norm for the Sobolev space and (3.5) implies that

$$\begin{aligned} \lim_{n\to\infty} \left\|(T_1-z)\overline{\partial}^i f_n^1 + T_2\overline{\partial}^i f_n^2\right\|_{2,D} &= 0, \\ \lim_{n\to\infty} \left\|(T_3-z)\overline{\partial}^i f_n^2\right\|_{2,D} &= 0 \end{aligned} \tag{3.6}$$

for $i = 1, 2, \ldots, 12+2k$.

We claim that the following equation holds for every $j = 0, 1, \ldots, k$

$$\lim_{n\to\infty} \left\|T_3^{k-j}\overline{\partial}^i f_n^2\right\|_{2,D_j} = 0 \tag{3.7}$$

for $i = 1, 2, \ldots, 2k-2j+12$, where $\sigma(T) \subset \overline{D_k} \subset \overline{D_{k-1}} \subset \overline{D_1} \subset D_0 = D$.

To prove our claim, we will apply the induction on j. Since $T_3^k = 0$. The equation (3.7) holds obviously when $j = 0$. Suppose that the claim is true for $j = s < k$. Then

$$\lim_{n\to\infty} \left\|T_3^{k-s}\overline{\partial}^i f_n^2\right\|_{2,D_s} = 0 \tag{3.8}$$

for $i = 1, 2, \ldots, 2k-2s+12$. By the second equation of (3.6) and (3.8), we get that

$$\lim_{n\to\infty} \left\|zT_3^{k-s-1}\overline{\partial}^i f_n^2\right\|_{2,D_s} = 0$$

for $i = 1, 2, \ldots, 2k-2s+12$. By applications of Lemma 3.2, we obtain

$$\lim_{n\to\infty} \left\|T_3^{k-s-1}\overline{\partial}^i f_n^2\right\|_{2,D_{s+1}} = 0$$

for $i = 1, 2, \ldots, 2k-2s+10$, where $\sigma(T) \subset \overline{D_{s+1}} \subset D_r$. Hence we complete the proof of our claim. From our claim with $j = k$, we have

$$\lim_{n\to\infty} \left\|\overline{\partial}^i f_n^2\right\|_{2,D_k} = 0 \tag{3.9}$$

for $i = 1, 2, \ldots, 12$, which implies that

$$\lim_{n\to\infty} \left\|z^*\overline{\partial}^i f_n^2\right\|_{2,D_k} = 0 \tag{3.10}$$

for $i = 1, 2, \ldots, 12$. By using Lemma 3.1 with the zero operator, we get from (3.10) that

$$\lim_{n\to\infty} \|(I-P_{\mathscr{K}})f_n^2\|_{2,D_k} = 0, \tag{3.11}$$

where $P_{\mathscr{K}}$ denotes the orthogonal projection of $L^2(D_k, \mathscr{K})$ onto $A^2(D_k, \mathscr{K})$. By (3.9), it holds that

$$\lim_{n\to\infty} \left\|T_2\overline{\partial}^i f_n^2\right\|_{2,D_k} = 0 \tag{3.12}$$

for $i = 1, 2, \ldots, 12$. By combining (3.12) with the first equation of (3.6), we obtain that

$$\lim_{n\to\infty} \left\|(T_1-z)\overline{\partial}^i f_n^1\right\|_{2,D_k} = 0$$

for $i = 1, 2, \ldots, 12$. Thus

$$\lim_{n\to\infty} \left\|(T_1^2-z^2)\overline{\partial}^i f_n^1\right\|_{2,D_k} = 0 \tag{3.13}$$

for $i = 1, 2, \ldots, 12$. Let $T_1^2 = U|T_1^2|$ and $\widehat{T_1^2} = V|\widehat{T_1^2}|$ be the polar decomposition of T_1^2 and $\widehat{T_1^2}$, respectively. Since $\widehat{T_1^2}|T_1^2|^{\frac{1}{2}} = |T_1^2|^{\frac{1}{2}}T_1^2$ and $\widehat{T_1^2}^{(2)}|\widehat{T_1^2}|^{\frac{1}{2}} = |\widehat{T_1^2}|^{\frac{1}{2}}\widehat{T_1^2}$, we have

$$\begin{aligned} &\lim_{n\to\infty}\left\|(\widehat{T_1^2} - z^2)\overline{\partial}^i|T_1^2|^{\frac{1}{2}}f_n^1\right\|_{2,D_k} = 0, \\ &\lim_{n\to\infty}\left\|(\widehat{T_1^2}^{(2)} - z^2)\overline{\partial}^i|\widehat{T_1^2}|^{\frac{1}{2}}|T_1^2|^{\frac{1}{2}}f_n^1\right\|_{2,D_k} = 0 \end{aligned} \tag{3.14}$$

for $i = 1, 2, \ldots, 12$. Since T_1 belongs to class A, it follows from [8] that T_1^2 is w-hyponormal operator, and so $\widehat{T_1^2}$ is semi-hyponormal and $\widehat{T_1^2}^{(2)}$ is hyponormal by the definition of a w-hyponormal operator and [3]. Hence, it follows from (3.14) that

$$\lim_{n\to\infty}\left\|(\widehat{T_1^2}^{(2)} - z^2)^*\overline{\partial}^i|\widehat{T_1^2}|^{\frac{1}{2}}|T_1^2|^{\frac{1}{2}}f_n^1\right\|_{2,D_{k,1}} = 0 \tag{3.15}$$

for $i = 1, 2, \ldots, 12$. By Theorem 3.1 of [13], there exists a constant $C_{D_{k,1}}$ such that

$$\begin{aligned} &\left\|(I - P_{\mathscr{H}})\overline{\partial}^i|\widehat{T_1^2}|^{\frac{1}{2}}|T_1^2|^{\frac{1}{2}}f_n^1\right\|_{2,D_{k,1}} \le \\ &\qquad C_D\sum_{j=2+i}^{4+i}\left\|(\widehat{T_1^2}^{(2)} - z^2)^*\overline{\partial}^i|\widehat{T_1^2}|^{\frac{1}{2}}|T_1^2|^{\frac{1}{2}}f_n^1\right\|_{2,D_{k,1}} \end{aligned} \tag{3.16}$$

for $i = 0, 1, \ldots, 8$, where $P_{\mathscr{H}}$ denotes the orthogonal projection of $L^2(D_{k,1}, \mathscr{H})$ onto $A^2(D_{k,1}, \mathscr{H})$. From (3.15) and (3.16), we obtain

$$\lim_{n\to\infty}\left\|(I - P_{\mathscr{H}})\overline{\partial}^i|\widehat{T_1^2}|^{\frac{1}{2}}|T_1^2|^{\frac{1}{2}}f_n^1\right\|_{2,D_{k,1}} = 0 \tag{3.17}$$

for $i = 1, \ldots, 8$. Thus, by (3.14) and (3.17),

$$\lim_{n\to\infty}\left\|(\widehat{T_1^2}^{(2)} - z^2)P\overline{\partial}^i|\widehat{T_1^2}|^{\frac{1}{2}}|T_1^2|^{\frac{1}{2}}f_n^1\right\|_{2,D_{k,1}} = 0 \tag{3.18}$$

for $i = 1, \ldots, 8$. Since $\widehat{T_1^2}^{(2)}$ is hyponormal, it has the property (β). Hence

$$\lim_{n\to\infty}\left\|P_{\mathscr{H}}\overline{\partial}^i|\widehat{T_1^2}|^{\frac{1}{2}}|T_1^2|^{\frac{1}{2}}f_n^1\right\|_{2,D_{k,1}} = 0 \tag{3.19}$$

for $i = 1, \ldots, 8$, where $\sigma(T) \subsetneq \overline{D_{k,1}} \subsetneq D$. From (3.17) and (3.19), we get

$$\lim_{n\to\infty}\left\||\widehat{T_1^2}|^{\frac{1}{2}}|T_1^2|^{\frac{1}{2}}\overline{\partial}^i f_n^1\right\|_{2,D_{k,1}} \tag{3.20}$$

for $i = 1, \ldots, 8$. Since $\widehat{T_1^2}|T_1^2|^{\frac{1}{2}} = |T_1^2|^{\frac{1}{2}}T_1^2$, from (3.14) and (3.20) we obtain

$$\lim_{n\to\infty}\left\|z^4\overline{\partial}^i f_n^1\right\|_{2,D_{k,1}} = 0 \tag{3.21}$$

for $i = 1, \ldots, 8$. By Theorem 3.1 of [13], there exists a constant $C_{D_{k,1}}$ such that

$$\left\|(I - P_{\mathscr{H}})f_n^1\right\|_{2,D_{k,1}} \le C_{D_{k,1}}\sum_{i=4}^{8}\left\|z^4\overline{\partial}^i f_n^1\right\|_{2,D_{k,1}}, \tag{3.22}$$

where $P_{\mathscr{H}}$ denotes the orthogonal projection of $L^2(D_{k,1}, \mathscr{H})$ onto the Bergman space $A^2(D_{k,1}, \mathscr{H})$. By (3.21) and (3.22), it follows that

$$\lim_{n\to\infty}\left\|(I - P_{\mathscr{H}})f_n^1\right\|_{2,D_{k,1}} = 0. \tag{3.23}$$

Combining (3.23) with (3.11), we have

$$\lim_{n\to\infty}\left\|(I - P_{\mathscr{H}})f_n^1\right\|_{2,D_{k,1}} = \lim_{n\to\infty}\left\|(I - P_{\mathscr{K}})f_n^2\right\|_{2,D_{k,1}} = 0. \tag{3.24}$$

Set $Pf_n = \begin{pmatrix} P_{\mathscr{H}} f_n^1 \\ P_{\mathscr{K}} f_n^2 \end{pmatrix}$. Combining (3.24) with (3.4)

$$\lim_{n\to\infty} \|(T-z)Pf_n + 1\otimes h_n\|_{2,D_{k,1}} = 0.$$

Let Γ be a curve in $D_{k,1}$ surrounding $\sigma(T)$. Then

$$\lim_{n\to\infty} \left\|Pf_n(z) + (T-z)^{-1}(1\otimes h_n)(z)\right\| = 0$$

uniformly for all $z \in \Gamma$. Applying the Riesz-Dunford functional calculus, we obtain that

$$\lim_{n\to\infty} \left\| \frac{1}{2\pi i} \int_\Gamma Pf_n(z)dz + h_n \right\| = 0.$$

But by Cauchy's theorem, $\frac{1}{2\pi i}\int_\Gamma Pf_n(z)dz = 0$ and so

$$\lim_{n\to\infty} \|h_n\| = 0.$$

Hence F is injective and has closed range. □

Now we are ready to show that every k-quasi-class A operator has a scalar extension.

Theorem 3.4. *Every k-quasi-class A operator is subscalar of order $2k+12$.*

Proof. Suppose that $T \in \mathscr{B}(\mathscr{H})$ is a k-quasi-class A operator. If $\mathscr{R}(T^k)$ is dense in $\mathscr{H}$, then T is class A, and so T is subscalar of order 12 by [10]. Now assume that $\mathscr{R}(T^k)$ is not dense in $\mathscr{H}$. By [20], we have the following matrix representation of T with respect to the decomposition $\mathscr{H} = \overline{\mathscr{R}(T^k)} \oplus \ker(T^{*k})$: $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$, where T_1 belongs to class A on $\overline{\mathscr{R}(T^k)}$ and T_3 is nilpotent of order k. Let D be a bounded open disk in $\mathbb{C}$ containing $\sigma(T)$. Define an operator $F : \mathscr{H} \to H(D)$ by $Fh = \widetilde{1\otimes h}$ as in Lemma 3.3, where

$$H(D) := W^{12+2k}(D, \overline{\mathscr{R}(T^k)}) \oplus W^{12+2k}(D, \ker(T^{*k}))/\overline{(T-z)W^{12+2k}(D,\overline{\mathscr{R}(T^k)}) \oplus W^{12+2k}(D,\ker(T^{*k}))}.$$

Then F is injective and has closed range. The class of a vector f or an operator S on $H(D)$ will be denoted by $\widetilde{f}$, respectively $\widetilde{S}$. Let M be the operator of multiplication by z on $W^{12+2k}(D,\overline{\mathscr{R}(T^k)}) \oplus W^{12+2k}(D,\ker(T^{*k}))$. Then M is a scalar operator of order $12+2k$ and has a spectral distribution ϕ. Since the range of $T-z$ is invariant under M, $\widetilde{M}$ can be well-defined. Moreover, consider the spectral distribution $\phi : C_0^{2k+12}(\mathbb{C}) \to \mathscr{B}(W^{12+2k}(D,\overline{\mathscr{R}(T^k)}) \oplus W^{12+2k}(D,\ker(T^{*k})))$ defined by the following relation: for $\phi \in C_0^{2k+12}(\mathbb{C})$ and $f \in W^{12+2k}(D,\overline{\mathscr{R}(T^k)}) \oplus W^{12+2k}(D,\ker(T^{*k}))$, $\psi(\phi)f = \phi f$. Then the spectral distribution ϕ of M commutes with $T-z$, and so $\widetilde{M}$ is still a scalar operator of order $2k+12$ with $\widetilde{\psi}$ as a spectral distribution. Since

$$FTh = \widetilde{1\otimes Th} = \widetilde{z\otimes h} = \widetilde{M}(\widetilde{1\otimes h}) = \widetilde{M}Fh$$

for all $h \in \mathscr{H} = \overline{\mathscr{R}(T^k)} \oplus \ker(T^{*k})$, $FT = \widetilde{M}F$. In particular, $\mathscr{R}(F)$ is invariant under $\widetilde{M}$. Since $\mathscr{R}(F)$ is closed, it is closed invariant subspace of the scalar operator $\widetilde{M}$. Since T is similar to the restriction $\widetilde{M}|_{\mathscr{R}(F)}$ is scalar of order $2k+12$, T is a subscalar operator of order $2k+12$. □

Next we give some applications of our main theorem.

Corollary 3.5. *Let $T \in \mathscr{B}(\mathscr{H})$ be a k-quasi-class A operator. If $\sigma(T)$ has nonempty interior in $\mathbb{C}$, then T has a nontrivial invariant subspace.*

Proof. The proof follows from Theorem 3.4 and [6]. □

Recall from [4] that an operator $T \in \mathscr{B}(\mathscr{H})$ is said to be power regular if $\lim\limits_{n\to\infty} \|T^n h\|^{1/n}$ exists for every $h \in \mathscr{H}$.

Corollary 3.6. *Let $T \in \mathscr{B}(\mathscr{H})$. then the following assertions hold:*

(a) *If T is a k-quasi-class A operator, then T has the property (β), the Dunford's property (C), the property (Q), and the single-valued extension property.*

(b) *If T is a k-quasi-class A operator, then T is power regular.*

Proof. (a) Let $T \in \mathscr{B}(\mathscr{H})$ be a k-quasi-class A operator. It suffices to prove that T has the property (β). Since the property (β) is transmitted from an operator to its restrictions to closed invariant subspaces, we are reduced by Theorem 3.4 to the case of a scalar operator of order $2k+12$. Since every scalar operator has the property (β) (see [16]), T has the property (β).
(b) Let $T \in \mathscr{B}(\mathscr{H})$ be a k-quasi-class A operator. Since T is subscalar of order $2k+12$ from Theorem 3.4, it is the restriction of a scalar operator of order $2k+12$ to one of its closed invariant subspaces. Since a scalar operator is power regular and all restrictions of power regular operators to their invariant subspaces clearly remain power regular, T is power regular. ☐

Example 3.7. Denote by $\omega := \{\omega_n\}_{n\in\mathbb{N}}$ a bounded sequence of positive real numbers. The corresponding unilateral weighted right shift operator on $\ell^q(\mathbb{N})$ for some $1 \le q < \infty$ with the canonical orthogonal basis $\{e_n\}_{n=0}^{\infty}$ is defined by

$$Tx := \sum_{n=0}^{\infty} \omega_n x_n e_n \quad \text{for all} \quad x = \{x_n\}_{n\in\mathbb{N}} \in \ell^q(\mathbb{N}).$$

Then T belongs to k-quasi-class A if and only if $\omega_k \le \omega_{k+1} \le \cdots$ (see [12]). Then it follows from Theorem 3.4 that T is a subscalar of order $2k+12$.

The following lemma is useful in the sequel

Lemma 3.8. [17] *If both T and S have Bishop's property (β) and if they are quasisimilar, then $\sigma(T) = \sigma(S)$ and $\sigma_e(T) = \sigma_e(S)$ hold.*

Corollary 3.9. *Let $T, S \in \mathscr{B}(\mathscr{H})$ be k-quasi-class A operators. If T and S are quasisimilar, then $\sigma(T) = \sigma(S)$ and $\sigma_e(T) = \sigma_e(S)$.*

Proof. The proof follows immediately from Corollary 3.6 and Lemma 3.8. ☐

Example 3.10. Let $\mathscr{H} = \bigoplus\limits_{n=0}^{\infty} \mathbb{C}^2$ and define an operator T on $\mathscr{H}$ by

$$T(\cdots \oplus x_{-2} \oplus x_{-1} \oplus x_0^{(0)} \oplus x_1 \oplus \cdots) = \cdots \oplus Ax_{-2} \oplus Ax_{-1} \oplus Bx_0 \oplus Bx_1 \oplus \cdots,$$

where $A = \frac{1}{4}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then T is of k-quasi-class A. In fact,

$$\left\langle T^{*k}\left((T^*|T|^2T)^{1/(2)} - |T|^2\right)T^k x, x\right\rangle = \left\langle A^k\left((ABA)^{1/2} - A^2\right)A^k x_{-1}, x_{-1}\right\rangle$$
$$= \left(\frac{1}{16}\right)^k \left\{\left(\frac{1}{32}\right)^{1/2} - \left(\frac{1}{16}\right)\right\} \left\| \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} x_{-1} \right\|^2 \ge 0$$

for each $x \in \mathscr{H}$ (see [19]). Hence it follows from Theorem 3.4 that T is a subscalar of order $2k+12$.

Next we study several spectral properties of the scalar extension of a k-quasi-class A operator.

Theorem 3.11. *Let $T \in \mathscr{B}(\mathscr{H})$ be a k-quasi-class A operator. Let $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ with respect to the decomposition $\mathscr{H} = \overline{\mathscr{R}(T^k)} \oplus \ker(T^{*k})$. Then the following assertions hold:*

(a) *$\sigma_{T_3}(x_2) \subset \sigma_T(x_1 \oplus x_2)$ and $\sigma_{T_1}(x_1) = \sigma_T(x_1 \oplus 0)$, where $x_1 \oplus x_2 \in \mathscr{H}$.*

(b) *$\mathcal{R}_{T_1}(\mathcal{F}) \oplus 0 \subset \mathscr{H}_T(\mathcal{F})$, where $\mathcal{R}_{T_1}(\mathcal{F}) := \{y \in \overline{\mathscr{R}(T^k)} : \sigma_{T_1}(y) \subset \mathcal{F}\}$ for any set $\mathcal{F} \subset \mathbb{C}$.*

Proof. Suppose that $T \in \mathscr{B}(\mathscr{H})$ is a k-quasi-class A operator. By [20], we have the following matrix representation of T with respect to the decomposition $\mathscr{H} = \overline{\mathscr{R}(T^k)} \oplus \ker(T^{*k})$: $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$, where T_1 belongs to class A on $\overline{\mathscr{R}(T^k)}$ and T_3 is nilpotent of order k.

(a) Let $x_1 \oplus x_2 \in \mathscr{H} = \overline{\mathscr{R}(T^k)} \oplus \ker(T^{*k})$. If $\mu_0 \in \rho_T(x_1 \oplus x_2)$, then there is an $\mathscr{H}$-valued analytic function f defined on a neighborhood U of μ_0 such that $(T - \mu)f(\mu) = x_1 \oplus x_2$ for all $\mu \in U$. We can write $f = f_1 \oplus f_2$ where $f_1 \in \mathcal{O}(U, \overline{\mathscr{R}(T^k)})$ and $f_2 \in \mathcal{O}(U, \ker(T^{*k}))$. Then we have

$$\begin{pmatrix} T_1 - \mu & T_2 \\ 0 & T_3 - \mu \end{pmatrix} \begin{pmatrix} f_1(\mu) \\ f_2(\mu) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Thus $(T_3 - \mu)f_2(\mu) \equiv x_2$. Hence $\mu_0 \in \rho_{T_3}(x_2)$.

On the other hand, if $\mu_0 \in \rho_T(x_1 \oplus 0$, then there is an $\mathscr{H}$-valued analytic function g defined on a neighborhood U of μ_0 such that $(T - \mu)g(\mu) = x_1 \oplus 0$ for all $\mu \in U$. If we set $g = g_1 \oplus g_2$ where $g_1 \in \mathcal{O}(U, \overline{\mathscr{R}(T^k)})$ and $g_2 \in \mathcal{O}(U, \ker(T^{*k}))$, then we have

$$\begin{pmatrix} T_1 - \mu & T_2 \\ 0 & T_3 - \mu \end{pmatrix} \begin{pmatrix} g_1(\mu) \\ g_2(\mu) \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}.$$

Thus $(T_1 - \mu)g_1(\mu) + T_2 g_2(\mu) \equiv x_1$ and $(T_3 - \mu)g_2(\mu) \equiv 0$. Since T_3 is nilpotent of order k, it has the single-valued extension property, which implies that $g_2(\mu) \equiv 0$. Thus $(T_1 - \mu))g_1(\mu) \equiv x_1$, and so $\mu_0 \in \rho_{T_1}(x_1)$. Conversely, let $\mu_0 \in \rho_{T_1}(x_1)$. Then there exists a function $g_1 \in \mathcal{O}(U, \overline{\mathscr{R}(T^k)})$ for some neighborhood U of μ_0 such that $(T_1 - \mu)g_1(\mu) \equiv x_1$. Then

$$(T - \mu)(g_1(\mu) \oplus 0) \equiv x_1 \oplus 0.$$

Therefore $\mu_0 \in \rho_T(x_1 \oplus 0)$.

(b) If $x_1 \in \mathcal{R}_{T_1}(\mathcal{F})$, then $\sigma_{T_1}(x_1) \subset \mathcal{F}$. Since $\sigma_{T_1}(x_1) = \sigma_T(x_1 \oplus 0)$ by (a), $\sigma_T(x_1 \oplus 0) \subset \mathcal{F}$. Thus $x_1 \oplus 0 \in \mathscr{H}_T(\mathcal{F})$, and hence $\mathcal{R}_{T_1}(\mathcal{F}) \oplus 0 \subset \mathscr{H}_T(\mathcal{F})$. □

Theorem 3.12. *Let $T \in \mathscr{B}(\mathscr{H})$ be a k-quasi-class A operator. With the same notations as in Theorem 3.4, the equality $\sigma_{\widetilde{M}}(Fh) = \sigma_T(h)$ holds for every $h \in \mathscr{H}$.*

Proof. Since T is similar to $\widetilde{M}|_{\overline{\mathscr{R}(F)}} = \widetilde{M}|_{\mathscr{R}(F)}$, it is clear that

$$\sigma_T(h) = \sigma_{\widetilde{M}|_{\mathscr{R}(F)}}(Fh) \supset \sigma_{\widetilde{M}}(Fh).$$

Let $\mu_0 \in \rho_{\widetilde{M}}(Fh)$. Then there exist a neighborhood U of μ_0 and an $H(D)$-valued analytic function $\widetilde{f} : U \ni \mu \to \widetilde{f}(\mu) \in H(D)$ such that $(\widetilde{M} - \mu)\widetilde{f}(\mu) = Fh$ for $\mu \in U$, where

$$H(D) = W^{12+2k}(D, \overline{\mathscr{R}(T^k)}) \oplus W^{12+2k}(D, \ker(T^{*k}))/\overline{(T - z)W^{12+2k}(D, \overline{\mathscr{R}(T^k)}) \oplus W^{12+2k}(D, \ker(T^{*k}))}.$$

Hence there exists an analytic function

$$f : U \ni \mu \to f(\mu) \in W^{12+2k}(D, \overline{\mathscr{R}(T^k)}) \oplus W^{12+2k}(D, \ker(T^{*k}))$$

such that

$$\widetilde{f}(\mu)=\widetilde{f(\mu)}=f(\mu)+\overline{(T-z)W^{12+2k}(D,\overline{\mathscr{R}(T^k)})\oplus W^{12+2k}(D,\ker(T^{*k}))}$$

for $\mu\in U$. Since $f(\mu)\in W^{12+2k}(D,\overline{\mathscr{R}(T^k)})\oplus W^{12+2k}(D,\ker(T^{*k}))$, we set

$$f(\mu,z)=(f(\mu))(z)\in\overline{\mathscr{R}(T^k)}\oplus\ker(T^{*k})$$

for $\mu\in U$ and $z\in D$. Let $M\in\mathscr{B}(D,W^{12+2k}(D,\overline{\mathscr{R}(T^k)})\oplus W^{12+2k}(D,\ker(T^{*k})))$ be the multiplication operator by z, i.e., $Mg(z)=zg(z)$ for $g\in W^{12+2k}(D,\overline{\mathscr{R}(T^k)})\oplus W^{12+2k}(D,\ker(T^{*k}))$ and $z\in D$. Let $\xi\in U$ be fixed. Then we have

$$Mf(\xi)-\xi f(\xi)-1\otimes h\in\overline{(T-z)W^{12+2k}(D,\overline{\mathscr{R}(T^k)})\oplus W^{12+2k}(D,\ker(T^{*k}))}.$$

This means that the function

$$D\ni z\to z(f(\xi))(z)-\xi(f(\xi))(z)-h\in\overline{\mathscr{R}(T^k)}\oplus\ker(T^{*k})$$

belongs to $\overline{(T-z)W^{12+2k}(D,\overline{\mathscr{R}(T^k)})\oplus W^{12+2k}(D,\ker(T^{*k}))}$. Note that

$$\begin{aligned}&\mathcal{O}(U)\hat{\otimes}\left(W^{12+2k}(D,\overline{\mathscr{R}(T^k)})\oplus W^{12+2k}(D,\ker(T^{*k}))\right)\\&\quad=\mathcal{O}(U,W^{12+2k}(D,W^{12+2k}(D,\overline{\mathscr{R}(T^k)})\oplus W^{12+2k}(D,\ker(T^{*k})))\end{aligned}$$

by Grothendiecks theorem in [14], where $\mathcal{O}(U)$ denotes the Fréchet space of all complex-valued analytic functions on U. Since the dense range property of a Hilbert space operator is preserved by the topological tensor product with a nuclear space, there is a sequence

$$\{g_n\}\subset\mathcal{O}(U,W^{12+2k}(D,W^{12+2k}(D,\overline{\mathscr{R}(T^k)})\oplus W^{12+2k}(D,\ker(T^{*k})))$$

such that

$$\lim_{n\to\infty}(h-(z-\xi)f(\xi,z)-(T-z)g_n(\xi,z))=0 \tag{3.25}$$

with respect to the Fréhet topology of the space

$$\mathcal{O}(U,W^{12+2k}(D,W^{12+2k}(D,\overline{\mathscr{R}(T^k)})\oplus W^{12+2k}(D,\ker(T^{*k}))),$$

where $g_n(\xi):D\to\overline{\mathscr{R}(T^k)}\oplus\ker(T^{*k})$ for $\xi\in U$ and $g_n(\xi,z):=(g_n(\xi))(z)$ for $z\in D$. Let U_0 be any open neighborhood of μ_0 which is relatively compact in U, and let

$$\begin{aligned}\mathfrak{m}:\ &\mathcal{O}(U)\hat{\otimes}\left(W^{12+2k}(D,\overline{\mathscr{R}(T^k)})\oplus W^{12+2k}(D,\ker(T^{*k}))\right)\\&\to W^{12+2k}(U_0,\overline{\mathscr{R}(T^k)})\oplus W^{12+2k}(U_0,\ker(T^{*k}))\end{aligned}$$

denote the unique continuous linear extension of the map $\phi\otimes\psi\to(\phi.\psi)|_{U_0}$. Then we get that

$$\mathfrak{m}(h-(z-\xi)f(\xi,z)-(T-z)g_n(\xi,z)=h-(T-z)f_n(z), \tag{3.26}$$

where $f_n(z):=g_n(z,z)$ for $z\in U_0$. Hence from (3.25) and (3.26) we obtain that

$$\lim_{n\to\infty}\|h-(T-z)f_n\|_{W^{12+2k}(U_0,\overline{\mathscr{R}(T^k)})\oplus W^{12+2k}(U_0,\ker(T^{*k}))}=0.$$

From some applications of the proof in Lemma 3.3, we have

$$\lim_{n\to\infty}\|(I-P)f_n\|_{2,U_1}=0,$$

where U_1 is a neighborhood of μ_0 which is relatively compact in U_0, and so

$$\lim_{n\to\infty}\|h-(T-z)Pf_n\|_{2,U_1}=0.$$

This implies

$$h\in\overline{(T-z)\mathcal{O}(U_2,\overline{\mathscr{R}(T^k)})\oplus\mathcal{O}(U_2,\ker(T^{*k}))},$$

where U_2 is a neighborhood of μ_0 which is relatively compact in U_1. Since T has the property (β) from Corollary 3.6, the operator $T-z$ has closed range on $\mathcal{O}(U_2, \overline{\mathscr{R}(T^k)}) \oplus \mathcal{O}(U_2, \ker(T^{*k}))$. Thus it follows that

$$h \in (T-z)\mathcal{O}(U_2, \overline{\mathscr{R}(T^k)}) \oplus \mathcal{O}(U_2, \ker(T^{*k})),$$

that is, $\mu_0 \in \rho_T(h)$. □

Corollary 3.13. *Let $T \in \mathscr{B}(\mathscr{H})$ be a k-quasiclass A operator. With the same notations as in Theorem 3.4, $\sigma(T) = \sigma(\widetilde{M})$. In particular, if $\sigma(\widetilde{M}) = \{0\}$, then T is nilpotent.*

Proof. Since $\sigma_T(h) \subset \sigma_{\widetilde{M}}(Fh)$ for all $h \in \mathscr{H}$ by Theorem 3.12, $\sigma_T(h) \subset \sigma(\widetilde{M})$ for all $h \in \mathscr{H}$. Hence $\bigcup\{\sigma_T(h) : h \in \mathscr{H}\}$. Since T has the single-valued extension property by Corollary 3.6, $\sigma(T) = \bigcup\{\sigma_T(h) : h \in \mathscr{H}\}$. Conversely, note that if $U \subset \mathbb{C}$ is any open disk containing $\sigma(T)$ and M is the multiplication operator by z on $W^{12+2k}(U, \overline{\mathscr{R}(T^k)}) \oplus W^{12+2k}(U, \ker(T^{*k}))$, then $\sigma(\widetilde{M}) \subset \sigma(M) \subset \overline{U}$ holds. From this property, if $\mu \in \rho(T)$, then we can choose an open disk D so that $\widetilde{M} - \mu$ is invertible. Since this algebraic property is independent of the choice of D, we get $\sigma(\widetilde{M}) \subset \sigma(T)$.

If $\sigma(\widetilde{M}) = \{0\}$, then $\sigma(T) = \{0\}$. Since $\sigma(T) = \sigma(T_1) \cup \{0\}$ from [20], $\sigma(T_1) = \{0\}$. Since T_1 belongs to class A, it holds that $\|T_1\| = \sup\{|\mu| : \mu \in \sigma(T_1)\} = 0$, and so $T_1 = 0$. Hence T is nilpotent. □

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Department of Mathematics, Faculty of Science P. O. Box(7), Mu'tah university, Al-Karak-Jordan.
E-mail address: malik_okasha@yahoo.com

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