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# SOME RESULTS ON ALMOST BANACH-SAKS OPERATORS

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Abstract. We introduce and study a new class of operators that we call almost Banach-Saks operators. We characterize Banach lattices under which each operator is almost Banach-Saks. Furthermore, we study the relationship between this class and other classes of operators, some other interesting results are also obtained.

## 1. Introduction

Let us recall that a sequence $(x_n)$ in a Banach space is said to be Cesáro convergent if its Cesáro means are norm-convergent and a Banach space $X$ is said to have the Banach-Saks property if every bounded sequence has a Cesáro convergent subsequence (i.e. every bounded sequence $(x_n)$ in $X$ admits a subsequence $(z_n)$ such that its Cesáro means $(\frac{1}{n}\sum_{k=1}^{n} z_k)$ are norm-convergent in $X$). A Banach space $X$ is said to have the weak Banach-Saks property (or the Banach-Saks-Rosenthal property) if every weakly null sequence has a Cesáro convergent subsequence (i.e. every weakly null sequence $(x_n)$ in $X$ admits a subsequence $(y_n)$ such that its Cesáro means $(\frac{1}{n}\sum_{k=1}^{n} y_k)$ are norm-convergent in $X$).

Flores and Ruiz [7] introduced the notion of disjointly Banach-Saks space (Banach lattice with the disjoint Banach-Saks property) as another version of the properties discussed above. Recall that a Banach lattice $E$ is said to have the disjoint Banach-Saks property, if for every bounded disjoint sequence $(x_n)$ in $E$, there exists a subsequence $(x_{n_k})_k$ of $(x_n)$ whose Cesáro means are norm convergent, as an examples of such Banach lattices we have $c_0$, $l_p$ with $1 < p < \infty$ and all uniformly convex Banach lattices (see [7]).

The aim of this paper is to introduce a new class of operators, that we call almost Banach-Saks operators. Our definition is based on disjoint Banach-Saks property (Definition 3.1). Mainly, we establish some characterizations of this class of operators and its relationship with other known classes of operators. We also obtain some results about duality and domination problems.

## 2. Preliminaries

We will use the term operator $T : X \longrightarrow Y$ between two Banach spaces to mean a bounded linear mapping. $T'$ will be the adjoint operator defined from $Y'$ into $X'$ by $T'(f)(x) = f(T(x))$ for each $f \in Y'$ and each $x \in X$. An operator $T$ between two Banach lattices $E$ and $F$ is positive if $T(x) \geq 0$ in $F$ whenever $x \geq 0$ in $E$. It is well known that each positive linear mapping on a Banach lattice is continuous. For terminology concerning Banach lattice theory and positive operators, we refer the reader to [1].

To state our results we need to fix some notations and recall some definitions:

- A Banach lattice is a Banach space $(E, \|.\|)$ such that $E$ is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$,

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we have $\|x\| \leq \|y\|$. Note that the topological dual $E'$, equipped with the dual norm and the dual order is also a Banach lattice.

- A norm $\|.\|$ of a Banach lattice $E$ is order continuous if for each generalized sequence $(x_\alpha)$ such that $x_\alpha \downarrow 0$, the sequence $(x_\alpha)$ converges to 0 for the norm $\|.\|$, where the notation $x_\alpha \downarrow 0$ means that the sequence $(x_\alpha)$ is decreasing, its infimum exists and $\inf x_\alpha = 0$.
- A vector lattice $L$ is Dedeking $\sigma$-complete if every majorized countable nonempty subset of $L$ has a supremum.
- A Banach lattice $E$ is said to be KB-space, if every increasing norm bounded sequence of $E^+$ is norm convergent.
- A vector lattice $L$ is said to be $\sigma$-laterally complete, if the supremum of every disjoint sequence of $L^+$ exists in $L$.
- A sequence $(x_n)$ in Banach space is said to be Cesáro convergent if its Cesáro means are norm-convergent.
- A Banach space $X$ is said to have the Banach-Saks property if every bounded sequence has Cesáro convergent subsequence (i.e. every bounded sequence $(x_n)$ in $X$ admits a subsequence $(z_n)$ such that its Cesáro means $(\frac{1}{n}\sum_{k=1}^{n} z_k)$ are norm-convergent in $X$.)
- A Banach space $X$ is said to have the weak Banach-Saks property (or the Banach-Saks-Rosenthal property) if every weakly null sequence has Cesáro convergent subsequence (i.e. every weakly null sequence $(x_n)$ in $X$ admits a subsequence $(y_n)$ such that its Cesáro means $(\frac{1}{n}\sum_{k=1}^{n} y_k)$ are norm-convergent in $X$ )
- A bounded subset $A$ of a Banach lattice $E$ is said to be L-weakly compact, if $\|x_n\| \to 0$ for every disjoint sequence $(x_n)$ in the solid hull of $A$.
- A subset $A$ of a Banach lattice $E$ is said to be almost order bounded, if for any $\epsilon > 0$ there exists $x \in E_+$ such that $A \subset [-x, x] + \epsilon B_E$. And from The Riesz decomposition property, it follows that $A \subset [-x, x] + \epsilon B_E$ if and only if $\sup_{u\in A} \|(u - |x|)^+\| \leq \epsilon$.
- We recall from [7] that a Banach lattice $E$ with an order continuous norm is said to have the subsequence splitting property, if for every bounded sequence $(x_n)$ in E, there are a subsequence $(x_{n_k})_k$ of $(x_n)$ and two sequences $(y_n)$, $(z_n)$ in E, with $|y_k| \wedge |z_k| = 0$ and $x_{n_k} = y_k + z_k$ such that;
  - $\{y_n; n \in \mathbb{N}\}$ is L-weakly compact;
  - $(z_n)$ is a disjoint sequence.

  As examples of Banach lattices satisfying this property we have the Banach lattices which does not uniformly contain copies of $\ell_n^\infty$, for all $n \in \mathbb{N}$ and the rearrangement invariant function space which contains no isomorphic copy of $c_0$ (see [7]).
- A net $(x_\alpha)$ in a vector lattice $L$ is said to be uo-converge to $x$, if $|x_\alpha - x| \wedge u \xrightarrow{o} 0$ for all $u \in E^+$, where the notation $\xrightarrow{o}$ means convergence in order, we mention that order convergence implies uo-convergence and they coincide for order bounded nets. The Reader can find more details about this notion of convergence in [8].
- A Banach lattice $E$ is said to have the Komlós property if, for each bounded sequence $(x_n)$ in E there exists a subsequence $(y_n)$ of $(x_n)$ and $y \in$ E such that Cesáro means of every subsequence of $(y_n)$ is uo-convergent to $y$.
- An operator $T : E \longrightarrow Y$ is called M-weakly compact, if $T(x_n)$ is norm-null for every bounded disjoint sequence $(x_n)$ in $E$.

## 3. Main results

**Definition 3.1.** *An operator $T : E \longrightarrow X$ from a Banach lattice $E$ into a Banach space $X$ is said to be almost Banach-Saks, if for each bounded disjoint sequence $(x_n)$ in $E$, $(T(x_n))$ has a subsequence whose Cesáro means are norm convergent in $X$.*

It results obviously that a Banach lattice $E$ has disjoint Banach-Saks property if, and only if, the identity operator $Id_E$ is almost Banach-Saks. It is easily observed that this class of operators contains compact and M-weakly compact operators.

**Proposition 3.2.** *Let $E$, $F$ be two Banach lattices and $X$, $Y$ be two Banach spaces. Then*

1. *If $T : E \to X$ is an almost Banach-Saks operator then, for each operator $S : X \to Y$ the composed operator $S \circ T$ is almost Banach-Saks.*
2. *If $T : E \to F$ is a disjointness preserving operator and $S : F \to Y$ is an almost Banach-Saks operator then, the composed operator $S \circ T$ is almost Banach-Saks.*

*Proof.* The proof is straightforward. □

As consequence of the above proposition we have the following:

**Corollary 3.3.** *Let $E$ be a Banach lattice. Then, the following statements are equivalent:*

1. *each operator $T : E \to E$ is almost Banach-Saks.*
2. *the identity operator $Id_E$ of $E$ is almost Banach-Saks.*
3. *$E$ has disjoint Banach-Saks property.*

We note that there exists a weak Banach-Saks operator which is not almost Banach-Saks. In fact, the identity operator $Id_{\ell^1}$ of $\ell^1$ is weak Banach-Saks (because $\ell^1$ has the weak Banach-Saks property), but fails to be almost Banach-Saks (because $\ell^1$ does not have disjoint Banach-Saks property). For the converse case, example 6.10 [8] shows that the Bochner space $L_p(c_0) = L_p([0.1], c_0)$ where $1 < p < \infty$ has disjoint Banach-Saks property, but fails to have the weak Banach-Saks property. This yields the identity operator $Id_{L_p(c_0)}$ of $L_p(c_0)$ almost Banach-Saks, but it is not a weak Banach-Saks operator.

We have the following proposition.

**Proposition 3.4.** *Let $X$ and $Y$ be two Banach spaces and let $G$ be a Banach lattice such that the norm of $G'$ is order continuous. If $T : X \longrightarrow Y$ is a weak Banach-Saks operator then, for each operator $S : G \longrightarrow X$, the composed operator $T \circ S$ is almost Banach-Saks.*

*Proof.* Let $(x_n)$ be a disjoint bounded sequence in $G$. Since the norm of $G'$ is order continuous, it follows from Theorem 2.4.14 [10] that $(x_n)$ is a weakly null sequence, and hence $(S(x_n))$ is a also a weakly null sequence in $X$ . As $T : X \longrightarrow Y$ is a weak Banach-Saks operator then, $(T \circ S(x_n))$ has a Cesáro convergent subsequence. Thus $T \circ S$ is an almost Banach-Saks operator. □

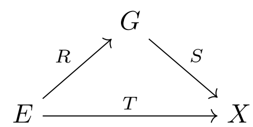
As consequence, we have the following result.

**Corollary 3.5.** *Let $E$ be a Banach lattice such that the norm of $E'$ is order continuous and $Y$ be a Banach space with weak Banach-Saks property. Then, each operator $T : E \longrightarrow Y$ is almost Banach-Saks.*

Now we are in position to give our first major result.

**Theorem 3.6.** *Let $E$ be a $\sigma$-laterally complete Banach lattice and $X$ a Banach space such that $c_0$ does not embed in $X$. Then, each operator $T : E \longrightarrow X$ is M-weakly compact and hence is almost Banach-Saks.*

*Proof.* Using Theorem 4.63 [1], the operator $T$ admits the following factorization:




where $G$ is a KB-space and $R$ is a lattice homomorphism. Now, let $(x_n)$ be a disjoint bounded sequence in $E$, since $E$ is $\sigma$-laterally complete, we infer that $(x_n)$ has a supremum, and hence it is order bounded, and thus $(R(x_n))$ is an order bounded disjoint sequence in the KB-space $G$. By Theorem 2.4.2 [10], $\|R(x_n)\| \underset{n\to+\infty}{\longrightarrow} 0$, hence $\|T(x_n)\| = \|S \circ R(x_n)\| \underset{n\to+\infty}{\longrightarrow} 0$, we conclude that $T$ is M-weakly compact, and therefore $T$ is almost Banach-Saks operator. □

In the following result, we give necessary conditions on Banach lattices $E$ and $F$ for which each operator $T : E \longrightarrow F$ is almost Banach-Saks.

**Theorem 3.7.** *Let $E$ be a Banach lattice and $Y$ be a Banach space. If each operator $T : E \longrightarrow Y$ is almost Banach-Saks then, $E'$ has order continuous norm or $Y$ has the Banach-Saks property.*

*Proof.* We proceed by contradiction. Assume that neither the norm of $E'$ is order continuous nor $Y$ has the Banach-Saks property. Then, by Theorem 2.4.14 and Proposition 2.3.11 [10] $E$ contains a complemented copy of $\ell^1$ and there exists a positive projection $p : E \longrightarrow \ell^1$, on the other hand since $Y$ does not have the Banach-Saks property then, there exists $(y_n)$ a bounded sequence in $Y$ with no Cesáro convergent subsequences.

We consider the following operator:

$$S : \begin{array}{ccc} \ell^1 & \longrightarrow & Y \\ (\lambda_n)_n & \longmapsto & \sum_{n=1}^{\infty} \lambda_n y_n \end{array},$$

$S$ is well defined.

Now, we consider the composed operator $T = S \circ P$. To end the proof we have to claim that $T$ is not an almost Banach-Saks operator. Otherwise, since the injection $i : \ell^1 \longrightarrow E$ is a lattice homomorphism then, $i \circ T$ will be an almost Banach-Saks operator, but by taking $(e_n)$ the unit basis of $\ell^1$ as a bounded and disjoint sequence, we have $i \circ T(e_n) = y_n$ with no Cesáro convergent subsequence, which is a contradiction. □

Note that the Banach-Saks property does not imply the Schur property, in fact $L_2[0.1]$ has the Banach-Saks property but does not have the Schur property and conversely, $\ell^1$ has the Schur property but it lacks the Banach-Saks property.

As consequence of the above theorem, we have the following result.

**Corollary 3.8.** *Let $E$ be a Banach lattice and $Y$ be a Banach space, such that $Y$ has the Schur property. The following statements are equivalent:*

1) *each operator $T : E \longrightarrow Y$ is almost Banach-Saks;*
2) *one of the following condition is holds:*
    a) *$E'$ has order continuous norm;*
    b) *$Y$ has the Banach-Saks property.*

*Proof.* **Sufficient condition**: $(2-a) \Rightarrow (1)$ Let $(x_n)$ be a bounded disjoint sequence in $E$. Since the norm of $E'$ is order continuous then, it follows from Theorem 2.4.14 [10] that $(x_n)$ is weakly null and hence $(T(x_n))$ is weakly null in $F$. By the Schur property, $(T(x_n))$ is norm-null which implies that $T$ is M-weakly compact and hence $T$ is almost Banach-Saks.

$(2-b) \Rightarrow (1)$ In this case, each operator $T : E \longrightarrow Y$ is Banach-Saks and hence is almost Banach-Saks.

**Necessary condition**: Follows from the theorem 3.7. □

In the following result we give some sufficient conditions under which order bounded operator will be almost Banach-Saks.

**Theorem 3.9.** *Let $E$ and $F$ be two Banach lattices such that $E$ is $\sigma$-laterally complete and $F$ is a KB-space. Then, each order bounded operator $T : E \longrightarrow F$ is almost Banach-Saks.*

*Proof.* Let $T : E \longrightarrow F$ be an order bounded operator and let $(x_n)$ be a bounded disjoint sequence in $E$. Since $E$ is $\sigma$-laterally complete, then $(x_n)$ is an order bounded sequence in $E$ and hence $T(x_n)$ will be an order bounded sequence in $F$. As $F$ is a KB-space, it follows from Corollary 5.14 [8] that $F$ has the Komlós property, hence there exist $(T(x_{n_k}))_k$ a subsequence of $(T(x_n))$ and $y \in F$ such that the Cesáro means of any subsequence of $(T(x_{n_k}))_k$ is uo-convergent to $y$. On the other hand, we have that the subsequence $(T(x_{n_k}))_k$ is order bounded and hence its Cesáro means $(\frac{1}{N}\sum_{k=1}^{N} T(x_{n_k}))_N$ are also order bounded, so $\frac{1}{N}\sum_{k=1}^{N} T(x_{n_k}) \xrightarrow{o} y$. By choosing any arbitrary subsequence of $(T(x_{n_k}))_k$, we can conclude from the order continuity of $F$ that the Cesáro means of the chosen subsequence must be norm convergent. Therefore, $T$ is almost Banach-Saks. □

**Remark 3.10.**
- *We can get the same result given in the previous theorem, if we replace the condition "$E$ is $\sigma$-laterally complete" by "$E$ has an order unit".*
- *In Theorem 3.9, if we change order bounded operator by positive operator and the condition on $F$ by the fact that $F$ has order continuous norm, then the operator $T$ will be M-weakly compact. Indeed, let $(x_n)$ be a bounded disjoint sequence in $E$, since $E$ is $\sigma$-laterally complete, it follows from Theorem 7.8* [2] *that $T$ is $\sigma$-order continuous, on the other hand $(x_n)$ will be order bounded and disjoint sequence in $E$, hence $x_n \xrightarrow{o} 0$ and by $\sigma$-order continuity of $T$, we have $T(x_n) \xrightarrow{o} 0$. Now, since $F$ has order continuous norm, it follows that $T(x_n) \xrightarrow{\|\cdot\|} 0$. Thus, $T$ is M-weakly compact (in particular, $T$ is an almost Banach-Saks operator).*

As it was studied, the problem of domination has been solved for many classes of operators between Banach lattices, such as Banach-Saks operators [7]. In the same direction, we present the following result which is a generalization of Lemma 2.5 [7].

**Proposition 3.11.** *Let $E$ and $F$ be two Banach lattices such that $E'$ has order continuous norm. If $T$ and $S$ are two operators from $E$ into $F$ satisfying $0 \leq S \leq T$ and $T$ is almost Banach-Saks, then $S$ is almost Banach-Saks.*

*Proof.* The proof of this proposition is the same of that in Lemma 2.5 [7]. □

**Remark 3.12.** *The assumption $E'$ has order continuous norm in Theorem 3.11 is essential. Indeed, the example 2.9 mentioned in* [7] *answer positively to this fact, since $\ell^1$ does not have disjoint Banach-Saks property.*

To obtain more characterization of almost Banach-Saks operators, we need the next two Lemmas, the first one represent variant of Kadeč-Pelczyński dichotomy.

**Lemma 3.13.** [8] *Let $E$ be a Banach lattice with order continuous norm and $(x_n)_n$ a bounded sequence in $E$. If $x_n \xrightarrow{uo} 0$ in $E$, then there exists a subsequence $(x_{n_k})_k$ of $(x_n)$ and a disjoint sequence $(d_k)$ in $E$ such that $\|x_{n_k} - d_k\| \longrightarrow 0$.*

**Lemma 3.14.** [8] *Let $E$ be a Banach lattice with order continuous norm and $(x_n)$ a bounded sequence in $E$. Suppose that every subsequence of $(x_n)$ has further subsequence* *whose Cesàro means are almost order bounded. Then, there exist a subsequence $(x_{n_k})_k$ of $(x_n)$ and a vector $x \in E$ such that the Cesàro means of any subsequence of $(x_{n_k})_k$ are uo-convergent and norm-convergent to $x$.*

**Proposition 3.15.** *Let $E$ be a Banach lattice with order continuous norm and $X$ a Banach space. Then for an operator $T : E \longrightarrow X$ the following statements are equivalent:*

(1) *$T$ is almost Banach-Saks;*
(2) *for each bounded sequence $(x_n)$ in $E$, such that $x_n \xrightarrow{uo} 0$, $(T(x_n))$ has a Cesàro convergent subsequence.*

*Proof.* (1) $\Rightarrow$ (2) Let $(x_n)$ be a bounded sequence in $E$ such that $x_n \xrightarrow{uo} 0$, since $E$ has an order continuous norm, it follows from Lemma 3.13 that there exist $(x_{n_k})_k$ a subsequence of $(x_n)$ and a disjoint sequence $(d_k)_k$ in $E$ such that $\|x_{n_k} - d_k\| \longrightarrow 0$, and hence $\|T(x_{n_k}) - T(d_k)\| \longrightarrow 0$. On the other hand, since $T$ is almost Banach-Saks operator, it follows that $(T(d_k))$ has a Cesàro convergent subsequence, and therefore $(T(x_{n_k}))_k$ has a Cesàro convergent subsequence.

(2) $\Rightarrow$ (1) Let $(x_n)$ be a bounded disjoint sequence in $E$ , by Corollary 3.6 [8], we have $x_n \xrightarrow{uo} 0$, thus $(T(x_n))$ has a Cesàro convergent subsequence and hence $T$ is almost Banach-Saks operator. □

**Proposition 3.16.** *Let $E$ and $F$ be two Banach lattices such that $F$ has an order continuous norm and let $T : E \longrightarrow F$ be an operator. Then, the following statements are equivalent:*

(1) *$T$ is almost Banach-Saks;*
(2) *for each bounded disjoint sequence $(x_n)$ in $E$, $(T(x_n))$ has a subsequence whose Cesàro means are almost order bounded.*

*Proof.* (1) $\Rightarrow$ (2) Follows from the fact that every norm-convergent sequence is almost order bounded.

(2) $\Rightarrow$ (1) Let $(x_n)$ be a bounded disjoint sequence in $E$. Since any subsequence $(x_{n_k})$ of $(x_n)$ is also a bounded and disjoint sequence in $E$, the assumption of (2) yields, that $(T(x_{n_k}))$ has a subsequence whose Cesàro means are almost order bounded. In particular, any subsequence of $(T(x_n))$ has a further subsequence whose Cesàro means are almost order bounded. Now, since $F$ has an order continuous norm, it follows from Lemma 3.14 that $(T(x_n))$ has a Cesàro norm-convergent subsequence, consequently $T$ is an almost Banach-Saks operator. □

We note that almost Banach-Saks operator need not be Banach-Saks operator in general. Indeed, the identity operator $Id_{c_0}$ of the Banach lattice $c_0$ is almost Banach-Saks but fails to be Banach-Saks.

In the following result, we give sufficient conditions under which each almost Banach-Saks operator is Banach-Saks.

**Theorem 3.17.** *Let $E$ be a Banach lattice with the subsequence splitting property. Then, each almost Banach-Saks operator from $E$ into an arbitrary Banach space $X$ is Banach-Saks.*

*Proof.* Let $T : E \longrightarrow X$ be an almost Banach-Saks operator and $(x_n)$ a bounded sequence in $E$, since $E$ has the subsequence splitting property, there exists a subsequence $(x_{\varphi(n)})$ of $(x_n)$, such that for all $n \in \mathbb{N}$

$$x_{\varphi(n)} = y_n + z_n,$$

where the sequence $(y_n)$ is L-weakly compact in $E$ and $(z_n)$ is a bounded disjoint sequence in $E$. By Proposition 3.6.2 [10] the sequence $(y_n)$ is almost order bounded, so the Cesàro means of every subsequence of $(y_n)$ are almost order bounded. By Lemma 3.14, there exist a subsequence $(y_{\psi(n)})$ of $(y_n)$ and a vector $x \in E$ such that the Cesàro means of any subsequence of $(y_{\psi(n)})$ are norm-convergent to $x$. We note that $x_{\varphi\circ\psi(n)} = y_{\psi(n)} + z_{\psi(n)}$. Since $(z_{\psi(n)})$ is also a bounded disjoint sequence, then by the fact that $T$ is an almost Banach-Saks operator, $(T(z_{\psi(n)})$ has a Cesàro convergent subsequence denoted by $(T(z_{\psi\circ\sigma(n)}))$. As, it was shown above $(y_{\psi\circ\sigma(n)})$ is Cesàro convergent to $x$ and hence $(T(y_{\psi\circ\sigma(n)}))$ is Cesàro convergent to $T(x)$. Combining the previous facts and the fact that $x_{\varphi\circ\psi\circ\sigma(n)} = y_{\psi\circ\sigma(n)} + z_{\psi\circ\sigma(n)}$, we conclude that $(T(x_{\varphi\circ\psi\circ\sigma(n)}))$ is a Cesàro convergent subsequence of $T(x_n)$, therefore $T$ is a Banach-Saks operator. Where $\varphi$, $\psi$ and $\sigma$ are increasing mappings from $\mathbb{N}$ to $\mathbb{N}$. □

We note that there exists weakly compact operator which is not almost Banach-Saks, in fact the identity operator of the Baerstein space constructed in [4] is weakly compact but fails to be almost Banach-Saks. And there exists almost Banach-Saks operator which is not weakly compact, indeed the identity operator $Id_{c_0}$ of the Banach lattice $c_0$ is almost Banach-Saks but is not weakly compact.

Using Lemma 1 [5] and the above theorem, we remark that if $E$ is a Banach lattice with the subsequence splitting property, then each almost Banach-Saks operator from $E$ into an arbitrary Banach space $X$ is weakly compact.

In the following proposition, we give sufficient condition under which each weakly compact operator is almost Banach-Saks;

**Proposition 3.18.** *Let $E$ be Banach lattice such that $E'$ has the positive Schur property and $X$ a Banach space. Then, each weakly compact operator from $E$ into $X$ is almost Banach-Saks.*

*Proof.* Assume that $T : E \longrightarrow X$ is weakly compact operator, since $E'$ has the positive Schur property, by Theorem 3.3 [6] we infer that $T$ is M-weakly compact and therefore is an almost Banach-Saks operator. □

**Proposition 3.19.** *Let $E$ be a Banach lattice and $X$ be a Banach space. If each weakly compact operator $T : E \longrightarrow X$ is almost Banach-Saks, then one of the following conditions is valid:*

(1) *$X$ has the weak Banach-Saks property;*
(2) *$E'$ has an order continuous norm.*

*Proof.* We suppose that the assertions (1) and (2) are not satisfied, then there exists $(y_n)$ a weakly null sequence in $X$ which does not contain any Cesáro convergent subsequence. We set the following operator:

$$S_1 : \begin{array}{ccc} \ell^1 & \longrightarrow & X \\ (\lambda_n)_n & \longmapsto & \sum_{n=1}^{\infty} \lambda_n y_n \end{array},$$

which is a weakly compact operator (see Theorem 5.26 [1]), and since $S_1(e_n) = y_n$, where $(e_n)$ is the unit basis sequence of $\ell^1$, then the operator $S_1$ cannot be almost Banach-Saks. On the other hand, since the norm of $E'$ is not order continuous then it follows from Theorem 2.4.14 [10] that $E$ contains a sublattice isomorphic to $\ell^1$, and by Proposition 2.3.11 [10] there exists a positive projection $P : E \longrightarrow \ell^1$.

We consider the composed operator $T = S_1 \circ P$. It is a weakly compact operator ( since $S_1$ is weakly compact) but it is not almost Banach-Saks, otherwise $T \circ i = S_1 \circ P \circ i = S_1$ will be almost Banach-Saks, where $i : \ell^1 \longrightarrow E$ is the canonical injection, and this is a contradiction. So, the proof is complete. □

Now, we are in position to present some results about duality property of almost Banach-Saks operators. Firstly, we note that direct duality property is not valid for this class of operators, indeed the identity operator $Id_{c_0}$ of the Banach lattice $c_0$ is almost

Banach-Saks, but its adjoint $Id_{\ell^1}$ which is the identity operator of the Banach lattice $\ell^1$ fails to be almost Banach-Saks. In our investigation for solving this problem, we obtain a necessary conditions as the following next result shows.

**Proposition 3.20.** *Let $E$ and $F$ be two Banach lattices such that $E$ is reflexive. If for each almost Banach-Saks operator $T : E \longrightarrow F$, the adjoint operator $T' : F' \longrightarrow E'$ is almost Banach-Saks. Then, one of the following conditions holds:*

a) *$E'$ has disjoint Banach-Saks property;*
b) *$c_0$ is not a closed sublattice of $F$;*
c) *there is no positive and onto projection from $F$ to $c_0$.*

*Proof.* Assume by way of contradiction that, $(a)$, $(b)$ and $(c)$ are not true, then there exists $(x_n^{'})$ a bounded disjoint sequence in $E'$ with no Cesáro convergent subsequence. Since $E$ is reflexive, then the norm of $E''$ is order continuous and hence it follows from Theorem 2.4.14 [10] that the sequence $(x_n^{'})$ is weakly null.

We consider the operator

$$S: \begin{array}{ccc} E & \longrightarrow & c_0 \\ x & \longmapsto & (x_n^{'}(x)) \end{array},$$

$S$ is well-defined. On the other hand, since the norm of $E'$ is order continuous and $c_0$ has the weak Banach-Saks property, it follows from corollary 3.5 that $S$ is an almost Banach-Saks operator. Now, with the fact that $c_0$ is a closed sublattice of $F$ and there exists a positive and onto projection $P : F \longrightarrow c_0$, it follows from Exercise 1.4.E4 page 43 [10] that $P' : \ell^1 \longrightarrow F'$ is a lattice isomorphism. We denote by $i : c_0 \longrightarrow F$ the canonical injection. It is clear that $T = i \circ S$ is an almost Banach-Saks operator, but we will show that $T' = S' \circ i'$ fails to be almost Banach-Saks. Otherwise, since $P' : \ell^1 \longrightarrow F'$ is a lattice isomorphism, then the composed operator $T' \circ P' = S' \circ i' \circ P' : \ell^1 \longrightarrow E'$ will be almost Banach-Saks, but by taking the unit basis $(e_n)_n$ of $\ell^1$ which is bounded and disjoint sequence , we infer that $T' \circ P'(e_n) = \sum_{n=1}^{\infty} e_n x_n^{'} = x_n^{'}$ and by our hypothesis the sequence $(x_n^{'})$ does not contain any Cesáro convergent subsequence, consequently $T'$ is not an almost Banach-Saks operator which makes a contradiction, and hence the proof is complete. □

We recall that a subset $A$ of a Banach lattice $E$ is called b-order bounded if it is order bounded in the topological bidual $E''$, and a Banach lattice $E$ is said to have the (b)-property if $A \subset E$ is order bounded in $E$ whenever it is order bounded in its topological bidual $E''$.

**Remark 3.21.** *If an addition the norm of $F$ is order continuous in the above proposition 3.20, then by using Lemma 2.1* [3] *we can change the two assertions (b) and (c) by the assertion "F has the (b)-property".*

**Theorem 3.22.** *Let* E *be a reflexive Banach lattice. Then, the following statements are equivalent:*

(1) *for each Banach lattice $F$, the adjoint operator $T'$ :* F′ $\longrightarrow$ E′ *of each almost Banach-Saks operator $T : E \longrightarrow F$ is almost Banach-Saks.*
(2) E′ *has weak Banach-Saks property.*

*Proof.* (1)⇒(2) Assume by way of contradiction, that E′ does not have the weak Banach-Saks property, then there exists a weakly null sequence $(f_n) \in$ E′ which does not have any Cesáro convergent subsequence. We consider the operator

$$T: \begin{array}{ccc} E & \longrightarrow & c_0 \\ x & \longmapsto & (f_n(x)) \end{array},$$

which is well-defined and almost Banach-Saks (see the proof of Proposition 3.20). But its adjoint operator $T' : \ell^1 \longrightarrow E'$ defined by $T'(\lambda_n) = \sum_{n=1}^{\infty} \lambda_n f_n$ fails to be almost Banach-Saks. Indeed, the sequence $(e_n)$ of unit basis is bounded and disjoint in $l^1$ and we have $T'(e_n) = f_n$ does not have any Cesáro convergent subsequence, this completes the proof of necessary conditions.
(2)⇒(1) Since E′ is reflexive and has the weak Banach-Saks property, then E′ has the Banach-Saks property and therefore any operator $T'$ : F′ ⟶ E′ is (Banach-Saks) almost Banach-Saks. □

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