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PROBLEM OF DETERMINING A MULTIDIMENSIONAL THERMAL MEMORY IN A HEAT CONDUCTIVITY EQUATION

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Abstract. We consider a multidimensional integro-differential equation of heat conductivity with time-convolution integral in the right hand-side. The direct problem is represented by the Cauchy problem of determining the temperature of the medium for a known initial distribution of heat. We study the inverse problem of determining the kernel, in the integral part, that depends on time and spatial variables, if a solution of the direct problem is known on the hyperplane $x_n = 0$ for $t > 0$. With a use of the resolvent of the kernel, this problem is reduced to a study of a more convenient inverse problem. The later problem is replaced with an equivalent system of integral equations with respect to the unknown functions and, using a contractive mapping, we prove that the direct and the inverse problems have unique solutions.

1. Introduction. Formulation of the problem

Constitutive relations in the linear non-homogeneous diffusion processes with thermal memory contain space-dependent memory kernel [1], [2]. Often, in practice, these kernels are unknown functions. Inverse problems to determine time- and space-dependent kernels in parabolic integro-differential equations with several additional conditions have been studied by many authors [1]–[9]. In these papers there were proved existence, uniqueness and stability theorems. In the works [10]–[15] the authors discussed the linear inverse source and nonlinear inverse coefficient problems for parabolic integro-differential equations. Here also has been applied a numerical approach for solving such problems.

It should be noted that nowadays there are few publications where the problems of determining multidimensional memory would be studied.

In the work [8] the inverse problem of determining of the kernel depending on the time variable t and the $(n-1)$-dimensional spatial variable $x' = (x_1, \dots, x_{n-1})$ was studied. While the main part of the considered integro-differential equation is n-dimensional heat conduction operator and the integral term has the form of convolution for unknown functions that are solutions of direct and inverse problems. However in applications it is of great interest to study problems of the kernel determining when it is present in the convolution with an elliptic operator of the solution to the direct problem. In this article we consider one of these integro-differential equations for which the inverse problem is posed.

More precisely, we study an inverse problem of determining the functions $u(x,t)$, $K(x',t)$, $x = (x_1, x_2, \dots, x_{n-1}, x_n) = (x', x_n) \in \mathbb{R}^n,\ t > 0$ from the following equations:

$$u_t - \triangle u = \int_0^t K(x', t-\tau) \triangle u(x,\tau) d\tau, \quad (x,t) \in \mathbb{R}^n_T, \tag{1.1}$$

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$$u\Big|_{t=0} = \varphi(x), \quad x \in \mathbb{R}^n, \tag{1.2}$$

$$u\Big|_{x_n=0} = f(x', t), \quad 0 \le t \le T, \quad f(x', 0) = \varphi(x', 0), \tag{1.3}$$

where $\triangle$ is the Laplace's operator with respect to the variables $x = (x_1, \dots, x_n)$: $\triangle = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$, $\mathbb{R}_T^n = \{(x,t)|x = (x', x_n) \in \mathbb{R}^n,\ 0 < t < T\}$, $T > 0$ is an arbitrary fixed number.

Everywhere in this article, we assume that

$$\varphi(x) \in H^{l+8}\left(\mathbb{R}^n\right), \quad f(x', t) \in H^{l+6,(l+6)/2}\left(\bar{\mathbb{R}}_T^{n-1}\right),$$

$$\bar{\mathbb{R}}_T^{n-1} = \left\{(x', t)\Big| x' \in \mathbb{R}^{n-1},\ 0 \le t \le T\right\}, \quad l \in (0,1),$$

the spaces $H^l(Q)$, $H^{l,l/2}(Q_T)$ and their norms are defined in [16, pp. 18–27]. In what follows, for the norm of a function in the space $H^{l,l/2}(Q_T)$ (in concrete cases $Q_T = \mathbb{R}_T^n$ or $Q_T = \mathbb{R}_T^{n-1}$) depending on spatial and time variables we use the notation $|\cdot|_T^{l,l/2}$, while for functions depending only on spatial variables we use $|\cdot|^l$ (in this case $Q = \mathbb{R}^n$ or $Q = \mathbb{R}^{n-1}$).

At the beginning we prove the following assertion.

Lemma 1.1. *Let be $K(x', t) \in H^{l+2,(l+2)/2}\left(\bar{\mathbb{R}}_T^{n-1}\right)$. Then problem (1.1)–(1.3) is equivalent to the problem of finding the functions $u(x,t)$, $R(x',t)$ from the equation*

$$u_t = \triangle u - \int_0^t R(x', t-\tau) u_\tau(x,\tau) d\tau, \tag{1.4}$$

with the initial and additional conditions (1.2), (1,3), respectively, where $R(x',t)$ is the resolvent of the kernel $K(x',t)$.

Proof. Let $u(x,t)$ be a solution of Cauchy problem (1.1), (1.2). We note that the equation (1.1), for x fixed, can be considered as a Volterra integral equation of the second kind with the kernel $K(x',t)$ with respect to the function $\triangle u(x,t)$

$$\triangle u = -\int_0^t K(x', t-\tau) \triangle u(x,\tau) d\tau + u_t.$$

It follows from the general theory of integral equations (see e.g. [18, pp. 39–44]) that a solution of this equation is given by

$$\triangle u = \int_0^t R(x', t-\tau) u_\tau(x,\tau) d\tau + u_t.$$

This leads to (1.4). In (1.4), the kernels $K(x',t)$ and $R(x',t)$ are connected with the relation

$$K(x',t) = -R(x',t) + \int_0^t R(x', t-\tau) K(x',\tau) d\tau. \tag{1.5}$$

Thus, if it is possible to define the solution $u(x,t)$, $R(x',t)$ to the problem (1.4), (1.2), (1.3), then the function $K(x',t)$ is as the solution to integral equation (1.5). The lemma is proved. □

2. An auxiliary problem

For simplicity, we denote by $h(x',t)$ the function $R_t(x',t)$, i.e., $h(x',t) = R_t(x',t)$.

Lemma 2.1. *Problem (1.6), (1.4), (1.5) is equivalent to the following auxiliary problem of determining the functions* $\vartheta(x,t),\ h(x',t)$:

$$\begin{aligned}\vartheta_t - \triangle\vartheta = &- R(x',0)\vartheta \\ &- h(x',t)\triangle\varphi_{x_nx_n}(x) - \int_0^t h(x',\tau)\vartheta(x,t-\tau)d\tau, \quad (x,t)\in\mathbb{R}^n_T,\end{aligned} \tag{2.1}$$

$$\vartheta|_{t=0} = \triangle^2\varphi_{x_nx_n}(x) - R(x',0)\triangle\varphi_{x_nx_n}(x), \quad x\in\mathbb{R}^n, \tag{2.2}$$

$$\begin{aligned}\vartheta|_{x_n=0} = &f_{ttt}(x',t) - \triangle_{x'}f_{tt}(x',t) + R(x',0)f_{tt}(x',t) + h(x',t)\triangle\varphi(x',0) \\ &- \int_0^t h(x',\tau)f_{tt}(x',t-\tau)d\tau, \quad (x',t)\in\mathbb{R}^{n-1}_T,\end{aligned} \tag{2.3}$$

where $\vartheta(x,t) = u_{ttx_nx_n}(x,t),\ \triangle_{x'} = \sum_{i=1}^{n-1}\frac{\partial^2}{\partial x_i^2}$,

$$R(x',0) = \frac{\triangle^2\varphi(x',0) - f_{tt}(x',0)}{\triangle\varphi(x',0)}. \tag{2.4}$$

Proof. Introducing new function $\vartheta^{(1)}(x,t) = u_t(x,t)$, we differentiate the equalities (1.4) and (1.3) with respect to t. As a result, one has the problem of finding the functions $\vartheta^{(1)}(x,t),\ h(x',t)$ from the following equations:

$$\vartheta^{(1)}_t = \triangle\vartheta^{(1)} - R(x',0)\vartheta^{(1)} - \int_0^t h(x',t-\tau)\vartheta^{(1)}(x,\tau)d\tau, \quad (x,t)\in\mathbb{R}^n_T, \tag{2.5}$$

$$\vartheta^{(1)}|_{t=0} = \triangle\varphi(x), \quad x\in\mathbb{R}^n, \tag{2.6}$$

$$\vartheta^{(1)}|_{x_n=0} = f_t(x',t), \quad (x',t)\in\mathbb{R}^{n-1}_T, \quad \triangle\varphi(x',0) = f_t(x',0). \tag{2.7}$$

Here, the initial condition (2.6) were obtained from equality (1.4) by setting $t = 0$. The next problem will be obtained from (2.5)–(2.7) for the functions $\vartheta^2(x,t) = \vartheta^1_t(x,t)$, $h(x',t)$ in an analogous way,

$$\begin{aligned}\vartheta^{(2)}_t = &\triangle\vartheta^{(2)} - R(x',0)\vartheta^{(2)} \\ &- h(x',t)\triangle\varphi(x) - \int_0^t h(x',\tau)\vartheta^{(2)}(x,t-\tau)d\tau, \quad (x,t)\in\mathbb{R}^n_T,\end{aligned} \tag{2.8}$$

$$\vartheta^{(2)}|_{t=0} = \triangle^2\varphi(x) - R(x',0)\triangle\varphi(x), \quad x\in\mathbb{R}^n, \tag{2.9}$$

$$\vartheta^{(2)}|_{x_n=0} = f_{tt}(x',t), \quad (x',t)\in\mathbb{R}^{n-1}_T, \quad \triangle\varphi(x',0) = f_t(x',0). \tag{2.10}$$

Demanding equality of (2.9), (2.10) at $t = 0$ and $x_n = 0$ we get a relation from which (2.4) follows. Further, the function $R(x',0)$ will be considered as known.

Now let us denote by $\vartheta(x,t)$ the function $\vartheta^{(2)}_{x_nx_n}(x,t)$. Differentiating (2.8) and (2.9) twice with respect to x_n we obtain the equations (2.1) and (2.2). To derive an additional condition for $\vartheta(x,t)$ at $x_n = 0$ we allocate the function $\vartheta^{(2)}_{x_nx_n}(x,t)$ in $\triangle\vartheta^{(2)}(x,t)$ of (2.8), i.e., $\triangle\vartheta^{(2)}(x,t) = \vartheta^{(2)}_{x_nx_n}(x,t) + \triangle_{x'}\vartheta^{(2)}(x,t)$. Taking into account this fact and substituting $x_n = 0$ in (2.8), after some calculations we have (2.3). Thus, problem (1.4), (1.2), (1.3) is reduced to problem (2.1)–(2.3). It is not difficult to show that inverse transformations take place [8]. The lemma is proved. □

3. Existence and uniqueness

In this section, existence and uniqueness for problem (2.1)–(2.3) is proved using the contraction mapping principle [17, pp. 87–97]. The idea is to write the integral equations for the unknown functions $\vartheta(x,t),\ h(x',t)$ as a system with a nonlinear operator, and prove that this operator is a contraction mapping operator. The existence and uniqueness then follow immediately.

Definition. Let F be an operator defined on a closed set Ω which is a subset of a Banach space. F is called a contraction mapping operator in Ω if it satisfies the following two properties:

1) if $y\in\Omega$, then $Fx\in\Omega$ (i.e. F maps Ω into itself);

2) if $y,z\in\Omega$, then $\|Fy-Fz\|\le\rho\,\|y-z\|$ with $\rho<1$ (ρ - is a constant independent of y and z).

Lemma (contraction mapping principle [17, pp. 87–97]). *If F is a contraction mapping operator from Ω to Ω, then the equation*

$$y=Fy$$

has a unique solution $y_0\in\Omega$.

Now we reduce the Cauchy problem (2.1) and (2.2) to an integral equation with respect to the function $\vartheta(x,t)$. For this purpose in accordance with Poisson's formula we have

(3.1)
$$\begin{aligned}\vartheta(x,t)=&\int_{\mathbb{R}^n}G(x-\xi;t)\left[\triangle^2\varphi_{\xi_n\xi_n}(\xi)-R(\xi',0)\triangle\varphi_{\xi_n\xi_n}(\xi)\right]d\xi\\&-\int_0^t d\tau\int_{\mathbb{R}^n}G(x-\xi;t-\tau)h(\xi',\tau)\triangle\varphi_{\xi_n\xi_n}(\xi)d\xi\\&-\int_0^t d\tau\int_{\mathbb{R}^n}G(x-\xi,t-\tau)\left[-R(\xi',0)\vartheta(\xi,\tau)-\int_0^t h(\xi',\alpha)\vartheta(\xi,\tau-\alpha)d\alpha\right]d\xi,\end{aligned}$$

where $G(x;t)=\frac{1}{(2\sqrt{\pi t})^n}e^{\frac{-|x|^2}{4t}}$ is a fundamental solution of the heat operator $\frac{\partial}{\partial t}-\triangle,\ \xi=(\xi_1,\dots,\xi_n),\ \xi'=(\xi_1,\dots,\xi_{n-1}),\ d\xi=d\xi_1\cdots d\xi_n,\ |x|^2=x_1^2+\cdots+x_n^2$.

The integral equation for $h(x',t)$ is obtained from (3.1) considering it at $x_n=0$ and using equality (2.3),

(3.2)
$$\begin{aligned}h(x',t)=&\frac{1}{\triangle\varphi(x',0)}\Big\{-f_{ttt}(x',t)+\triangle_{x'}f_{tt}(x',t)-R(x',0)f_{tt}(x',t)\\&+\int_{\mathbb{R}^n}G(x'-\xi',\xi_n;t)\left[\triangle^2\varphi_{\xi_n\xi_n}(\xi)-R(\xi',0)\triangle\varphi_{\xi_n\xi_n}(\xi)\right]d\xi\Big\}\\&+\frac{1}{\triangle\varphi(x',0)}\Big\{\int_0^t h(x',\tau)f_{tt}(x',t-\tau)d\tau\\&+\int_0^t d\tau\int_{\mathbb{R}^n}G(x'-\xi',\xi_n;t-\tau)h(\xi',\tau)\triangle\varphi_{\xi_n\xi_n}(\xi)d\xi\Big\}\\&+\frac{1}{\triangle\varphi x',0)}\int_0^t d\tau\int_{\mathbb{R}^n}G(x'-\xi',\xi_n;t-\tau)\\&\times\left[-R(\xi',0)\vartheta(\xi,\tau)-\int_0^\tau h(\xi',\alpha)\vartheta(\xi,\tau-\alpha)d\alpha\right]d\xi,\end{aligned}$$

where $G(x'-\xi',\xi_n;t-\tau)=G(x-\xi;t-\tau)\Big|_{x_n=0}$.

Theorem (existence and uniqueness). *If the conditions $\varphi(x)\in H^{l+8}\left(\mathbb{R}^n\right),\ |\triangle\varphi(x',0)|\ge const>0,\ f(x',t)\in H^{l+6,(l+6)/2}\left(\bar{\mathbb{R}}^{n-1}_T\right),\ l\in(0,1),\ f(x',0)=\varphi(x',0),\ f_t(x',0)=$*

$\triangle\varphi(x',0)$ are met, then there exists a sufficiently small number $T>0$ such that a solution to the integral equations (3.1), (3.2) in the class of functions $\vartheta(x,t)\in H^{l+2,(l+2)/2}\left(\bar{\mathbb{R}}^n_T\right)$, $h(x',t)\in H^{l,l/2}\left(\bar{\mathbb{R}}^n_T\right)$ exists and is unique. Thus, there is a unique classical solution to the problem (2.1)–(2.3).

Proof. The system of equations (3.1), (3.2) is a closed system for the unknown functions $\vartheta(x,t),\ h(x',t)$ in the domain $\mathbb{R}^n_T$. It can be rewritten as a nonlinear operator equation,

$$\psi = A\psi, \tag{3.3}$$

where $\psi=(\psi_1,\ \psi_2)^*=(\vartheta(x,t),\ h(x',t))^*$, $*$ is the symbol of transposition, and according to the equations (3.1), (3.2), the operator $A\psi=[(A\psi)_1,\ (A\psi)_2]$ has the form

$$\begin{aligned}(A\psi)_1 &= \psi_{01}(x,t)-\int_0^t d\tau\int_{\mathbb{R}^n}G(x-\xi;t-\tau)\psi_2(\xi',\tau)\triangle\varphi_{\xi_n\xi_n}(\xi)d\xi\\ &-\int_0^t d\tau\int_{\mathbb{R}^n}G(x-\xi,t-\tau)\left[-R(\xi',0)\psi_1(\xi,\tau)-\int_0^t\psi_2(\xi',\alpha)\psi_1(\xi,\tau-\alpha)d\alpha\right]d\xi,\end{aligned} \tag{3.4}$$

$$\begin{aligned}(A\psi)_2 &= \psi_{02}(x',t)+\frac{1}{\triangle\varphi(x',0)}\Big\{\int_0^t\psi_2(x',\tau)f_{tt}(x',t-\tau)d\tau\\ &+\int_0^t d\tau\int_{\mathbb{R}^n}G(x'-\xi',\xi_n;t-\tau)\psi_2(\xi',\tau)\triangle\varphi_{\xi_n\xi_n}(\xi)d\xi\Big\}\\ &+\frac{1}{\triangle\varphi(x,0)}\int_0^t d\tau\int_{\mathbb{R}^n}G(x'-\xi',\xi_n;t-\tau)\times\\ &\times\left[-R(\xi',0)\psi_1(\xi,\tau)-\int_0^\tau\psi_2(\xi',\alpha)\psi_1(\xi,\tau-\alpha)d\alpha\right]d\xi.\end{aligned} \tag{3.5}$$

In (3.4) and (3.5) we introduced the notations

$$\begin{aligned}\psi_{01}(x,t) &= \int_{\mathbb{R}^n}G(x-\xi;t)\left[\triangle^2\varphi_{\xi_n\xi_n}(\xi)-R(\xi',0)\triangle\varphi_{\xi_n\xi_n}(\xi)\right]d\xi,\\ \psi_{02}(x',t) &= \frac{1}{\triangle\varphi(x',0)}\Big\{-f_{ttt}(x',t)+\triangle_{x'}f_{tt}(x',t)-R(x',0)f_{tt}(x',t)\\ &+\int_{\mathbb{R}^n}G(x'-\xi',\xi_n;t)\left[\triangle^2\varphi_{\xi_n\xi_n}(\xi)-R(\xi',0)\triangle\varphi_{\xi_n\xi_n}(\xi)\right]d\xi\Big\}.\end{aligned}$$

Denote $|\psi|^l_T=\max\left(|\psi_1|^l_{T_0},\ |\psi_2|^l_{T_0}\right)$, $T<T_0$ and consider in the space $H^{l,l/2}\left(\mathbb{R}^n_T\right)$ the set $S(T)$ of functions $\psi(x,t)$, which obey the inequality

$$|\psi-\psi_0|^l_T\le|\psi_0|^l_{T_0}, \tag{3.6}$$

where $\psi_0=(\psi_{01},\ \psi_{02})$ and $|\psi_0|^l_{T_0}=\max\left(|\psi_{01}|^l_{T_0},\ |\psi_{02}|^l_{T_0}\right)$.

It can be shown that for a sufficiently small T the operator A is a contraction mapping operator in $S(T)$. The theorem of existence and uniqueness then follows immediately from the contraction mapping principle.

First it is shown that A has the first property of a contraction mapping operator. Let $\psi\in S(T),\ T<T_0$. Then from the inequality (3.6), we have

$$|\psi_i|^l_T\le 2\,|\psi_0|^l_{T_0},\quad i=1,2.$$

It is easy to see that

$$
\begin{aligned}
|(A\psi)_1 - \psi_{01}|_T^l = \Bigg| &- \int_0^t d\tau \int_{\mathbb{R}^n} G(x-\xi; t-\tau)\psi_2(\xi',\tau)\triangle\varphi_{\xi_n\xi_n}(\xi)d\xi \\
&- \int_0^t d\tau \int_{\mathbb{R}^n} G(x-\xi, t-\tau)\Big[-R(\xi',0)\psi_1(\xi,\tau) \\
&- \int_0^\tau \psi_2(\xi',\alpha)\psi_1(\xi,\tau-\alpha)d\alpha\Big]d\xi\Bigg|_T^l \\
&\le 2\alpha_0(T)\left[\left|\triangle^2\varphi\right|^l + |R(x',0)|^l\right]|\varphi_0|_T^l + 4\alpha_1(T)\left(|\varphi_0|_T^l\right)^2,
\end{aligned}
$$

$$
\begin{aligned}
|(A\psi)_2 - \psi_{02}|_T^l = \Bigg| &\frac{1}{\triangle\varphi(x',0)}\left\{\int_0^t \psi_2(x',\tau)f_{tt}(x',t-\tau)d\tau\right. \\
&\left. + \int_0^t d\tau \int_{\mathbb{R}^n} G(x'-\xi',\xi_n;t-\tau)\psi_2(\xi',\tau)\triangle\varphi_{\xi_n\xi_n}(\xi)d\xi\right\} \\
&+ \frac{1}{\triangle\varphi(x',0)}\int_0^t d\tau \int_{\mathbb{R}^n} G(x'-\xi',\xi_n;t-\tau) \\
&\times\left[-R(\xi',0)\psi_1(\xi,\tau) - \int_0^\tau \psi_2(\xi',\alpha)\psi_1(\xi,\tau-\alpha)d\alpha\right]d\xi\Bigg|_T^l \le \left|(\triangle\varphi(x',0))^{-1}\right|^l \\
&\times\left[2\alpha_0(T)\left(|f_{tt}|_T^l + \left|\triangle^2\varphi\right|^l + |R(x',0))\,|\varphi_0|_T^l\right|^l + 4\alpha_1(T)\left(|\varphi_0|_T^l\right)^2\right],
\end{aligned}
$$

where $\alpha_i(T) \to 0$ at $T \to 0$, $i = 0, 1$. Therefore, if we choose T $(T < T_0)$ so that the following inequalities will be satisfied:

$$
\begin{aligned}
&2\alpha_0(T)\left(\left|\triangle^2\varphi\right|^l + |R(x',0)|^l\right) + 4\alpha_1(T)\left(|\varphi_0|_T^l\right) \le 1, \\
&2\left|(\triangle\varphi(x',0))^{-1}\right|^l\left[\alpha_0(T)\left(|f_{tt}|_T^l + \left|\triangle^2\varphi\right|^l + |R(x',0)|\right) + 2\alpha_1(T)\left(|\varphi_0|_T^l\right)\right] \le 1,
\end{aligned}
\tag{3.7}
$$

then the operator A has the first property of a contraction mapping operator, i.e., $A\psi \in S(T)$.

Consider next the second property of a contraction mapping operator for A. Let $\psi^{(1)} = \left(\psi_1^{(1)},\ \psi_2^{(1)}\right) \in S(T)$, $\psi^{(2)} = \left(\psi_1^{(2)},\ \psi_2^{(2)}\right) \in S(T)$. Then we have

$$
\begin{aligned}
&\left|\left(A\psi^{(1)} - A\psi^{(2)}\right)_1\right|_T^l \\
&= \Bigg| - \int_0^t d\tau \int_{\mathbb{R}^n} G(x-\xi;t-\tau)\left[\psi_2^{(1)}(\xi',\tau) - \psi_2^{(2)}(\xi',\tau)\right]\triangle^2\varphi(\xi)d\xi \\
&- \int_0^t d\tau \int_{\mathbb{R}^n} G(x-\xi,t-\tau)\Big\{-R(\xi',0)\left[\psi_1^{(1)}(\xi,\tau) - \psi_1^{(2)}(\xi,\tau)\right] \\
&- \int_0^\tau \left[\psi_2^{(1)}(\xi',\alpha)\psi_1^{(1)}(\xi,\tau-\alpha) - \psi_2^{(2)}(\xi',\alpha)\psi_1^{(2)}(\xi,\tau-\alpha)\right]d\alpha\Big\}d\xi\Bigg|_T^l.
\end{aligned}
$$

Here the integrand in the last integral can be estimated as follows:

$$\left|\psi_2^{(1)}\psi_1^{(1)} - \psi_2^{(2)}\psi_1^{(2)}\right|_T^l = \left|\left(\psi_2^{(1)} - \psi_2^{(2)}\right)\psi_1^{(1)} + \psi_2^{(2)}\left(\psi_1^{(1)} - \psi_1^{(2)}\right)\right|_T^l$$
$$\leq 2\left|\psi^{(1)} - \psi^{(2)}\right|_T^l \max\left(\left|\psi_1^{(1)}\right|_T^l, \ \left|\psi_2^{(2)}\right|_T^l\right) \leq 4\left|\varphi_0\right|_T^l \left|\psi^{(1)} - \psi^{(2)}\right|_T^l.$$

Therefore

$$\left|\left(A\psi^{(1)} - A\psi^{(2)}\right)_1\right|_T^l$$
$$\leq \left[2\alpha_0(T)\left(\left|\triangle^2\varphi\right|^l + \left|R(x',0)\right|^l\right) + 8\alpha_1(T)\left|\varphi_0\right|_T^l\right]\left|\psi^{(1)} - \psi^{(2)}\right|_T^l.$$

The second component $A\psi$ can be estimated in an analogous way,

$$\left|\left(A\psi^{(1)} - A\psi^{(2)}\right)_2\right|_T^l \leq \left|\left(\triangle\varphi(x',0)\right)^{-1}\right|^l$$
$$\times\left[2\alpha_0(T)\left(\left|f_{tt}\right|_T^l + \left|\triangle^2\varphi\right|^l + \left|R(x',0)\right|\right) + 8\alpha_1(T)\left|\varphi_0\right|_T^l\right]\left|\psi^{(1)} - \psi^{(2)}\right|_T^l.$$

Hence, $\left|\left(A\psi^{(1)} - A\psi^{(2)}\right)\right|_T^l < \rho\left|\psi^{(1)} - \psi^{(2)}\right|_T^l$, where $\rho < 1$, if T satisfies the conditions

$$2\alpha_0(T)\left(\left|\triangle^2\varphi\right|^l + \left|R(x',0)\right|^l\right) + 8\alpha_1(T)\left|\varphi_0\right|_T^l \leq \rho < 1,$$
$$\left|\left(\triangle\varphi(x',0)\right)^{-1}\right|^l\left[2\alpha_0(T)\left(\left|f_{tt}\right|_T^l + \left|\triangle^2\varphi\right|^l + \left|R(x',0)\right|\right) + 8\alpha_1(T)\left|\varphi_0\right|_T^l\right] \leq \rho < 1, \tag{3.8}$$
$$T < T_0.$$

It is not difficult to see that from fulfilling the inequalities (3.8) it follows that inequalities (3.7) hold true. This indicates that at any T satisfying conditions (3.8), A satisfies both properties of a contraction mapping operator, i.e., A realizes a contracted mapping of the set $S(T)$ onto itself. Then, according to Banach theorem (see, for instance, [17, pp. 87–97]), in the set $S(T)$ there exists only one fixed point of transformations, i.e., there exists only one solution to (3.3). Hence, solving system of (3.1), (3.2), for example, by the method of successive approximations, we uniquely find the functions $\vartheta(x,t)$, $h(x',t)$ which belong to $H^{l,l/2}\left(\mathbb{R}_T^n\right)$ and $H^{l,l/2}\left(\mathbb{R}_T^{n-1}\right)$, respectively. Moreover, it follows from the general theory of parabolic equations [19, pp. 380–384] (see also [8]), under the conditions of the theorem, the function $\vartheta(x,t)$, as a solution to integral equation (3.1), belongs to $H^{l+2,(l+2)/2}\left(\mathbb{R}_T^n\right)$. The theorem is proved. □

Since $h(x',t) = R_t(x',t)$, the obtained function $h(x',t)$ will be used to determine the function $R(x',t)$ using the formula

$$R(x',t) = R(x',0) - \int_0^t h(x',\tau)d\tau, \quad (x',t) \in \mathbb{R}_T^{n-1},$$

where $R(x',0)$ is the known function determined by (2.4). Then solving, at every fixed x', the integral equation (1.5) we uniquely find $K(x',t)$. Due to the proved theorem we conclude that the problem (1.1)–(1.3) has a unique solution such that $\vartheta(x,t) \in H^{l+6,(l+6)/2}\left(\mathbb{R}_T^n\right)$, $K(x',t) \in H^{l+2,(l+2)/2}\left(\mathbb{R}_T^{(n-1)}\right)$.

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