Methods of Functional Analysis and Topology
Vol. 25 (2019), no. 3, pp. 197–204

SPACE OF CONFIGURATIONS AND THE SPECIAL MEASURES ON IT

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Abstract. The article is devoted to an exact account of initial results about configurations and measures on them, starting from the concept of a unique topologization of the space of all configurations, including both finite and infinite cases (not as it is made in the classical works).

1. Introduction

Configurations are the sets of points of a locally compact (but not compact) space X and which are represented by sequences $\gamma = [x_1, x_2, \dots]$ of different points x_n of the space X and having the following property: in every compact set Λ of the space X, the number of points from γ is finite. Configurations are used in many domains of mathematics, in particular, in mathematical physics, functional analysis, probability theory, etc. (see, for example, [9, 5, 6, 7, 10, 4, 8]).

By tradition, it is usual to consider the space $\Gamma_0(X)$ of finite configurations (i.e. finite sets γ) and the space $\Gamma(X)$ of all configurations with fixed X. In these spaces, different topologies are introduced, different σ-algebras of Borel sets in $\Gamma_0(X)$ and $\Gamma(X) \setminus \Gamma_0(X)$ are constructed, and different measures are investigated, etc.

The author, in a joint article with V. A. Tesko [2], which was devoted to the moment problem related to Bogoliubov functionals, found that, in some questions (for example, in some questions of spectral theory), it is necessary and convenient to introduce a unique topology into $\Gamma(X)$, namely the topology of weak convergence of linear functionals on the space of finite continuous functions on X, and consider the corresponding relative topology on $\Gamma_0(X)$.

The corresponding facts were stated in Section 2 of the article [2]. However the traditional "classical" point of view, which essentially distinguishes the spaces $\Gamma_0(X)$ and $\Gamma(X)$, was dominant in the paper [2]. It did not permit to give simple and clear account of the results obtained by the authors. These results were correct, but their presentation was somewhat cumbersome.

The article [1], devoted to a study and generalization of the classical Poisson measure, was written in a more clear way, since the author by that time had a more precise understanding of the differences between the two cases.

In this short article, we propose an account of initial results about configurations and measures on them, starting from the proposed in [2] concept of introducing topology only for the whole space $\Gamma(X)$. This article, from my point of view, finalizes the discussion of some results of the work [2].

2010 *Mathematics Subject Classification.* 44A60, 47A57.
Key words and phrases. Space of configurations and its topologization, measure.

2. Topologization of the space of configurations and corresponding Borel σ-algebras

1. Definition of configuration. Let X be a connected non-compact Riemannian manifold, which is a union of its compact balls with some fixed center, $\rho(x, y)$, $x, y \in X$, is a corresponding metric.

Denote by $\Gamma(X)$ all subsets γ of X, consisting of distinct points and such that

$$\Gamma(X) = \Big\{\gamma \subset X \,\Big|\, |\gamma \cap \Lambda| < \infty \text{ for every compact } \Lambda \subset X\Big\}, \tag{2.1}$$

where $|\cdot|$ means the cardinality of the set.

Such a subset $\gamma \in \Gamma(X)$ is called a configuration. So, each $\gamma \in \Gamma(X)$ consists of different points; the number of points from $\gamma \cap \Lambda$, where Λ is an arbitrary compact subset of X, is finite.

Understandably, two configurations γ_1, γ_2 by definition are different, if the sets γ_1 and γ_2 are different sets of points. Thus, if the sets $\gamma_1 \subset \gamma_2$ and $\gamma_2 \setminus \gamma_1$ are not empty, then we have two different configurations. let us underline that configurations γ_1 and γ_2 are different, if the sets γ_1 and γ_2 are different. So, different configurations may have a common finite or a countable subset of points from X.

For configurations γ_1, γ_2, regarded as subsets of the space X, it is possible to introduce the usual operations, $\gamma_1 \cup \gamma_2$, $\gamma_1 \cap \gamma_2$, $\gamma_1 \setminus \gamma_2$. The same can be done for more than two subsets. A countable or finite union γ of different configurations is also a configuration iff this union γ consists of different points and set $\gamma \cap \Lambda$ is finite (Λ is an arbitrary compact set).

Of course, the simplest configuration consists of a single point x from X, therefore $X \subset \Gamma(X)$.

Denote by $\Gamma_0(X)$ the set of all configurations consisting of a finite number of points (finite configurations) and by $\Gamma^{(n)}(X)$, $n \in \mathbb{N} = \{1, 2, \dots\}$, the set of all configurations from $\Gamma_0(X)$ consisting of n points. Obviously, we have

$$\Gamma_0(X) = \bigsqcup_{n=0}^{\infty} \Gamma^{(n)}(X) \subset \Gamma(X), \qquad \Gamma^{(0)}(X) := \varnothing. \tag{2.2}$$

Let $Y \subset X$ be some subspace of X endowed with relative topology. Then $\Gamma(Y) \subset \Gamma(X)$. In particular, $Y = \Lambda$ can be some compact subset of X. Then $\Gamma(\Lambda)$ consists only of finite configurations. Therefore we have, instead of (2.2), the equality

$$\Gamma(\Lambda) = \Gamma_0(\Lambda) = \bigsqcup_{n=0}^{\infty} \Gamma^{(n)}(\Lambda), \qquad \Gamma^{(0)}(\Lambda) = \varnothing. \tag{2.3}$$

2. Introducing topology into the set $\Gamma(X)$. At first, denote by $C_{fin}(X)$ the linear set of real-valued continuous functions $X \ni x \mapsto f(x) \in \mathbb{R}$ which are finite, i. e. are equal to zero, if x is outside of some compact subset of X. This space with uniform convergence on compact sets is a linear topological space.

Recall that uniformly finite convergence means the following: $C_{fin}(X) \ni f_m \to f \in C_{fin}(X)$, $m \to \infty$, where $f_m \to f$ uniformly on X and $f_m(x) = 0$ for $x \in X \setminus \Lambda$, where Λ is some compact set.

Let μ be a fixed locally finite non-negative measure on the space X, defined on σ-algebra $\mathcal{B}(X)$ of Borel subsets α of X.

So, the integral

$$l_\mu(f) := \int_X f(x)\, d\mu(x) \tag{2.4}$$

exists for every $f \in C_{fin}(X)$ and is a linear continuous functional $l_\mu \in (C_{fin}(X))' =: C'_{fin}(X)$.

We interpret the δ-function δ_{x_0} in a point $x_0 \in X$ as the corresponding measure $\mu_{x_0} \in \mathcal{B}(X)$ and by $l_{\mu_{x_0}}$ the corresponding functional. So, in our case the equality (2.4) means that

$$l_{\mu_{x_0}}(f) = f(x_0) = \delta_{x_0}(f) := l_{x_0}, \qquad f \in C_{fin}(X). \tag{2.5}$$

Every configuration $\gamma \in \Gamma(X)$, i. e., the set $\gamma = [x_1, x_2, \dots]$, consists of a finite or infinite number of different points. For γ we introduce the measure on $\mathcal{B}(X) \ni \alpha$ as follows: $\mu_\gamma(\alpha) = \sum\limits_{x_j \in \alpha} \mu_{x_j}$.

Using (2.5) we can write

$$\gamma \leftrightarrow [x_1, x_2, \dots] \leftrightarrow \{\delta_{x_1}, \delta_{x_2}, \dots\} \leftrightarrow \{l_{x_1}, l_{x_2}, \dots\}$$

or

$$\gamma \leftrightarrow \delta_\gamma \leftrightarrow \mu_\gamma \leftrightarrow l_\gamma, \qquad \gamma \in \Gamma(X). \tag{2.6}$$

We have (2.6) $\Gamma(X) \ni \gamma \leftrightarrow l_\gamma \in C'_{fin}(X)$, and therefore it is possible to introduce into $\Gamma(X)$ the corresponding weak topology, $\Gamma(X) \ni \gamma^{(m)} \xrightarrow[m\to\infty]{} \gamma \in \Gamma(X)$, $m = 1, 2, \dots$, if and only if $C'_{fin}(X) \ni l_{\gamma^{(m)}} \to l_\gamma \in C'_{fin}(X)$.

This weak topology in $\Gamma(X)$ is also called the "vague topology". The vague topology can also be understood as a relative topology on $\Gamma(X)$, using the weak topology of $C'_{fin}(X)$ and including (see (2.6))

$$\Gamma(X) \subset C'_{fin}(X). \tag{2.7}$$

The relations (2.6) and (2.7) are essential for us, — we introduce a topology on $\Gamma(X)$ with the help of the weak topology on $C'_{fin}(X)$.

The set of finite configurations $\Gamma_0(X)$, as a part of $\Gamma(X)$, can also be endowed with the vague topology, but now we consider the classical ordinary topology on $\Gamma_0(X)$.

3. Ordinary topologization of the set $\Gamma_0(X)$. The n-point configuration is, by definition, a non-ordered set $\gamma = [x_1, \dots, x_n]$ of points $x_1, \dots, x_n \in X$, $x_j \neq x_k$ if $j \neq k$. The set of all such finite configurations is denoted by $\Gamma^{(n)}(X)$. It is clear that

$$\Gamma^{(n)}(X) = \left\{ \gamma \subset X \,\middle|\, |\gamma| = n \right\}.$$

The topology into $\Gamma^{(n)}(X)$ is introduced as the image of topology in the space

$$\widehat{X}^n := \left\{ (x_1, \dots, x_n) \in X^n \,\middle|\, x_j \neq x_k, \text{ if } j \neq k \right\}$$

under the mapping

$$\widehat{X}^n \ni (x_1, \dots, x_n) \mapsto [x_1, \dots, x_n] = \gamma \in \Gamma^{(n)}(X).$$

Thus, a sequence $\gamma^{(m)} = [x_1^{(m)}, \dots, x_n^{(m)}]$ converges to $\gamma = [x_1, \dots, x_n]$ as $m \to \infty$ in the topology of $\Gamma^{(n)}(X)$ if and only if $x_1^{(m)} \to x_1, \dots, x_n^{(m)} \to x_n$ as $m \to \infty$ (the coordinate-wise convergence).

Let us stress that the space $\Gamma_0(X)$ of finite configurations satisfies (2.2). The ordinary topology in $\Gamma_0(X)$ is introduced in the following way. By (2.2) this space is represented as a disjoint sum of $\Gamma^{(n)}(X) \subset X^n$, endowed with the relative topology from X^n, $\Gamma^{(0)}(X) = \varnothing$. So, convergence in the space (2.2) is uniformly finiteness and coordinate-wise convergence for every coordinate $f_n \in \Gamma^{(n)}(X)$ of the vector

$$f = (f_0, f_1, \dots) \in \bigsqcup_{n=0}^{\infty} \Gamma^{(n)}(X).$$

4. A study of Borel σ-algebra on the space $C'_{fin}(X)$. Denote this σ-algebra by $\mathcal{B}(C'_{fin}(X))$ and investigate properties of some of its subsets. The following theorem is a consequence of results from [3].

Theorem 2.1. *Let $\Lambda \subset X$ be a compact set. Then we have that*

$$\Gamma(\Lambda) \in \mathcal{B}(C'_{fin}(X)). \tag{2.8}$$

5. A further study of $\Gamma(X)$ and $\mathcal{B}(C'_{fin}(X))$. Since X is a separable locally compact space, there exists a sequence of its compact subspaces Λ_n, $n \in \mathbb{N}$, such that

$$\Lambda_1 \subset \Lambda_2 \subset \dots \quad \text{and} \quad X = \bigcup_{n=1}^{\infty} \Lambda_n. \tag{2.9}$$

We have the following decomposition of the space X:

$$\begin{gathered} X = \Lambda_1 \cup (\Lambda_2 \setminus \Lambda_1) \cup (\Lambda_3 \setminus \Lambda_2) \cup \dots = K_1 \cup K_2 \cup K_3 \cup \cdots, = \bigsqcup_{n=1}^{\infty} K_n \\ K_n = \Lambda_n \setminus \Lambda_{n-1} \subset \Lambda_n, \qquad n \in \mathbb{N} \qquad (\Lambda_0 := \varnothing), \end{gathered} \tag{2.10}$$

where the sets $K_1, K_2, K_3, \dots$ are pairwise disjoint and have compact closures. Let $\gamma \in \Gamma(X)$ be an arbitrary configuration, i. e., some subset of points from X. Then representation (2.10) gives that

$$\begin{aligned} \gamma &= (\gamma \cap \Lambda_1) \cup (\gamma \cap (\Lambda_2 \setminus \Lambda_1)) \cup (\gamma \cap (\Lambda_3 \setminus \Lambda_2)) \cup \dots \\ &= (\gamma \cap K_1) \cup (\gamma \cap K_2) \cup (\gamma \cap K_3) \cup \dots \\ &= \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \dots, \qquad \gamma_n = \gamma \cap K_n, \qquad n \in \mathbb{N}, \end{aligned} \tag{2.11}$$

i. e., we have the following equality for the subspaces γ, γ_n of the space X:

$$\Gamma(X) \ni \gamma = \bigcup_{n=0}^{\infty} (\gamma \cap K_n) = \bigcup_{n=0}^{\infty} \gamma_n, \qquad \gamma_n = \gamma \cap K_n \in \Gamma(K_n), \qquad n \in \mathbb{N}. \tag{2.12}$$

Moreover, (2.11) gives, that if γ_n is an arbitrary configuration from $\Gamma(K_n)$, $n \in \mathbb{N}$, then

$$\bigcup_{n=0}^{\infty} \gamma_n =: \gamma$$

belongs to $\Gamma(X)$ (see Subsection 1 of this section).

To be more specific, we prove the following result.

Lemma 2.1. *We have the following decomposition of the set $\Gamma(X)$:*

$$\Gamma(X) = \{\gamma : \gamma = \bigcup_{n=1}^{\infty} \gamma_n, (\gamma_1, \gamma_2, \dots) \in \prod_{n=1}^{\infty} \Gamma(K_n)\} \equiv \sum_{n=1}^{\infty}{}^{\bullet} \Gamma(K_n), \tag{2.13}$$

where $\Gamma(K_n)$, as usual, denotes the set of all configurations constructed from the space K_n, and $\sum_{n=1}^{\infty}{}^{\bullet}$ denotes the direct sum of commutative semigroups $\Gamma(K_n)$ where the algebraic operation is the union of configurations, which are subsets of the space X.

Proof. Denote by $A(X)$ the collection of all subsets of X, including X and the empty set. If $Y \subset X$ is some subspace of X, with the relative topology, then $A(Y) \subset A(X)$.

If (2.10) holds, then

$$A(X) = \sum_{n=1}^{\infty}{}^{\bullet} A(K_n). \tag{2.14}$$

Let us prove (2.14). Let α be some set from X, then $\alpha_n = \alpha \cap K_n$ belongs to $A(K_n)$, $n \in \mathbb{N}$, and $\alpha = \bigcup_{n=1}^{\infty} \alpha_n$. Conversely, if for every $n \in \mathbb{N}$ $\beta_n \in A(K_n)$, then $\bigcup_{n=1}^{\infty} \beta_n =: \beta$ belongs to $\sum_{n=1}^{\infty}{}^{\bullet} A(K_n)$. For every n $A(K_n) \subset A(X)$, therefore the transition from β to β_n is such that for α to α_n: $\beta_n = \beta \cap K_n$. Hence (2.14) takes place.

After this general formula (2.14) consider the case where, instead of $A(X)$, we consider the set $\Gamma(X)$ of all configurations on this space X.

By definition (2.1), the set γ from X is a configuration, if γ satisfies the two conditions: 1) γ consists of different points from X and 2) for every compact subset Λ of the space X, the number of points in $\gamma \cap \Lambda$ is finite.

Therefore (2.13) follows from (2.14) if we prove that if $\gamma = \bigcup_{n=1}^{\infty} \gamma_n$ is an expansion of the set $\gamma \subset X$ into sets $\gamma_n \subset K_n$, then γ satisfies conditions 1), 2) if and only if every γ_n, $n \in \mathbb{N}$, satisfies these conditions.

So, let every γ_n satisfy conditions 1), 2). Then γ consists of different points from X, because points of γ from K_n are different since $\gamma_n \in \Gamma(K_n)$.

Verify condition 2) for γ. Let Λ be some compact subset of X. Then from (2.9) it follows that $\Lambda \subset \Lambda_m$ with some fixed $m = m(\Lambda)$. Therefore $\Lambda \cap K_n = \emptyset$, when $n = m+1, m+2, \ldots$, and $\gamma = \bigcup_{n=1}^{m} \gamma_n$. Every γ_n consists, by (2.1), of a finite number of points from K_n; therefore the number of points from our α, which belong to Λ, is also finite.

Conversely, let γ satisfy conditions 1) and 2). Then γ_n, as a part of the set γ, also satisfies these conditions. $\square$

Now we prove a simple supplement to the equalities (2.11) and (2.12).

Lemma 2.2. *Let Λ_1 and Λ_2 be some compact subsets of X, $\Lambda_1 \subset \Lambda_2$. Then*

$$\Gamma(\Lambda_2) = \Gamma(\Lambda_1) \dot{+} \Gamma(\Lambda_2 \setminus \Lambda_1), \tag{2.15}$$

where $\dot{+}$ denotes the direct sum.

Proof. Denote $K_1 = \Lambda_2 \setminus \Lambda_1$, then we have $\Lambda_2 = \Lambda_1 \cup K_1$ and for every $\gamma \in \Gamma(\Lambda_2)$,

$$\gamma = (\gamma \cap \Lambda_1) \cup (\gamma \cap K_1) = \gamma_1 \cup \gamma_2,$$

where

$$\gamma_1 = \gamma \cap \Lambda_1 \in \Gamma(\Lambda_1), \qquad \gamma_2 = \gamma \cap K_1 \in \Gamma(K_1).$$

Conversely, if $\gamma_1 \in \Gamma(\Lambda_1)$ and $\gamma_2 \in \Gamma(K_1)$, then $\gamma_1 \cup \gamma_2 =: \gamma$ belongs to Λ_2. As in the proof of Lemma 2.1, from these conclusions we have that the equality (2.15) is true. $\square$

Example. Let the set Λ_1 contain one point x_1 and $\Lambda_2 \setminus \Lambda_1$ contain one point x_2. Then $\Gamma(\Lambda_1) = (\emptyset, \{x_1\})$, $\Gamma(\Lambda_2 \setminus \Lambda_1) = (\emptyset, \{x_2\})$. The equality (2.15) gives

$$\begin{aligned}\Gamma(\Lambda_2) &= \Gamma(\Lambda_1) \dot{+} \Gamma(\Lambda_2 \setminus \Lambda_1) = (\emptyset, \{x_1\}) \dot{+} (\emptyset, \{x_2\}) \\ &= (\emptyset \cup \emptyset, \emptyset \cup \{x_2\}, \{x_1\} \cup \emptyset, \{x_1\} \cup \{x_1\}) \\ &= (\emptyset, \{x_1\}, \{x_2\}, \{x_1, x_2\}).\end{aligned}$$

Our nearest aim is to extend the number of sets from $\Gamma(X)$, which belong to $\mathcal{B}(C'_{fin}(X))$.

Conclusion 2.1. *Let Λ, Λ_1, Λ_2 be some compact subsets of X; $\Lambda_1 \subset \Lambda_2$. Then*

$$\Gamma(\Lambda),\ \Gamma(\Lambda_2 \setminus \Lambda_1),\ \Gamma(X),\ \Gamma^{(n)}(\Lambda),\ \Gamma(K_n), \quad n \in \mathbb{N} \quad (K_n \textit{ have the form } (2.10)), \tag{2.16}$$

belong to $\mathcal{B}(C'_{fin}(X))$.

This conclusion follows from Theorem 2.1, Lemmas 2.1, 2.2.

It is possible to investigate, instead of the σ-algebra $\mathcal{B}(C'_{fin}(X))$, another σ-algebra $\mathcal{B}(\Gamma(X))$, where $\Gamma(X)$ is endowed with the relative topology as a set from $C'_{fin}(X)$, topologized with the weak topology (i. e. $\Gamma(X)$ is topologized with the vague topology). It is interesting to know what is the collections of sets from $\mathcal{B}(\Gamma(X))$, what are their properties, etc. But we will not investigate these questions in detail and consider only the vague topologization of the set of finite configurations $\Gamma_0(X)$.

6. Vague topologization of the set $\Gamma_0(X)$. We give some details of such a topologization of $\Gamma_0(X)$ as a subset of $\Gamma(X)$.

As follows from (2.4), (2.5) and (2.6), a sequence $\left(\gamma^{(m)}\right)_{m=1}^{\infty}$ of finite configurations $\gamma^{(m)} = [x_{m,1}, \dots, x_{m,n_m}]$, where $x_{m,1}, \dots, x_{m,n_m}$ are different points from X, tends to an n-points configuration $\gamma = [x_1, \dots, x_n] \in \Gamma_0(X)$ if and only if for every $f \in C_{fin}(X)$

$$l_{\gamma^{(m)}}(f) = \int\limits_X f(x) d\mu_{l_{\gamma^{(m)}}}(x) = \sum_{j=1}^{n_m} f(x_{m,j}) \xrightarrow[m\to\infty]{} \sum_{j=1}^{n} f(x_j). \tag{2.17}$$

Thus, it is necessary that the set $[x_{m,1}, \dots, x_{m,n_m}] \subset X$ "tends" to the set $[x_1, \dots, x_n] \subset X$ of different points. Therefore it is possible that two (or more than two) points $x_{m,j}$ "paste together" into one point, and the initial amount of n_m points $x_{m,1}, \dots, x_{m,n_m}$ are reduced. Thus, the reduction of the number of points is possible.

The second purpose of such a reduction is a departure of a point "to infinity": see (2.17), every $f \in C_{fin}(X)$ is a finite function. But we consider only γ from $\Gamma_0(X)$, therefore such a cause is impossible.

3. The measure on the space of configurations

We have the space of configurations $\Gamma(X)$ topologized with the vague topology, i. e., the the weak topology of the space $C'_{fin}(X)$. The space $\Gamma(X)$ belongs to the σ-algebra $\mathcal{B}(C'_{fin}(X))$ (see Conclusion 2.1). This conclusion gives that sufficiently many sets from $\Gamma(X)$ are sets from $\mathcal{B}(C'_{fin}(X))$.

Suppose we have a positive finite measure σ on the σ-algebra $\mathcal{B}(C'_{fin}(X))$, i. e.,

$$\mathcal{B}(C'_{fin}(X)) \ni \alpha \mapsto \sigma(\alpha) \geq 0. \tag{3.1}$$

Starting from representation (2.13) we can consider a special class of measures on the space of configurations $\Gamma(X)$.

Definition 3.1. We say that a measure σ on the space of configurations $\Gamma(X)$ is special, if there exists a decomposition, $X = \bigcup\limits_{n=1}^{\infty} K_n$ of the form (2.10) and a constant $C \geq 1$ such that the inequality

$$\sigma(\Gamma(X)) \leq C \sum_{n=1}^{\infty} \sigma(\Gamma(K_n)) \tag{3.2}$$

is true.

Theorem 3.1. *Let σ be a nontrivial measure on $\Gamma(X)$, i.e., $\sigma(\Gamma(X)) > 0$. This measure is special if and only if there exist $n \in \mathbb{N}$ and a compact set $\Lambda \subset X$ such that*

$$\sigma(\Gamma^{(n)}(\Lambda)) > 0, \tag{3.3}$$

where $\Gamma^{(n)}(\Lambda)$ is the corresponding set of finite configurations.

Proof. Since $\sigma(\Gamma(X)) > 0$ and the measure σ satisfy (3.2), there exists $n_0 \in \mathbb{N}$ such that $\sigma(\Gamma(K_{n_0})) > 0$. Taking into account the inclusion $K_{n_0} \subset \Lambda_{n_0}$ (see (2.10)), we conclude that $\Gamma(K_{n_0}) \subset \Gamma(\Lambda_{n_0})$, whence $\sigma(\Gamma(\Lambda_{n_0})) > 0$. Using the latter inequality, the equality $\Gamma(\Lambda) = \Gamma_0(\Lambda)$ for any compact set $\Lambda \subset X$, and (2.3), we obtain the following relation:

$$0 < \sigma(\Gamma(K_{n_0})) \leq \sigma(\Gamma(\Lambda_{n_0})) = \sigma(\Gamma_0(\Lambda_{n_0}))$$
$$= \sigma\left(\bigcup_{n=1}^{\infty} \Gamma^{(n)}(\Lambda_{n_0})\right) = \sum_{n=1}^{\infty} \sigma(\Gamma^{(n)}(\Lambda_{n_0})).$$

So, $\sigma(\Gamma^{(n)}(\Lambda_{n_0})) > 0$ at least for one $n \in \mathbb{N}$. Putting in the formulation of the theorem $\Lambda = \Lambda_{n_0}$, we arrive at (3.3), which confirms the validity of the necessary condition.

Let us pass now to the proof of the sufficient condition (3.2) for a measure σ to be special. Assume that there exist Λ and $n \in \mathbb{N}$ such that (3.2) holds true, and choose a decomposition $X = \sigma\left(\bigcup_{n=1}^{\infty} K_n\right)$ so that $K_1 = \Lambda$. We set $C = \frac{\sigma(\Gamma(X))}{\sigma(\Gamma^{(n)}(\Lambda))}$. Then, in view of the relation $\Gamma(K_1) = \Gamma(\Lambda) \supset \Gamma^{(n)}(\Lambda)$, we have

$$\sigma(\Gamma(X)) = C\sigma(\Gamma^{(n)}(\Lambda)) \leq C \sum_{n=1}^{\infty} \sigma(\Gamma(K_n)).$$

Thus, $\sigma(\Gamma(K_1)) \geq \sigma(\Gamma^{(n)}(\Lambda))$. This means, by Definition 3.1, that the measure σ is special. □

Theorem 3.2. *In order that a nontrivial measure σ on the space of configurations $\Gamma(X)$ be special in accordance with Definition 3.1, it is necessary and sufficient that $\sigma(\Gamma_0(X)) > 0$.*

Proof. If a measure σ is special, then, by Theorem 3.1, inequality (3.2) is fulfilled and so, we have

$$\sigma(\Gamma_0(X)) \geq \sigma(\Gamma^{(n)}(\Lambda)) > 0.$$

Assume now that $\sigma(\Gamma_0(X)) \geq \sigma(\Gamma^{(n)}(\Lambda)) > 0$ and consider the decomposition $X = \bigcup_{n=1}^{\infty} K_n$, $K_n = \Lambda_n \setminus \Lambda_{n-1}$ of the form (2.2), where $0 < \sigma(\Gamma_0(X)) \leq \sum_{n=1}^{\infty} \sigma(\Gamma_0(\Lambda_n))$. Since the sum is positive, at least one of its summands is positive. Suppose $\sigma(\Gamma_0(\Lambda_{n_0})) > 0$. Then $0 < \sigma(\Gamma_0(\Lambda_n)) \leq \sum_{k=1}^{\infty} \sigma(\Gamma^{(k)}(\Lambda_{n_0}))$ and, hence, one of the summands is positive. Let, for example, $\sigma(\Gamma^{(k_0)}(\Lambda_{n_0})) > 0$. It follows from Theorem 3.1 that the measure σ is special. □

Let us give some examples of special measures supported on the set $\Gamma^{(2)}(X)$ of all two-point configurations.

Example 3.1. Let $X = [0, \infty)$ be the right semi-axis with the natural topology. Every two-point configuration $\gamma \in \Gamma^{(2)}(X)$ is determined uniquely by two real numbers x and y $(0 \leq x \leq \infty, 0 < y < \infty)$, if we set $\gamma(x, y) = \{x, x + y\}$. If $A \subset \Gamma^{(2)}(X)$ is some set of two-point configurations, then we obtain on the plane (x, y) a set of the points $A(x, y) = \{(x, y) : \gamma(x, y) \in A\}$. Let us assume that A is a measurable set, and define on it a measure σ by the equality

$$\sigma(A) = \iint \chi_A(x, y)e^{-(x+y)}\, dxdy, \tag{3.4}$$

where $\chi_A(x, y)$ is the characteristic function of the above set $A(x, y)$. It is clear that $\sigma(\Gamma^{(2)}(X)) = 1$, and the measure σ is special according to Definition 3.1. The inequality (3.2) holds for any decomposition of $X = [0, \infty)$ with a constant $C > 1$ depending on the choice of the decomposition.

Example 3.2. Let $X = N$ be the set of all natural numbers. In this case, every two-point configuration is determined uniquely by two natural numbers n and m: $\gamma(n,m) = \{n, n+m\}$. Define a measure on $\Gamma^{(2)}(N)$, by the equality

$$\sigma(\gamma(n,m)) = 2^{-(n+m)}, \tag{3.5}$$

Then $\sigma(\Gamma^{(2)}(N)) = 1$. This measure is special. The inequality (3.2) holds for each decomposition of $X = N$ except for the single-point one, where $X = \bigcup_{n=1}^{\infty} K_n, K_n = \{n\}$.

Let us remark in conclusion that every finite measure σ on the space of configurations $\Gamma(X)$ admits a unique representation as a sum of two measures σ_0 and σ_∞:

$$\sigma(A) = \sigma_0(A) + \sigma_\infty(A),$$

where $\sigma_0(A) = \sigma(A \bigcap \Gamma_0(X))$ is a special measure supported on $\Gamma_0(X))$, while $\sigma_\infty(\Gamma_0(X)) = 0$.

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Received 14/08/2018