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SCATTERING PROBLEM FOR DIRAC SYSTEM WITH NONLOCAL POTENTIALS

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Dedicated to the 70th anniversary of A. N. Kochubei

Abstract. For a Dirac system on the half-axis, we obtain an explicit expression for the scattering operator in terms of a nonlocal potential.

1. Introduction

Scattering theory is a well-developed part of the spectral theory of operators [21, 7, 10]. A large number of particular problems have been worked out, and this makes an important contribution to modern mathematical physics, see the encyclopedic monograph [20]).

Scattering problems consist in matching the two problems. One of which describes free wave propagation, and the other one contains a perturbation that leads to wave scattering. From the operator point of view, this leads to considering two operators. One of the operators, A_0, describes the free propogation, and the other one, A, describes the wave scattering. If the operators A_0 and A are selfadjoint, then the scattering problem corresponds to continuous spectrum of the operators. The are several approaches to formulation of the scattering problem. These are a nonstationary and stationary approaches [21, 7, 10], as well as the Lax-Phillips approach [14].

Inverse scattering problems are important for scattering problems. These problems consist in constructing scattering data, a proof that the operator A can be uniquely constructed from the scattering data, and finding an effective algorithm for determining all unknown parameters that define the operator A. Inverse scattering problems (ISP) are important for quantum mechanics, since they provide effective means to study patterns of the microcosm. The ISP are also important in other applications including geophysics exploration, radar technologies, tomography, as well as in other branches of mechanics, engineering, biology, and medicine [20]. Important applications of the ISP are obtained in soliton theory [1] for integration of nonlinear evolution equations. These applications have significantly increased interest to the ISP, regardless of its physical origin, since the algorithm used in its solution can be applied to study a more interesting problem in the theory of solitons.

In this paper we deal with the scattering problem for a Dirac system on half-axis with a nonlocal potential.

Models for quantum mechanics operators with nonlocal potentials were proposed in [3, 4]. These are exact solvable models that contain not only coupling constants, as opposed to models with point interactions [2], but also nonlocal potentials. This leads to new results concerning the spectrum and solution of inverse scattering problems [5, 8, 13, 17, 18, 19]. In this paper, we find all quantities for a Dirac system with nonlocal potential that enter various approaches to the scattering problem in an explicit form, as this is

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done for an exact solvable model. In particular, we explicitly find the scattering matrix in terms of the Fourier transform of the nonlocal potential. We discuss a number of new problem statements for that ISP and ways to solve them. Let us remark that inverse scattering problems for a one-dimensional Schrödinger system and a Dirac system with usual potentials have been extensively studied [15, 16].

2. Scattering problem

Let us consider the following boundary-value problem for a Dirac system with nonlocal potentials on the half-axis:

$$
\begin{aligned}
&i\frac{d\psi_1(x)}{dx} + v_1(x)\psi_+ = \lambda\psi_1(x),\\
&-i\frac{d\psi_2(x)}{dx} + v_2(x)\psi_+ = \lambda\psi_2(x), \qquad 0 \le x < +\infty.
\end{aligned} \tag{2.1}
$$

The constant ψ_+ in system (2.1) depends on the solution ψ_1, ψ_2,

$$
\psi_+ = \frac{1}{2}\left[\psi_1(0) + \psi_2(0)\right]. \tag{2.2}
$$

A solution of system (2.1) should satisfy the nonlocal boundary-value condition

$$
\psi_1(0) - \psi_2(0) - i\int_0^\infty \left[\psi_1(x)\overline{v_1(x)} + \psi_2(x)\overline{v_2(x)}\right] dx = 0. \tag{2.3}
$$

In this section, we assume that the nonlocal potentials v_1 and v_2 are complex-valued functions and belong to $L_2(0,\infty)$ and $L_1(0,\infty)$,

$$
v_1, v_2 \in L_2 \bigcap L_1. \tag{2.4}
$$

Lemma 2.1. *Let the potentials v_1 and v_2 satisfy (2.4). Then, for any real λ and any ψ_+, the solution ψ_1, ψ_2 of system (2.1) are bounded functions on the half-axis, $0 \le x < +\infty$, and admit the representation*

$$
\begin{aligned}
\psi_1(x;\lambda) &= ae^{-i\lambda x} - i\int_x^\infty e^{-i\lambda(x-s)}v_1(s)ds \cdot \psi_+,\\
\psi_2(x;\lambda) &= be^{i\lambda x} + i\int_x^\infty e^{i\lambda(x-s)}v_2(s)ds \cdot \psi_+,
\end{aligned} \tag{2.5}
$$

where the numbers a and b are positive.

Proof. By substituting (2.5) into (2.1) we see that the functions in (2.5) give a solution of system (2.1). Uniqueness of this solutions follows from the fact that, if $\psi_+ = 0$, then only the functions $\psi_1 = \widehat{a}e^{-i\lambda x}$ and $\psi_2 = \widehat{b}e^{i\lambda x}$ make a solution of system (2.1) with arbitrary constants $\widehat{a}$ and $\widehat{b}$. □

It follows from (2.5) that any solution of system (2.1) satisfying conditions (2.4) is a pair of uniformly bounded functions ψ_1, ψ_2 that have the asymptotics

$$
\psi_1(x;\lambda) = ae^{-i\lambda x} + o(1), \qquad \psi_2(x;\lambda) = be^{i\lambda x} + o(1), \qquad x \to \infty. \tag{2.6}
$$

The functions $ae^{-i\lambda x}$ and $be^{i\lambda x}$ have the physical meaning of an incoming and a scattered (reflected) waves. The numbers a and b are amplitudes of these waves. The relation $\frac{b}{a} = S(\lambda)$ is called a scattering matrix. In this particular case, $S(\lambda)$ is the scattering coefficient. The scattering problem for system (2.1) with conditions (2.2) and (2.3) consists in constructing solution (2.5) of problem (2.1) — (2.3) from a given amplitude a of the incoming wave $ae^{-i\lambda x}$.

Theorem 2.1. *Let the potentials v_1 and v_2 in problem (2.1) — (2.3) satisfy condition (2.4). Then there exists a solution of the scattering problem (2.1) — (2.3) with a given amplitude a of the incoming wave. It is unique if and only if the real number λ is not a zero of two functions $p(\lambda)$ and $\chi(\lambda)$ of the form*

$$
\begin{aligned}
p(\lambda) &= 1 + \frac{i}{2}\widetilde{v}_1(\lambda) - \frac{i}{2}\widetilde{v}_2(-\lambda), \\
\chi(\lambda) &= 1 + i\widetilde{v}_1(\lambda) + i\widetilde{v}_2^*(-\lambda) - \frac{1}{2}\widetilde{v}_1(\lambda)\cdot\widetilde{v}_2^*(-\lambda) + \omega(\lambda),
\end{aligned} \tag{2.7}
$$

where $\widetilde{v}_j(\lambda) = \int\limits_0^\infty e^{i\lambda s} v_j(s)ds$ is the Fourier transform of the potentials, $j = 1, 2$, and

$$
\omega(\lambda) = \frac{1}{2}\left\{ \int\limits_0^\infty \overline{v_1(x)} \left[\int\limits_x^\infty e^{-i\lambda(x-s)} v_1(s)ds \right] dx + \int\limits_0^\infty \overline{v_2(x)} \left[\int\limits_0^x e^{i\lambda(x-s)} v_2(s)ds \right] dx \right\}.
$$

In the case where $p(\lambda) \neq 0$ and $\chi(\lambda) \neq 0$, the scattering operator $S(\lambda)$ can be represented as

$$
S(\lambda) = \frac{\chi^*(\lambda)}{\chi(\lambda)}. \tag{2.8}
$$

Proof. For solution (2.5) of system (2.1) to be a solution of the scattering problem (2.1) — (2.4), it is necessary and sufficient that solution (2.5) would satisfy conditions (2.2) and (2.3). This leads to a linear system for ψ_+ and b with the amplitude a of the incoming wave being given,

$$
\begin{aligned}
&\left[1 + \frac{i}{2}\widetilde{v}_1(\lambda) - \frac{i}{2}\widetilde{v}_2(-\lambda)\right]\psi_+ - \frac{1}{2}b = \frac{1}{2}a, \\
&\left[i\widetilde{v}_1(\lambda) + i\widetilde{v}_2^*(-\lambda) + K(\lambda)\right]\psi_+ + \left[1 + i\widetilde{v}_2^*(-\lambda)\right]b = \left[1 - i\widetilde{v}_1^*(\lambda)\right]a,
\end{aligned} \tag{2.9}
$$

where

$$
K(\lambda) = \int\limits_0^\infty \overline{v_1(x)} \int\limits_x^\infty e^{-i\lambda(x-s)} v_1(s)dsdx - \int\limits_0^\infty \overline{v_2(x)} \int\limits_x^\infty e^{i\lambda(x-s)} v_2(s)dsdx.
$$

The determinant of system (2.9) is the function $\chi(\lambda)$ given by (2.7). Hence, system (2.9) has a solution for any a if and only if $\chi(\lambda) \neq 0$ for the considered λ.

Uniqueness of the solution ψ_+ and b can be reduced to the condition that $a = 0$ and $b = 0$ would imply that $\psi_+ = 0$. The first equation of system (2.9) yields that $p(\lambda) \neq 0$ for the considered λ. System (2.9) implies that $b = Sa$, where $S(\lambda)$, can be represented as in (2.8). □

3. Associated operators

Equation (2.1) can be associated with a maximal operator $A_{\max}$ on the space $L_2((0,\infty);C^2)$ of two-component vector-valued functions that are square integrable on the half-axis $(0,+\infty)$. The domain of the operator $A_{\max}$ is the whole Sobolev space $W_2^1((0,\infty);C^2) \subset L_2((0,\infty);C^2)$. The action of the operator $A_{\max}$ is given by the left hand-side of equation (2.1),

$$
A_{\max}\psi = A_{\max}\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} = \begin{pmatrix} i\frac{d\psi_1(x)}{dx} + v_1(x)\frac{1}{2}(\psi_1(0) + \psi_2(0)) \\ -i\frac{d\psi_2(x)}{dx} + v_2(x)\frac{1}{2}(\psi_1(0) + \psi_2(0)) \end{pmatrix} \tag{3.1}
$$

The main operator A that corresponds to problem (2.1) — (2.3) is the restriction of the operator $A_{\max}$ to the set of functions satisfying the boundary-value condition (2.3),

$$
D(A) = \left\{\psi : \psi \in W_2^1((0,\infty);C^2), \psi_1(0) - \psi_2(0) - i(\psi, v) = 0\right\}. \tag{3.2}
$$

Theorem 3.1. *Let the nonlocal potential* $v = \mathrm{col}(v_1, v_2)$ *satisfy conditions* (2.4). *Then the operator* A *is selfadjoint on the space* $L_2((0,\infty); C^2)$.

Proof. For the operator $A_{\max}$, we have

$$\langle A_{\max}\psi, \varphi\rangle - \langle \psi, A_{\max}\varphi\rangle = -i\left[\Gamma_1\psi \cdot \overline{\Gamma_1\varphi} - \Gamma_2\psi \cdot \overline{\Gamma_2\varphi}\right], \tag{3.3}$$

on its domain $D(A_{\max}) = W_2^1((0,\infty); C^2)$, where $\Gamma_1\psi = \psi_1(0) - \frac{i}{2}(\psi, v)$, $\Gamma_2\psi = \psi_2(0) + \frac{i}{2}(\psi, v)$. $(\psi, v) = (\psi_1, v_1)_{L_2} + (\psi_2, v_2)_{L_2}$.

One can show in the same way as for a Dirac system with nonlocal potential on a bounded interval [8] that the minimal operator A_0, which is a restriction of the operator $A_{\max}$ to $D(A_0) = \{\psi : \psi \in D(A_{\max}), \Gamma_1\psi = 0, \Gamma_2\psi = 0\}$, is a densely defined symmetric operator on the space $L_2((0,\infty); C^2)$, and that $A_0^* = A_{\max}$.

Moreover, the operator (Γ_1, Γ_2) maps $D(A_{\max})$ onto the whole space C^2. Thus $(C^2, \Gamma_1, \Gamma_2)$ is a boundary triple for the operator $A_{\max} = A_0^*$. It follows from a general theorem [9, 11, 12] that the restriction of the operator $A_{\max}$ onto the set $D^\alpha = \left\{\psi : \psi \in D(A), \Gamma_1\psi = e^{i\alpha}\Gamma_2\psi\right\}$, where α is a real number, is a selfadjoint operator $A^{(\alpha)}$. If $\alpha = 0$, then the operator $A^{(\alpha)}$ coincides with the operator A. Another proof that the operator A is selfadjoint is given in Theorem 4.1. □

Let us consider the structure of eigenvalues and eigenfunctions of the operators $A_{\max}$ and A.

Theorem 3.2. *Let the nonlocal potential* $v = \mathrm{col}(v_1, v_2)$ *satisfy the conditions* $v_1, v_2 \in L_2(0,\infty) \bigcap L_1(0,\infty)$ *and* $\int_x^\infty |v_j(s)|ds \in L_2(0,\infty)$. *Then a real number* λ_0 *is an eigenvalue of the operator* $A_{\max}$ *if and only if* λ_0 *is a zero of the function* $p(\lambda)$,

$$p(\lambda) = 1 - \frac{i}{2}\int_0^\infty e^{i\lambda s} v_1(s)ds + \frac{i}{2}\int_0^\infty e^{-i\lambda s} v_2(s)ds. \tag{3.4}$$

A real number λ_0 *is an eigenvalue of the operator* A *if and only if*

$$p(\lambda_0) = 0, \qquad \chi(\lambda_0) = 0, \tag{3.5}$$

where the function $\chi(\lambda)$ *is given by* (2.7).

Proof. Functions of the form (2.5) will be eigenfunctions corresponding to an eigenvalue λ for the operator $A_{\max}$ if and only if $a = 0$, $b = 0$ and $\psi_+ = \frac{1}{2}\left[\psi_1(0) + \psi_2(0)\right] \neq 0$ in (2.5). This is possible for $\lambda = \lambda_0$ if and only if $p(\lambda_0) = 0$. Since the operator A is a restriction of the operator $A_{\max}$, eigenfunctions of the operator A are eigenfunctions of the operator $A_{\max}$ and, hence, satisfy the boundary-value conditioin (2.3). This gives conditions (3.5). □

Together with the operators $A_{\max}$ and A associated with problem (2.1) — (2.3), let us also consider two unperturbed operators $A_\pm$ for the potentials being zero. The operators $A_\pm$ are given on the space $L_2((0,\infty); C^2)$, defined on the functions from the Sobolev space $W_2^1((0,\infty); C^2)$, which satisfy the conditions $\psi_1(0) + \psi_2(0) = 0$ for the operator A_+, and the conditions $\psi_1(0) - \psi_2(0) = 0$ for the operator A_-. The action of the operators $A_\pm$ on functions $\psi(x) = \mathrm{col}(\psi_1(x), \psi_2(x))$ is given by the unperturbed Dirac operator $A_\pm\psi = \mathrm{col}(i\frac{d\psi_1(x)}{dx}, -i\frac{d\psi_2(x)}{dx})$. It is well known that the operators $A_\pm$ are selfadjoint on the space $L_2((0,\infty); C^2)$, have absolutely continuous spectrum that fills the whole real axis. Eigenfunctions of the operators $A_\pm$ corresponding to an eigenvalue λ are the functions

$$\varphi_\pm(x;\lambda) = \mathrm{col}(e^{-i\lambda x}, \mp e^{i\lambda x}).$$

Expansions with respect to these generalized functions and the Parseval identity are the same as for the usual Fourier transform [6].

4. Resolvent of the main operator

Let us represent problem (2.1) — (2.3) for the Dirac problem on half-axis as an equivalent problem for the moment operator on the whole axis with one point nonlocal potential. Instead of the two functions ψ_1 and ψ_2 defined on the positive half-axis, let us consider one function $\psi(x) = \theta(x)\psi_1(x) + \theta(-x)\psi_2(-x)$ defined on the whole axis. Instead of the two potentials v_1 and v_2, let us consider one potential $v(x) = \theta(x)v_1(x) + \theta(-x)v_2(-x)$. Then there is the problem

$$i\frac{d\psi(x)}{dx} + v(x)\psi_+ - z\psi(x) = h(x), \qquad x \neq 0, \tag{4.1}$$

where

$$\psi_+ = \frac{1}{2}\left[\psi(+0) + \psi(-0)\right], \tag{4.2}$$

and a solution $\psi(x)$ satisfies the nonlocal boundary-value condition

$$\psi(+0) - \psi(-0) - i\int_{-\infty}^{\infty} \psi(x)\overline{v(x)}dx = 0. \tag{4.3}$$

To problem (4.1) — (4.3) on the Hilbert space $L_2(-\infty,\infty)$, associate the main operator A, domain of which, $D(A)$, consists of all functions ψ of the Sobolev space $W_2^1((-\infty,\infty)\setminus\{0\})$ satisfying the boundary-value condition (4.3). The action of the operator $A - zI$ is given by the left hand-side of (4.1). Problem (4.1) — (4.3) has a solution for an arbitrary right hand-side $h \in L_2$ if and only if the operator $(A - zI)^{-1}$ exists on the space L_2 and is bounded. In the case where $v \equiv 0$, the operator A is the free moment operator $L = i\frac{d}{dx}$, which is selfadjoint on the space $L_2(-\infty,\infty)$ and has absolutely continuous spectrum filling the entire axis. The resolvent $(L - zI)^{-1}$, if $\operatorname{Im} z \neq 0$, is the integral operator

$$(L - zI)^{-1} h = \int g_z(x-s)h(s)ds, \tag{4.4}$$

where the kernel of the integral operator is Green's function

$$g_z(x) = i\,\operatorname{sign}(\operatorname{Im} z)\theta(-\operatorname{Im} zx)e^{-izx}. \tag{4.5}$$

Note that

$$\begin{gathered}\left\|(L - zI)^{-1}\right\| \leq \frac{1}{|\operatorname{Im} z|}, \qquad \|g_z(\cdot)\|_{L_2} = \frac{1}{\sqrt{2|\operatorname{Im} z|}},\\ [g_z(+0) + g_z(-0)] = i\,\operatorname{sign}(\operatorname{Im} z), \qquad g_z(+0) - g_z(-0) = -i.\end{gathered} \tag{4.6}$$

Represent a solution of system (4.1) as

$$\psi = (L - zI)^{-1}\left[h(x) - v(x)\psi_+\right] + \beta g_z(x) \tag{4.7}$$

with two constants ψ_+ and β.

For a function (4.7) to be a solution of problem (4.1) — (4.3) it is necessary and sufficient that this function would satisfy two equations (4.2) and (4.3). This leads to the following linear system for the numbers ψ_+ and β:

$$\begin{cases}\left(1 + \langle v, g_{\overline{z}}\rangle\right)\psi_+ - \frac{i}{2}\operatorname{sign}(\operatorname{Im} z)\beta = \langle h, g_{\overline{z}}\rangle\\ -\left\langle (L - zI)^{-1} v, v\right\rangle \psi_+ + \left(1 + \langle g_z, v\rangle\right)\beta = -\left\langle h, (L - \overline{z}I)^{-1} v\right\rangle.\end{cases} \tag{4.8}$$

The determinant of system (4.8) is

$$\chi(\lambda) = \left(1 + \langle v, g_{\overline{z}}\rangle\right)\left(1 + \langle g_z, v\rangle\right) - \frac{i}{2}\operatorname{sign}(\operatorname{Im} z)\left\langle (L - zI)^{-1} v, v\right\rangle. \tag{4.9}$$

Let us mention important properties of the function $\chi(z)$,

$$\overline{\chi(z)} = \chi(\overline{z}), \qquad |\chi(z)| \geq \frac{3}{8} \text{ for } |\operatorname{Im} z| \geq 8\|v\|_{L_2}^2. \tag{4.10}$$

Theorem 4.1. *Let the nonlocal potential satisfy $v \in L_2(-\infty,\infty)$. Then the main operator A is selfadjoint on the space $L_2(-\infty,\infty)$ and its spectrum fills the entire axis. The resolvent $(A-zI)^{-1}$, for $\operatorname{Im} z \neq 0$, is an integral operator,*

$$(A-zI)^{-1}h = (L-zI)^{-1}h + \frac{1}{\chi(z)} \sum_{j,k=1}^{2} a_{jk}(z) e_j(x;z) \langle h, e_k(\cdot,\overline{z})\rangle, \tag{4.11}$$

where $e_1(x;z) = g_z(x)$, $e_2(x;z) = \int g_z(x-s)v(s)ds$, $a_{11}(z) = \left\langle (L-zI)^{-1} v, v \right\rangle$, $a_{22}(z) = -\frac{i}{2}\operatorname{sign}(\operatorname{Im} z)$, $a_{12}(z) = -(1+\langle v, g_{\overline{z}}\rangle)$, $a_{21}(z) = -(1+\langle g_z, v\rangle)$, and the function $\chi(z)$ is defined by (4.9).

In other words, the resolvent of the main operator A differs from the resolvent of the selfadjoint free moment operator $L = i\frac{d}{dx}$ by a bounded rank 2 operator, and $\left[(A-zI)^{-1}\right]^* = (A-\overline{z}I)^{-1}$, which is equivalent to the operator A being selfadjoint.

Proof. An explicit form of a solution of system (4.8) with respect to ψ_+ and β, after being substituted into (4.7), yields (4.11) and gives an explicit form of the functions $a_{jk}(z)$ given in (4.11). Since (4.10) are satisfied, for $|\operatorname{Im} z| \geq 8\|v\|_{L_2}^2$, the resolvent $(A-zI)^{-1}$ exists and is a bounded operator, whereas the properties (4.9) lead to the identity $\left[(A-zI)^{-1}\right]^* = (A-\overline{z}I)^{-1}$. This means that the operator A is selfadjoint, hence problem (4.1) — (4.3) has a solution for any z, $\operatorname{Im} z \neq 0$. The latter holds if and only if $\chi(\lambda) \neq 0$ for $\operatorname{Im} z \neq 0$. Hence, identity (4.10) is true for all $\operatorname{Im} z \neq 0$. Spectrum of the operator A, since it is a rank 2 perturbation of the operator L, coincides with the whole axis. ∎

5. Inverse scattering problem

The inverse scattering problem (ISP) consists in constructing scattering data sufficient for a description of the diffuser, and also in constructing an effective algorithm for recovering unknown parameters of the diffuser. For the ISP for a Schrödinger operator with the usual potential, the condition for the operator to be selfadjoint implies that the potential is real. This facilitates the search for a solution. If the potentials are nonlocal, it could happen that they take complex values even if the main operator is selfadjoint. This leads to some difficulties in solving the ISP. Since the scattering problem with nonlocal potential for a Dirac system on half-axis is equivalent to a rank 2 perturbation of the free moment operator $L = i\frac{d}{dx}$, let us look at the difficulties that appear in the ISP by considering a scattering problem with rank 1 perturbation of the operator L. Consider the scattering problem for the equation

$$i\frac{d\psi}{dx} + v(x)\langle\psi, v\rangle_{L_2} = \lambda\psi, \qquad -\infty < x < \infty. \tag{5.1}$$

We assume that $v \in L_2(-\infty,\infty) \bigcap L_1(-\infty,\infty)$. Then any bounded solution of equation (5.1) has the form $\psi(x) = ae^{-i\lambda x} + o(1)$ for $x \to +\infty$, $\psi(x) = be^{-i\lambda x} + o(1)$ as $x \to -\infty$, where a is the amplitude of the incoming harmonic wave and the number b is the amplitude of the scattered wave. By repeating the calculations performed in Sections 2 and 4, we get the following result for the scattering problem (5.1).

Theorem 5.1. *Let the function v in equation* (5.1) *belong to the space $L_2(-\infty,\infty)$. Then, for any real λ there exists a unique solution of the scattering problem* (5.1) *for any* *value of the amplitude a of the incoming wave. This solution can be represented as*

$$\psi(x;\lambda) = ae^{-i\lambda x} - i\int_x^\infty e^{-i\lambda(x-s)}v(s)ds\,\langle\psi, v\rangle_{L_2}\,. \tag{5.2}$$

For $x \to -\infty$, the asymptotics of this solution is $\psi(x;\lambda) = be^{-i\lambda x} + o(1)$. We also have that $b = S(\lambda)a$, where

$$S(\lambda) = \frac{f^*(\lambda)}{f(\lambda)}, \qquad f(\lambda) = 1 - \frac{1}{2\pi}\int' \frac{|\widetilde{v}(p)|^2}{x-p}dp + \frac{i}{2}|\widetilde{v}(x)|^2, \tag{5.3}$$

and $\widetilde{v}(x) = \int\limits_{-\infty}^{\infty} e^{i\lambda s}v(s)ds$ and the integral $\int'$ in (5.3) is understood as its principal value.

Hence, identity (5.3) for the function $S(\lambda)$ shows that the ISP, considered as a problem of determining the function $v(x)$ in (5.1) from $S(\lambda)$, can not have unique solutions. The function $S(\lambda)$ only depends on $|\widetilde{v}(\lambda)|^2$. Thus, for small v, the first order approximation gives $S(\lambda) = 1$.

One encounters similar problems in ISP for the Dirac system (2.1) — (2.4) with nonlocal potential. Indeed, if the nonlocal are small, then for the first order approximation we have $S(\lambda) = 1 - 2i\operatorname{Re}\widetilde{v}_1(\lambda) - 2i\operatorname{Re}\widetilde{v}_2(-\lambda)$. If $\operatorname{Re}\widetilde{v}_1(\lambda) + \operatorname{Re}\widetilde{v}_2(-\lambda) = 0$, then $S(\lambda) = 1$, and the first order approximations of the potentials v_1 and v_2 can not be determined. To overcome this difficulty, one can supplement the scattering data E for the ISP with other observables, e.g., $\psi_+(\lambda) = \frac{1}{2}(\psi_1(0;\lambda) + \psi_2(0;\lambda) \stackrel{d}{=} S_+(\lambda)a$ and/or $\psi_1(0;\lambda) - \psi_2(0;\lambda) \stackrel{d}{=} S_-(\lambda)a$. Then, having supplemented $S(\lambda)$, $S_+(\lambda)$, and $S_-(\lambda)$, the first order approximations become $S_+(\lambda) = 1 - i\operatorname{Re}\widetilde{v}_1(\lambda) - i\operatorname{Re}\widetilde{v}_2(-\lambda) - \frac{i}{2}\widetilde{v}_1(\lambda) + \frac{i}{2}\widetilde{v}_2(-\lambda)$, $S_-(\lambda) = i\,[\widetilde{v}_1^*(\lambda) + \widetilde{v}_2^*(-\lambda)]$. Thus, by giving $S(\lambda)$ and $S_+(\lambda)$, or only $S_-(\lambda)$, for the first order approximation of $v_1(x)$ and $v_2(x)$ we get

$$v_1(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x}\varphi_\pm(\lambda)d\lambda, \qquad v_2(x) = \mp\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x}\varphi_\pm(\lambda)d\lambda,$$

where $\varphi_+(\lambda) = i\,[2S_+(\lambda) - S(\lambda) - 1]$, and $\varphi_-(\lambda) = iS_-^*(\lambda)$.

The problem of a correct setting for the ISP for a Dirac system o half-axis with nonlocal potential needs a detailed special study.

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