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OPERATORS PRESERVING ORTHOGONALITY ON HILBERT $\boldsymbol{K}(H)$-MODULES

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Abstract. In this paper, we study the class of orthogonality preserving operators on a Hilbert *K(H)*-module W and show that an operator T on W is orthogonality preserving if and only if it is orthogonality preserving on a special dense submodule of W. Then we apply this fact to show that an orthogonality preserving operator T is normal if and only if T^* is orthogonality preserving.

1. Introduction and preliminaries

The concept of orthogonality in an inner product space $(H, \langle\cdot,\cdot\rangle)$, as expected, is defined by its inner product. Actually, $x \perp y$ if and only if $\langle x, y\rangle = 0$, where x, y belong to H. Extending this definition to normed linear spaces has been studied by many mathematicians since 1934. These studies lead to several versions of orthogonality such as:
Roberts orthogonality(1934): $\|x - \lambda y\| = \|x + \lambda y\|$, for every $\lambda \in \mathbb{R}$;
Birkhoff orthogonality(1935): $\|x\| \leq \|x + \lambda y\|$, for every $\lambda \in \mathbb{R}$;
Isosceles orthogonality(1945): $\|x - y\| = \|x + y\|$ (due to James);
Singer orthogonality(1957): $x = 0$ or $y = 0$ or $\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\| = \|\frac{x}{\|x\|} - \frac{y}{\|y\|}\|$.

Also, there are some other versions of orthogonality. For more details, see [15] and [1, 2] where Sikorska, Alonso, and Benitez listed and compared these orthogonalities and investigated algebraic properties such as being homogenous, additive and symmetric.

The usual definition of orthogonality in a complex normed linear space $(X, \|.\|)$ is Birkhoff-James orthogonality which says that x is orthogonal to y in X (and in this case, we write $x \perp y$), if for each $\lambda \in \mathbb{C}$, $\|x\| \leq \|x + \lambda y\|$.

An operator $T : H \to H$ is said to be orthogonality preserving (OP in short), if $Tx \perp Ty$ whenever $x \perp y$, and T is called strongly orthogonality preserving (SOP in short) when $Tx \perp Ty \Leftrightarrow x \perp y$. Also, we say that T is a similarity if there exists a positive scalar γ such that $\|Tx\| = \gamma\|x\|$ for all $x \in H$. Indeed, a similarity is a positive scalar multiple of an isometry.

In general, an OP operator need not be linear or continuous (see [8, Examples 1.1 and 1.2]). Under the assumption of linearity, Koldobsky [10] for an OP operator T on a real normed space, and then Chmieliński [8] for an OP mapping S between (real or complex) inner product spaces have proved that T and S are not only continuous, but also similarities. In the setting of complex normed spaces, Blanco and Turnšek [7] have generalized the theorem with the orthogonality in the sense of B.J. Actually, they have shown that if $T : X \longrightarrow Y$ is a linear OP mapping between normed linear spaces, then T is a similarity. Later, using the fact that a C^*-algebra A is a subalgebra of $B(H)$ for some Hilbert space H, Ilišević and Turnšek [9] have proved that every A-linear orthogonality preserving operator on a Hilbert A-module is a similarity when A contains the C^*-algebra $K(H)$ of all compact operators on a Hilbert space H.

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In a more general aspect of orthogonality preserving property, considering the standard definition of the angle between two elements of a real inner product space, Sal Moslehian, Zamani, and Frank have obtained some interesting results about angle preserving mappings (see [16]).

In this paper, we study orthogonality preserving operators acting on Hilbert $K(H)$-modules and prove that an operator T on a Hilbert $K(H)$-module W is OP if and only if the restriction of T to $W_{\mathcal{HS}}$ (a dense submodule of W) is an OP operator. Applying this fact, we confirm the similarity of an OP operator on a Hilbert A-module (see [9, Proposition 2.3]) when $A = K(H)$ with a completely different approach. Finally, we prove that $T \in B_K(W)$ is a normal operator if and only if its adjoint T^* is OP, where $B_K(W)$ denotes all bounded $K(H)$-linear operators on W.

Recall that, a left A-module W is called an inner product C^*-module when there exists an A-valued inner product $\langle\cdot,\cdot\rangle : W \times W \longrightarrow A$ satisfying the following conditions:

(i) $\langle x,x\rangle \geq 0$, and $\langle x,x\rangle = 0$ if and only if $x = 0$,

(ii) $\langle \lambda x + \mu y, z\rangle = \lambda\langle x,z\rangle + \mu\langle y,z\rangle$,

(iii) $\langle ax,y\rangle = a\langle x,y\rangle$,

(iv) $\langle x,y\rangle^* = \langle y,x\rangle$

for all $x,y,z \in W,\ a \in A,\ \lambda,\mu \in \mathbb{C}$.

The norm defined by $\|x\| = \|\langle x,x\rangle\|^{\frac{1}{2}}$ makes W into a normed space and W is called a Hilbert C^*-module when $(W, \|.\|)$ is complete.

An H^*-algebra $\mathcal{A}$ is a Banach $*$-algebra whose underlying space is a Hilbert space. For instance, $\mathcal{HS}$ the space of all Hilbert-Schmidt operators on a Hilbert space H is an H^*-algebra. There exists a continuous linear form tr on $\tau(\mathcal{A}) := \{ab : a,b \in \mathcal{A}\}$ satisfying $tr(ab) = tr(ba) = \langle a, b^*\rangle$ for all $a,b \in \mathcal{A}$. An inner product H^*-module is an $\mathcal{A}$-module W equipped with the $\tau(\mathcal{A})$-valued inner product $[\cdot,\cdot] : W \times W \longrightarrow \tau(\mathcal{A})$ such that:

(i) $[\lambda x + y, z] = \lambda[x,z] + [y,z]$;

(ii) $[x,y]^* = [y,x]$;

(iii) $[x,x] = aa^*,\ for\ some\ a \in \mathcal{A}$;

(iv) $(W, (\cdot,\cdot))$ is an inner product space where $(x,y) := tr([x,y])$.

W is a Hilbert H^*-module when it is complete with the norm $\|x\| = (x,x)^{\frac{1}{2}}$.

Now, one can extract a Hilbert H^*-module from every Hilbert $K(H)$-module W as follows:

Let $W^0_{\mathcal{HS}}$ be the linear span of the set $\mathcal{HS}W$. The submodule $W^0_{\mathcal{HS}}$ of W can be made into a pre-Hilbert H^*-module over the H^*-algebra $\mathcal{HS}$ with the inner product $(x,y) = tr(\langle x,y\rangle)$. Let us denote by $\|.\|_{hs}$ the resulting norm $\|x\|_{hs} = \sqrt{tr\langle x,x\rangle}$ for every $x \in W^0_{\mathcal{HS}}$. The completion of $W^0_{\mathcal{HS}}$ in the norm $\|.\|_{hs}$ is a Hilbert H^*-module denoted by $W_{\mathcal{HS}}$ which is a dense submodule in W with respect to the original norm (for more details see [5]).

2. Orthogonality preserving maps on Hilbert $K(H)$-modules

Theorem 2.2 in [3] shows that every OP operator $T \in B(H)$ is normal if and only if T^* is OP. We are going to express this theorem for bounded $K(H)$-linear operators on Hilbert $K(H)$-modules. To achieve it, we prove Theorem 2.4 as the main of this section which plays a key role in the proof of our aim. At first, we recall some useful lemmas.

Lemma 2.1. *Let W be a Hilbert* K(H)*-module. Then, the new norm $\|.\|_{hs}$ dominates the original one i.e., $\|x\| \leq \|x\|_{hs}$ for all $x \in W_{\mathcal{HS}}$.*

Proof. See [4]. □

Note that, a Hilbert $K(H)$-module has two norms $\|.\|$ and $\|.\|_{hs}$. In fact, $\|.\|$ comes from the $K(H)$-valued inner product $\langle\cdot,\cdot\rangle$ and $\|.\|_{hs}$ from the $\mathbb{C}$-valued inner product

$(\cdot,\cdot)$. Actually, $\|x\| = \|\langle x,x\rangle\|^{\frac{1}{2}}$ and $\|x\|_{hs} = (x,x)^{\frac{1}{2}}$. It is known that every H^*-algebra has a minimal projection e, that means e is self-adjoint, $e^2 = e$ and $eAe = \mathbb{C}e$. Moreover:

Lemma 2.2. *Let W be a Hilbert $\mathrm{K}(H)$-module and e be a minimal projection in $\mathrm{K}(H)$. Then*

(i) $(W_{\mathcal{HS}})_e := \{ew : w \in W_{\mathcal{HS}}\}$ is a closed subspace of the Hilbert space $(W_{\mathcal{HS}}, (\cdot,\cdot))$ and also a dense submodule of $W_{\mathcal{HS}}$ i.e., $\overline{(W_{\mathcal{HS}})_e} = W_{\mathcal{HS}}$ with respect to the norm $\|.\|_{hs}$.

(ii) $\|ew\| = \|ew\|_{hs}$ for all $w \in W_{\mathcal{HS}}$.

Proof. See [5, Lemma 2] and [4, Proposition 2]. □

Remember that from the structure of a Hilbert H^*-module, we have a $\tau(\mathcal{A})$-valued function $[\cdot,\cdot]$ and then a $\mathbb{C}$-valued inner product $(\cdot,\cdot)$. There is a good relation between these two maps among the minimal projections which plays an essential role in the proof of the main theorem of this section. The next lemma expresses this relation.

Lemma 2.3. *Let W be a Hilbert $\mathrm{K}(H)$-module and e be a minimal projection in $\mathrm{K}(H)$. Then, $[x,y] = \frac{1}{\|e\|^2}(x,y)e$ for all $x,y \in (W_{\mathcal{HS}})_e$.*

Proof. See [5, Lemma 3]. □

According to Lemma 2.1, boundedness in the norm $\|.\|_{hs}$ implies boundedness in the original norm $\|.\|$.

Theorem 2.4. *Let W be a Hilbert $\mathrm{K}(H)$-module, $T \in B_{\mathrm{K}}(W)$ and $\widehat{T}$ be the restriction of T to $W_{\mathcal{HS}}$. Then, the following are equivalent:*

(i) $T : (W, \langle\cdot,\cdot\rangle) \longrightarrow (W, \langle\cdot,\cdot\rangle)$ is SOP.

(ii) $\widehat{T} : (W_{\mathcal{HS}}, (\cdot,\cdot)) \longrightarrow (W_{\mathcal{HS}}, (\cdot,\cdot))$ is SOP.

Proof. $(i) \Rightarrow (ii)$: Let $x,y \in E := W_{\mathcal{HS}}$ and $(x,y) = 0$. We have to show that $(\widehat{T}x, \widehat{T}y) = 0$. Since $\overline{E_e}^{\|.\|_{hs}} = E$, there are sequences (x_n) and (y_n) in E_e such that $x_n \longrightarrow x$ and $y_n \longrightarrow y$ in the norm $\|.\|_{hs}$. Therefore $(x_n,y_n) \longrightarrow (x,y) = 0$. Using Lemma 2.3, $\langle x_n,y_n\rangle = \frac{1}{\|e\|^2}(x_n,y_n)e$, where e is a minimal projection of W. So, $\langle x_n,y_n\rangle \longrightarrow 0$. At the same time, $\langle x_n,y_n\rangle \longrightarrow \langle x,y\rangle$. Thus $\langle x,y\rangle = 0$ and so $\langle Tx,Ty\rangle = 0$. From the continuity of T and by Lemma 2.1, we can write

$$\langle Tx_n, Ty_n\rangle \longrightarrow \langle Tx, Ty\rangle = 0.$$

Hence, $(Tx_n,Ty_n) \longrightarrow 0$. But we have

$$(Tx_n, Ty_n) = (\widehat{T}x_n, \widehat{T}y_n) \longrightarrow (\widehat{T}x, \widehat{T}y),$$

which means that $(\widehat{T}x, \widehat{T}y) = 0$.

Conversely, $(ii) \Rightarrow (i)$: Let $x,y \in W$ and $\langle x,y\rangle = 0$. We must show that $\langle Tx,Ty\rangle = 0$. Since $\overline{E} = W$ in the original norm $\|.\|$, there are sequences $(x_n),(y_n)$ in E such that $x_n \longrightarrow x$ and $y_n \longrightarrow y$. But $\overline{E_e} = E$ in the norm $\|.\|_{hs}$, and $x_n,y_n \in E$ $(\forall n \in \mathbb{N})$. Therefore, $\forall n \in \mathbb{N}$, $\exists (x_n^m) \subseteq E_e, \exists (y_n^m) \subseteq E_e$ such that $(x_n^m) \longrightarrow x_n$ and $(y_n^m) \longrightarrow y_n$, when $m \longrightarrow +\infty$.

Actually, in the norm $\|.\|$:

$$x_1, x_2, x_3, \ldots, x_n, \ldots \longrightarrow x \quad \textit{and} \quad y_1, y_2, y_3, \ldots, y_n, \ldots \longrightarrow y$$

and in the norm $\|.\|_{hs}$:

$$\begin{array}{ccc} x_1^1, x_1^2, x_1^3, \ldots, x_1^m, \ldots \longrightarrow x_1 & , & y_1^1, y_1^2, y_1^3, \ldots, y_1^m, \ldots \longrightarrow y_1 \\ x_2^1, x_2^2, x_2^3, \ldots, x_2^m, \ldots \longrightarrow x_2 & , & y_2^1, y_2^2, y_2^3, \ldots, y_2^m, \ldots \longrightarrow y_2 \\ & \cdots & \\ x_n^1, x_n^2, x_n^3, \ldots, x_n^m, \ldots \longrightarrow x_n & , & y_n^1, y_n^2, y_n^3 \ldots, y_n^m, \ldots \longrightarrow y_n. \end{array}$$

But $\widehat{T}$ is *SOP* on $(W_{\mathcal{HS}}, \|.\|_{hs})$ equivalently on $(W_{\mathcal{HS}}, (\cdot,\cdot))$, and so it is a scalar multiple of an isometry. So, there exists $\gamma > 0$ such that

$$\langle x_n^m, y_n^m\rangle = \frac{1}{\|e\|^2}(x_n^m, y_n^m)e = \frac{\gamma^2}{\|e\|^2}(\widehat{T}x_n^m, \widehat{T}y_n^m)e = \gamma^2\langle Tx_n^m, Ty_n^m\rangle.$$

On the other hand, $\lim_{m\to+\infty}\langle x_n^m, y_n^m\rangle = \langle x_n, y_n\rangle$ in $(W, \|.\|)$ because $x_n^m \longrightarrow x_n$ in the norm $\|.\|_{hs}$ and $\|.\| \leq \|.\|_{hs}$. Therefore,

$$\lim_{n\to+\infty}(\lim_{m\to+\infty}\langle x_n^m, y_n^m\rangle) = \lim_{n\to+\infty}\langle x_n, y_n\rangle = \langle x, y\rangle = 0.$$

Hence, $\lim_{n\to+\infty}\gamma^2\langle Tx_n, Ty_n\rangle = 0$. Since $\gamma \neq 0$, we have $\lim_{n\to+\infty}\langle Tx_n, Ty_n\rangle = 0$. Thus $\langle Tx, Ty\rangle = 0$. In the same way, $\langle Tx, Ty\rangle = 0$ implies $\langle x, y\rangle = 0$. □

In [9], it has been proved that if A is a C^*-algebra such that $A \supseteq K(H)$, then every A-linear orthogonality preserving operator on a Hilbert A-Module is a scalar multiple of an isometry. Now, as a result of Theorem 2.4, we confirm this fact for SOP operators when $A = K(H)$, with a completely different approach.

Corollary 2.5. *Let W be a Hilbert $\mathrm{K}(H)$-module. Then, for a bounded $\mathrm{K}(H)$-linear operator T on W, the following are equivalent:*

(i) $T : (W, \langle\cdot,\cdot\rangle) \longrightarrow (W, \langle\cdot,\cdot\rangle)$ is SOP.

(ii) $\|Tx\| = \gamma\|x\|$, for some $\gamma > 0$ and every $x \in W$.

Proof. Every scalar multiple of an isometry is obviously SOP. So, $(ii) \Rightarrow (i)$ is clear. Conversely, let T be a bounded $K(H)$-linear SOP operator on W and let (e_α) be a bounded approximate identity of minimal projections in $K(H)$. By Theorem 2.4, $\widehat{T}$ is SOP on $(W_{\mathcal{HS}}, (\cdot,\cdot))$ and since $(\cdot,\cdot$ is a $\mathbb{C}$-valued inner product, we have $\|\widehat{T}x\|_{hs} = \gamma\|x\|_{hs}$, for some $\gamma > 0$ and every $x \in W_{\mathcal{HS}}$.

Now, let $w \in W$. Since $\overline{W_{\mathcal{HS}}} = W$ in the original norm, there exists $(x_n) \subseteq W_{\mathcal{HS}}$ such that $x_n \longrightarrow w$. Since $\|e_\alpha x\| = \|e_\alpha x\|_{hs}$ for all $x \in W_{\mathcal{HS}}$ we can write

$$\begin{aligned} T(e_\alpha w)\| &= \|T(e_\alpha(\lim x_n))\| = \lim \|T(e_\alpha x_n) \\ &= \lim \|e_\alpha T(x_n)\| = \lim \|e_\alpha T(x_n)\|_{hs} \\ &= \|e_\alpha T(\lim x_n)\|_{hs} = \|e_\alpha T(w)\|_{hs}. \end{aligned}$$

Thus,

$$\|T(e_\alpha w)\| = \|e_\alpha T(w)\| = \|e_\alpha T(w)\|_{hs} = \|T(e_\alpha w)\|_{hs} = \gamma\|e_\alpha w\|_{hs} = \gamma\|e_\alpha w\|.$$

Therefore,

$$\lim_{\alpha\to+\infty}\|T(e_\alpha w)\| = \lim_{\alpha\to+\infty}\gamma\|e_\alpha w\|,$$

which implies that $\|Tw\| = \gamma\|w\|$. □

Unlike in Hilbert spaces, it is not necessary that every bounded A-linear operator has an adjoint on a Hilbert A-module. But, in the case $A = K(H)$, we will see that every bounded $K(H)$-linear operator on a Hilbert $K(H)$-module is adjointable. Then, applying this fact we prove the main theorem (Theorem 2.9) of this section. The proof of Proposition 2.8, is a mix of the following lemma and the theorem proved by Bakić and Saworotnow in [5] and [12], respectively. In what follows, we mean by $B_{\mathcal{HS}}(W_{\mathcal{HS}})$ the set of all bounded $\mathcal{HS}$-linear operators on $W_{\mathcal{HS}}$.

Lemma 2.6. *Let W be a Hilbert $\mathrm{K}(H)$-module. For each $T \in B_{\mathcal{HS}}(W_{\mathcal{HS}})$, there exists a unique operator $\widetilde{T} \in B_{\mathrm{K}}(W)$ which extends T.*

Proof. See [5, Lemma 3]. □

Theorem 2.7. *Let $T \in B_{\mathcal{HS}}(W_{\mathcal{HS}})$. Then T is adjointable and $T^* \in B_{\mathcal{HS}}(W_{\mathcal{HS}})$.*

Proof. See [12, Theorem 4]. □

Proposition 2.8. *Every bounded* $\mathrm{K}(H)$*-linear operator on* W *is adjointable.*

Proof. Let $T \in B_K(W)$ and e be a minimal projection in $K(H)$. We claim that T is continuous on $(W_{\mathcal{HS}}, \|.\|_{hs})$. The map $\psi : B_{\mathcal{HS}}(W_{\mathcal{HS}}) \longrightarrow B((W_{\mathcal{HS}})_e)$ by $\psi(R) = \widetilde{R}$ is an isomorphism of C^*-algebras (see [4, Theorem 1]). Thus, it is enough to show that $\widetilde{\widehat{T}}$ is continuous. Let $(ex_n) \subseteq (W_{\mathcal{HS}})_e$ and $ex_n \longrightarrow 0$ in the norm $\|.\|_{hs}$. Therefore, by Lemma 2.1, $ex_n \longrightarrow 0$ in the norm $\|.\|$. Since T is continuous on $(W, \|.\|)$, we have $\|T(ex_n)\| \longrightarrow 0$. Thus $\|eT(x_n)\| \longrightarrow 0$. Note that $x_n \in W_{\mathcal{HS}}$ implies $eT(x_n) \in (W_{HS})_e$. We have, by Lemma 2.2,

$$\|\widetilde{\widehat{T}}(ex_n)\|_{hs} = \|T(ex_n)\|_{hs} = \|eT(x_n)\|_{hs} = \|eT(x_n)\| \longrightarrow 0,$$

that means $\widehat{T}$ is continuous and therefore there exists $(\widehat{T})^* \in B_{\mathcal{HS}}(W_{\mathcal{HS}})$. Applying Theorem 2.7, there is $\widetilde{(\widehat{T})^*} \in B_K(W)$ which extends $(\widehat{T})^*$. Now, define $T^* := \widetilde{(\widehat{T})^*}$. It is straightforward to check that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle.$$

This completes the proof. □

Now, we are ready to characterize the bounded orthogonality preserving $K(H)$-linear operators whose adjoint is also orthogonality preserving.

Theorem 2.9. *Let* $T \in B_{\mathrm{K}}(W)$ *and* T *be OP. Then,* T^* *is OP if and only if* T *is a normal operator.*

Proof. Let $T, T^* \in B_K(W)$ and both are OP.

By Theorem 2.4, $\widehat{T}, \widehat{T^*} : (W_{\mathcal{HS}}, (\cdot,\cdot)) \longrightarrow (W_{\mathcal{HS}}, (\cdot,\cdot))$ are OP. $(W_{\mathcal{HS}}, (\cdot,\cdot))$ is a Hilbert space, So $\widehat{T}$ is normal in $B(W_{\mathcal{HS}})$ i.e. $\widehat{T}(\widehat{T})^* = (\widehat{T})^*\widehat{T}$. But $(\widehat{T})^* = \widehat{T^*}$ on $W_{\mathcal{HS}}$, because for every $x \in W_{\mathcal{HS}}$: $\widehat{T^*}(x) = T^*(x) = (\widehat{T})^*(x)$. The first equality comes from the fact that $\widehat{T^*}$ is a restriction of T^*. For the second equality, note that $T^* = \widetilde{(\widehat{T})^*}$, where $\widetilde{(\widehat{T})^*}|_{W_{\mathcal{HS}}} = (\widehat{T})^*$. So, $\widehat{T}\widehat{T^*} = \widehat{T^*}\widehat{T}$. But $\widehat{T}\widehat{T^*} = \widehat{TT^*}$ on $W_{\mathcal{HS}}$, because

$$\widehat{T}\widehat{T^*}(x) = \widehat{T}(\widehat{T^*}x) = \widehat{T}(T^*x) = TT^*(x) = \widehat{TT^*}(x).$$

Therefore, $\widehat{TT^*} = \widehat{T^*T}$, which implies that T^*T is the extension of $\widehat{TT^*}$ and $\widehat{T^*T}$. It is clear that TT^* is an extension of $\widehat{TT^*}$. According to Theorem 2.7, by the uniqueness of such extension, we have $TT^* = T^*T$.

Conversely, let T be a normal OP operator in $B_K(W)$, $x, y \in W$ and $\langle x, y\rangle = 0$. We have

$$\langle T^*x, T^*y\rangle = \langle TT^*x, y\rangle = \langle T^*Tx, y\rangle = \langle Tx, Ty\rangle = 0,$$

which means that T^* is also OP. □

As we mentioned, Ilišević and Turnšek [9] have proved that every OP operator T on a Hilbert C^*-module W is a similarity, where the C^*-algebra contains $K(H)$. More exactly, there is a positive scalar γ such that $\|Tx\| = \gamma\|x\|$ for all $x \in W$. We are going to determine the value of γ exactly. Note that, If $T \in B_K(W)$ and T is an OP operator, then T^*T and $|T|$ are also OP. Because if $x, y \in W$ and $\langle x, y\rangle = 0$, then $\langle x, T^*Ty\rangle = \langle Tx, Ty\rangle = 0$ and so $\langle T^*Tx, T^*Ty\rangle = \langle Tx, TT^*Ty\rangle = 0$. Therefore, T^*T is OP.

Again, it follows from $\langle x, T^*Ty\rangle = 0$ that

$$\langle |T|x, |T|y\rangle = \langle |T|^*x, |T|y\rangle = \langle x, |T|^2y\rangle = \langle x, T^*Ty\rangle = 0.$$

Which means $|T|$ is also OP. Moreover, T and $|T|$ have the same γ as two similarities.

Now, it is easy to calculate the positive scalar γ appearing for a $K(H)$-linear OP operator T in $B_K(W)$.

Corollary 2.10. *Let* $T \in B_{\mathrm{K}}(W)$ *and* T,T^* *are both OP. Then* $\gamma = r(T)$.

Proof. From Theorem 2.9, T is normal. Since $B_K(W)$ is a C^*-algebra, $\|T\| = r(T)$. Now we have

$$r(T) = \|T\| = \sup\{\|Tx\| : \|x\| \leq 1\} = \sup\{\gamma\|x\| : \|x\| \leq 1\} = \gamma.$$

□

In a more general aspect, for an arbitrary OP operator $T \in B_K(W)$, we can obtain γ as $r(|T|)$:

Corollary 2.11. *Let $T \in B_{\mathrm{K}}(W)$ and T be OP. Then $\gamma = r(|T|)$.*

Proof. The operator $|T|$ is OP, because T is OP. Note that $\||T|\| = \|T\|$. Now applying Corollary 2.10, we have

$$r(|T|) = \||T|\| = \gamma = \|T\|.$$

□

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