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LIMITED AND DUNFORD-PETTIS OPERATORS ON BANACH LATTICES

KHALID BOURAS, ABDENNABI EL ALOUI, AND AZIZ ELBOUR

Abstract. This paper is devoted to investigation of conditions on a pair of Banach lattices $E; F$ under which every positive Dunford-Pettis operator $T : E \to F$ is limited. Mainly, it is proved that if every positive Dunford-Pettis operator $T : E \to F$ is limited, then the norm on E' is order continuous or F is finite dimensional. Also, it is proved that every positive Dunford-Pettis operator $T : E \to F$ is limited, if one of the following statements is valid:

1. The norm on E' is order continuous, and F' has weak* sequentially continuous lattice operations.
2. The topological dual $E'$is discrete and its norm is order continuous.
3. The norm of E' is order continuous and the lattice operations in $E^{'}$ are weak* sequentially continuous.
4. The norms of E and of E' are order continuous.

1. Introduction

Throughout this paper $X,\ Y$ will denote Banach spaces, and $E,\ F$ will denote Banach lattices. B_X is the closed unit ball of X. We will use the term operator between two Banach spaces to mean a bounded linear mapping. We refer to [1] for unexplained terminology of Banach lattice theory and positive operators. Let us recall that an operator $T : X \longrightarrow Y$ is said to be:

- a Dunford-Pettis operator if T carries weakly convergent sequences to norm convergent sequences;
- a limited operator if T' carries weakly* convergent sequences in Y' to norm convergent sequences in X'.

Note that a Dunford-Pettis operator is not neccessarily limited and a limited operator is not neccessarily Dunford-Pettis. In fact, the identity operator $Id_{\ell^1} : \ell^1 \to \ell^1$ is Dunford-Pettis but not limited and the inclusion operator $i : c_0 \to \ell^\infty$ is limited but not Dunford-Pettis.

Recall from [9] that an operator T from a Banach lattice E to a Banach space Y is said to be almost Dunford-Pettis if the sequence $(\|T(x_n)\|)$ converges to 0 for every weakly null sequence (x_n) consisting of pairewise disjoint elements of E.

Also, we recall from [6] that an operator T from a Banach space X into a Banach lattice E is called almost limited if $T(B_X)$ is an almost limited set in E, equivalently, $\|T'(f_n)\| \to 0$ for every disjoint weak* null sequence (f_n) in E'.

In this paper, we investigate conditions under which Dunford-Pettis operators must be limited and conversely. We prove that if every positive Dunford-Pettis operator $T : E \to F$ is limited, then, at least, one of the following statements is valid:

(1) the norm on E' is order continuous;

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(2) F is finite dimensional.

Also, it is proved that each positive Dunford-Pettis operator T from E into F is limited if one of the following assertions is valid:

(1) the norm on E' is order continuous, and F' has weak* sequentially continuous lattice operations;
(2) the topological dual E' is discrete, and its norm is order continuous;
(3) the norm of E' is order continuous and the lattice operations in $E^{'}$ are weak* sequentially continuous;
(4) the norms of E and of E' are order continuous.

As a consequence, we give a characterization of a Banach lattice E for which every Dunford-Pettis operator $T : E \to c_0$ is limited (Corollary 2.4).

For the converse, we give some necessary conditions under which a positive limited operator must be Dunford-Pettis (Theorem 2.5).

2. Main results

To obtain our principal result, we need the following lemma:

Lemma 2.1. *Let F be an infinite dimentional Banach lattice. Then there exist a sequence (f_n) in F' converging weakly* to zero, a sequence (y_n) in B_F^+ and some $\varepsilon > 0$ such that $|f_n(y_n)| \geq \varepsilon$ for all n.*

Proof. Assume that F is infinite dimensional. So by Josefson-Nissenzweig Theorem [3, Chapter XII], there exists a sequence (f_n) in F' converge weakly* to zero such that $\|f_n\| \nrightarrow 0$. We may assume that $\|f_n\| > 2\varepsilon > 0$ for some $\varepsilon > 0$ and for all n. For each n, there exists some $y_n \in F$ such that $\|y_n\| \leq 1$ and $|f_n(y_n)| > 2\varepsilon$. We know that $y_n = y_n^+ - y_n^-$. Note that $|f_n(y_n^+)| > \varepsilon$ or $|f_n(y_n^-)| > \varepsilon$, otherwise we can have: $|f_n(y_n)| = |f_n(y_n^+) - f_n(y_n^-)| \leq |f_n(y_n^+)| + |f_n(y_n^-)| \leq 2\varepsilon$, which contradicts $|f_n(y_n)| > 2\varepsilon$. By replacing y_n with y_n^+ or y_n^-, we may assume that for all n, there exists $y_n \in B_F^+$ such that $|f_n(y_n)| > \varepsilon$, and the proof of the lemma is finished. □

We are now in a position to establish our first major result, it gives necessary conditions under which a positive Dunford-Pettis operator between Banach lattices must be limited.

Theorem 2.2. *Let E and F be Banach lattices. If every positive Dunford-Pettis operator $T : E \to F$ is limited, then one of the following statements is valid:*

1. *The norm on $E^{'}$ is order continuous.*
2. *F is finite dimensional.*

Proof. Assume by way of contradiction that E' does not have an order continuous norm and F is infinite dimensional. We must then build a positive Dunford-Pettis operator $T : E \to F$ that is not limited. Since $E^{'}$ does not have an order continuous norm, it follows from theorem 4.14 of [1] that there exists an order bounded disjoint sequence (x'_n) in $(E')^+$ satisfying $\|x'_n\| = 1$ for all n. Pick some $0 \leq x' \in E'$ such that $x'_n \in [0; x']$ for all n. Moreover, since F is finite dimensional, according to the previous lemma, there are a sequence (f_n) in F' converges weakly* to zero, a sequence (y_n) in B_F^+ and some $\varepsilon > 0$ satisfying $|f_n(y_n)| \geq \varepsilon$ for all n.

Now define two positive operators $P : E \to \ell^1$ and $S : \ell^1 \to F$ by

$$P(x) = (x_n^{'}(x))_{n=1}^{\infty},$$

$$S((\lambda_n)_{n=1}^{\infty}) = \sum_{n=1}^{\infty} \lambda_n y_n.$$

Since $\sum_{n=1}^{\infty}(|x'_n(x)|) \leq \sum_{n=1}^{\infty}(x'_n(|x|)) \leq x'(|x|)$ holds for all $x \in E$, the operator P is well defined and positive. Similarly, as $\sum\limits_{n=1}^{\infty} \|\lambda_n y_n\| \leq \sum\limits_{n=1}^{\infty} |\lambda_n| < \infty$ for all $(\lambda_n)_{n=1}^{\infty} \in \ell^1$, the operator S is well defined and is positive.

Now, we consider the operator $T = S \circ P : E \to \ell^1 \to F$ defined by

$$T(x) = \sum_{n=1}^{\infty} x'_n(x) y_n.$$

We claim that T is a Dunford-Pettis operator. Indeed, if (x_n) converges weakly to zero in E, then $(P(x_n))$ converges weakly to zero in ℓ^1 and since ℓ^1 has the Schur property, we deduce $\|P(x_n)\|_{\ell^1} \to 0$, from which it follows that $\|T(x_n)\| = \|S(P(x_n))\| \to 0$, as desired. However T is not limited. Assume that (f_n) converges weakly* to zero in F'. Then

$$|T'(f_n)| = |\sum_{i=1}^{\infty} f_n(y_i) x'_i| = \sum_{i=1}^{\infty} |f_n(y_i)|\ x'_i \geq |f_n(y_n)| x'_n$$

(as (x'_n) is a positive disjoint sequence), and hence

$$\|T'(f_n)\| \geq \|f_n(y_n) x'_n\| = |f_n(y_n)| \geq \varepsilon.$$

So, $\|T'(f_n)\| \nrightarrow 0$, which proves that T is not limited. □

Recall that the lattice operations in E (resp. in E') are called weak (resp. weak*) sequentially continuous if for every sequence (x_n) in E (resp. (f_n) in E'), the sequence $(|x_n|)$ (resp. $(|f_n|)$) converges to 0 in the weak (resp. weak*) topology, whenever the sequence (x_n) (resp. (f_n)) converges weakly (resp. weak*) to 0 in E (resp. in E').

The following result gives some sufficient conditions under which every Dunford-Pettis operator $T : E \to F$ is limited.

Theorem 2.3. *Let E and F be two Banach lattices. Then every positive Dunford-Pettis operator $T : E \to F$ is limited, if one of the following statements is valid:*

1. *The norm on E' is order continuous, and F' has weak* sequentially continuous lattice operations.*
2. *The topological dual $E'$is discrete and its norm is order continuous.*
3. *The norm of E' is order continuous and the lattice operations in E' are weak* sequentially continuous.*
4. *The norms of E and of E' are order continuous.*

Proof. Let $T : E \to F$ be a positive Dunford-Pettis operator.

(1) Suppose that the norm of E is continuous for the order and the lattice operations in F' are sequentially weak* continuous.

We show that T is limited, that is: for any sequence (f_n) in F' that converges weakly* to zero, we have $\|T'(f_n)\| \to 0$. According to ([4], Corollary 2.7), it suffices to show that:

i) $|T'(f_n)| \to 0$ for $\sigma(E', E)$.
ii) $(T'(f_n))(x_n) \to 0$ for any disjoint norm-bounded sequence (x_n) in E^+.

Let $x \in E^+$. Then

$$|T'(f_n)|(x) = \sup\{|T'(f_n)(z)|; |z| \le x\} = \sup\{|f_n(T(z))|; |z| \le x\} \le |f_n|(T(x)).$$

Since $f_n \to 0$ for $\sigma(E', E)$ and F' has weak* sequentially continuous lattice operations, so $|f_n| \to 0$ for $\sigma(E', E)$, which gives $|f_n|(T(x)) \to 0$. Therefore $|T'(f_n)| \to 0$ for $\sigma(E', E)$.

On the other hand, let (x_n) be a norm-bounded disjoint sequence of E^+. Since the norm of E' is order continuous, then according to ([4], Corollary 2.9), we have $x_n \to 0$ for $\sigma(E, E')$, and as T is Dunford-pettis, so $\|T(x_n)\| \to 0$. Pick some $k > 0$ such that $\|f_n\| \le k$ for all n. From

$$|(T'(f_n))(x_n)| = |f_n(T(x_n))| \le k\|T(x_n)\| \to 0$$

we conclude that $(T'(f_n))(x_n) \to 0$. Therefore, T is limited.

For (2), (3) and (4), since the operator T is Dunford-Pettis, by ([2], Theorem 3.8) it can be deduced that T is compact and therefore it is limited. □

Since c_0 has weak* sequentially continuous lattice operations, the following result is a consequences of Theorem 2.3.

Corollary 2.4. *Let E be a Banach lattice. Then the following assertions are equivalent:*

(1) *Each Dunford-Pettis operator $T : E \to c_0$ is limited.*
(2) *The norm of E' is order continuous.*

Now, we study the converse situation. We give some necessary conditions under which a positive limited operator must be Dunford-Pettis.

Theorem 2.5. *Let E and F be Banach lattices and F' has weak* sequentially continuous lattice operations. If every positive limited operator $T : E \to F$ is Dunford-Pettis, then, at least, one of the following statements is verified:*

(1) *The norm on F is order continuous.*
(2) *E has weak sequentially continuous lattice operations.*

Proof. Suppose that neither (l) not (2) holds. Since the lattice operations of E are not weakly sequentially continuous, using arguments from the proof of ([8], Theorem 2), one can obtain that there exist a weakly null sequence (x_n) of E, $f \in (E')^+$, $g, g_n \in [-f, f]$ satisfying $g_n \to g$ (weak*) and $g_n(x_n) \ge \varepsilon$ for all $n \in \mathbb{N}$.

Since F does not have an order continuous norm, we know (Theorem 4.14 in [1] or Corollary 2.4.2 in [7]) that there is an order bounded disjoint positive sequence in F which does not converge to zero in norm. By extracting a subsequence, we obtain a disjoint positive sequence (y_n), which is bounded away from 0 in norm and which is bounded from above by $y \in F^+$ and $\|y_n\| = 1$.

Now define two operators $R : E \to c_0$ and $S : c_0 \to F$ by:

$$R(x) = (f_n(x))_n,$$

$$S((\lambda_n)) = \sum_{n=1}^{\infty} \lambda_n y_n,$$

where $f_n(x) = g_n(x) - g(x)$. Using arguments from the proof of ([10], Theorem 117.1), one can obtain that the positive operator S is well defined and $S(B_{c_0}) \subseteq [-y, y]$.

Now, we consider the operator $T = S \circ R : E \to F$ defined by

$$T(x) = \sum_{n=1}^{\infty} f_n(x) y_n$$

for all $x \in E$. We have to show that T is limited. To this end, note that for every $x \in B_E$ we have

$$\|R(x)\|_\infty = \sup_n |f_n(x)| \leq 2\|f\|.$$

So $\frac{R(x)}{2\|f\|} \in B_{c_0}$ and hence $S\left(\frac{R(x)}{2\|f\|}\right) \in [-y, y]$ (as $S(B_{c_0}) \subseteq [-y, y]$). Therefore,

$$T(B_E) = S(R(B_E)) \subseteq \alpha[-y, y]$$

(where $\alpha = 2\|f\|$).

On the other hand, it follows from Proposition 3.1 of [5] that the order interval $[-y, y]$ is limited. So $T(B_E)$ is limited and hence that T is limited, as desired. But T is not Dunford-Pettis since although $x_n \to 0$ (weakly) we have

$$\|T(x_n)\| = \|\sum_{k=1}^{\infty} f_k(x_n)y_k\| \geqslant |f_n(x_n)|\|y_n\| = |g_n(x_n) - g(x_n)|\|y_n\|.$$

The right-hand side of the latter expression certainly does not converge to 0 as $\|y_n\|$, $|g_n(x_n) - g(x_n)|$ are both eventually bounded away from zero (in the latter case because $g(x_n) \to 0$ whilst $g_n(x_n) \geqslant \varepsilon > 0$) and the proof is complete. □

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Department of Mathematics, Faculty Polydisciplinary of Larache, Abdelmalek Essaadi University, P.O. Box 745, Larache 92004, Morocco
E-mail address: k.bouras@uae.ac.ma

Department of Mathematics, Faculty Polydisciplinary of Larache, Abdelmalek Essaadi University, P.O. Box 745, Larache 92004, Morocco
E-mail address: dehmanabdennabi@gmail.com

Department of Mathematics, Faculty of Science and Technology, Moulay Ismail University, P.O. Box 509, Boutalamine 52000, Errachidia, Morocco
E-mail address: azizelbour@hotmail.com

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