Methods of Functional Analysis and Topology
Vol. 25 (2019), no. 3, pp. 236–247

UNBOUNDED TRANSLATION INVARIANT OPERATORS ON COMMUTATIVE HYPERGROUPS

VISHVESH KUMAR, N. SHRAVAN KUMAR, AND RITUMONI SARMA

Abstract. Let K be a commutative hypergroup. In this article, we study the unbounded translation invariant operators on $L^p(K)$, $1 \leq p \leq \infty$. For $p \in \{1,2\}$, we characterize translation invariant operators on $L^p(K)$ in terms of the Fourier transform. We prove an interpolation theorem for translation invariant operators on $L^p(K)$ and we also discuss the uniqueness of the closed extension of such an operator on $L^p(K)$. Finally, for $p \in \{1,2\}$, we prove that the space of all closed translation invariant operators on $L^p(K)$ forms a commutative algebra over the field of complex numbers. We also prove Wendel's theorem for densely defined closed linear operators on $L^1(K)$.

1. Introduction

Gelfand pairs and the theory of spherical functions associated to them play an important role in the theory of Lie groups, and they have been studied in that context. However people have become interested in studying the properties in a wider context of hypergroups without a differential structure on them. Jewett introduced in [10] the notion of hypergroup (convo), and showed that many of the properties of the group algebras still continue hold in this context. Many researchers have contributed to the theory of hypergroups showing that the classical methods of locally compact groups extend to the context of hypergroups.

This article will contribute to the theory of multipliers on commutative hypergroups. In particular, we deal with closed translation invariant operators (also known as unbounded multipliers) on L^p-spaces on hypergroups. A good amount of results has been observed in the last few decades. It started with the works of Lasser [14], in which he characterized the bounded translation invariant operators in terms of the Fourier transform and proved the well-known Wendel's theorem for L^1-spaces on commutative hypergroups. In 2007, Pavel studied multipliers on the L^p-spaces on hypergroups [16]. In 2012, Degenfeld-Schonburg studied bounded multipliers on commutative hypergroups [6] (see also [7]). Recently, the authors, of this paper have studied the vector-valued version of Wendel's theorem on commutative hypergroups and compact hypergroups [13, 17].

It is obvious from the above discussion that the theory of bounded translation invariant operators on hypergroups got enough attention. Also, it is worth noting that the class of unbounded translation invariant operators contains very important operators like Pseudo differential operators. Therefore, it becomes natural and important to consider unbounded translation invariant operators on hypergroups.

One of the most celebrated theorems of the last century is the Wendel's theorem on the characterization of multipliers on locally compact abelian groups. In section 3 of this paper, we prove the Wendel's theorem for densely defined closed linear operators

2010 *Mathematics Subject Classification.* Primary 43A62, 43A22; Secondary 43A30, 47L60.
Key words and phrases. Unbounded multipliers, translation invariant operators, unbounded operators, hypergroups, Fourier transform.

on $L^1(K)$. Theorem 3.2 and Corollary 3.3 presents the Wendel's theorem analogue for unbounded multipliers on hypergroups.

In Section 4, we deal with unbounded multipliers on $L^1(K)$ and $L^2(K)$. We first prove that the domain of a translation invariant operator on $L^1(K)$ is a dense ideal of $L^1(K)$. After that, we characterize such operators in terms of the Fourier multiplication operators.

In Section 5, we consider translation invariant operators on $L^p(K)$, where we prove a kind of an interpolation result. We go on further, to prove that the space of all closed translation invariant operators on $L^p(K)$, $p \in \{1, 2\}$ forms a commutative algebra over the field of complex numbers.

We begin with some of the required preliminaries in the next section.

2. Preliminaries

In this section, we present some notations and concepts of commutative hypergroups and unbounded operators that need in the sequel.

For a locally compact Hausdorff space Ω, the space of continuous functions on Ω will be denoted by $C(\Omega)$. The subspace of $C(\Omega)$ consisting of all compactly supported functions on Ω will be denoted by $C_c(\Omega)$.

We first give some basics of commutative hypergroups. One can refer to [5, 10] for more details. In [10], Jewett refers to hypergroups as convos.

Definition 2.1. [5, 10] A nonempty locally compact Hausdorff space K is said to be a *hypergroup* if there exists a binary operation $*$ on $M(K)$, the space of all complex valued bounded regular measures on K, satisfying the following conditions.

(i) $(M(K), *)$ is a complex associative algebra.
(ii) For every $x, y \in K$, $\delta_x * \delta_y$ is a probability measure with compact support and the mapping $(x, y) \mapsto \delta_x * \delta_y$ is continuous from $K \times K$ to $M(K)$, where δ_x is the point mass measure at x.
(iii) There exists a unique element $e \in K$ such that for all $x \in K$, $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$.
(iv) There exists a unique homeomorphism $x \mapsto \check{x}$ of K such that
  (a) $\check{\check{x}} = x$ for all $x \in K$,
  (b) if $\check{\mu}$ is defined by $\int_K f(x)\ d\check{\mu}(x) = \int_K f(\check{x})\ d\mu(x)$ for all $f \in C_c(K)$, then $(\delta_x * \delta_y)\check{} = \delta_{\check{y}} * \delta_{\check{x}}$ for all $x, y \in K$,
  (c) $e \in \operatorname{spt}(\delta_x * \delta_y)$ if and only if $y = \check{x}$.
(v) The mapping $(x, y) \mapsto \operatorname{spt}(\delta_x * \delta_y)$ is continuous from $K \times K$ to $\mathcal{C}(K)$, where $\mathcal{C}(K)$ denotes the space of all nonempty compact subsets of K equipped with the Michael topology.

We say that a hypergroups $(K, *)$ is *commutative* if $\delta_x * \delta_y = \delta_y * \delta_x$ for all $x, y \in K$.

If f is a Borel function on K and $x, y \in K$, the *left translate* f_y (also denoted $L_y(f)$) is defined by

$$f_y(x) = L_y(f)(x) = \int_K f\ d(\delta_y * \delta_x).$$

Similarly, the right translate f^y of f is defined. If K is commutative, then there is no distinction between left and right translate and so we simply call f^y or f_y by translate of f. We shall also denote this by $f(y * x)$, although $y * x$ may not represent a point in K.

Let f be a Borel function on K. We define the function $\check{f}$ as $\check{f}(x) = f(\check{x})$. At times, κ will denote the map $f \mapsto \check{f}$, i.e., $\kappa(f) = \check{f}$.

A *Haar measure* on a commutative hypergroup K is a non zero regular Borel measure m such that $\delta_x * \lambda = \lambda * \delta_x = \lambda$ for all $x \in K$. It well-known that a commutative hypergroup always possesses a Haar measure which is unique up to a scalar multiple [10].

From now onward, K will denote a commutative hypergroup with λ the Haar measure.

For $1 \leq p \leq \infty$, $L^p(K, \lambda)$ will denote the usual L^p-space defined on the hypergroup K with respect to the Haar measure λ. We write $L^p(K)$ for $L^p(K, \lambda)$ if no confusion arises.

If f and g are Borel functions, then their convolution $f * g$ is defined by

$$f * g(x) = \int_K f(x * y) g(\check{y})\, d\lambda(y) = \int_K f(x * y)\, (\kappa g)(y)\, d\lambda(y)$$

whenever it makes sense.

For a commutative hypergroup K, $L^1(K)$ is a commutative Banach algebra with respect to convolution of functions.

Denote the space of all continuous bounded complex-valued functions defined on K by $C^b(K)$. The dual of K, denoted $\widehat{K}$, is defined by

$$\widehat{K} = \{\chi \in C^b(K) : \chi(x * y) = \chi(x)\chi(y), \chi(\check{x}) = \overline{\chi(x)} \text{ and } \chi(e) = 1\ \forall\ x, y \in K\}.$$

Equip $\widehat{K}$ with the compact-open topology so that $\widehat{K}$ is a locally compact Hausdorff space. The structure space on $L^1(K)$ can be identified with $\widehat{K}$. In general, $\widehat{K}$ may not have a naturally defined hypergroup structure. The Fourier transform of $f \in L^1(K)$ is defined as

$$(\mathcal{F}f)(\chi) = \widehat{f}(\chi) = \int_K f(x)\, \overline{\chi(x)}\, d\lambda(x), \quad \forall\ \chi \in \widehat{K}.$$

There exists a unique positive Borel measure π_K on $\widehat{K}$, called *Plancherel measure*, such that

$$\int_K |f(x)|^2\, dx = \int_{\widehat{K}} |\widehat{f}(\chi)|^2\, d\pi_K(\chi), \quad \forall\, f \in L^2(K) \cap L^1(K).$$

Note that the support $\mathcal{S}$ of π_K, unlike the group case, need not be the whole of $\widehat{K}$ [5, Example 2.2.49]. The extension of the Fourier transform from $L^1(K) \cap L^2(K)$ to $L^2(K)$ is called as the *Plancherel transform.* The Fourier transform is an isometric isomorphism from $L^2(K, \lambda)$ onto $L^2(\mathcal{S}, \pi_K)$. A commutative hypergroup is said to be a strong hypergroup if $\widehat{K}$ is also a commutative hypergroup and in this case $\mathcal{S} = \widehat{K}$. Further, $\widehat{\widehat{K}}$ and K are isomorphic as hypergroups.

Definition 2.2.

(i) Let $1 \leq p < \infty$. A densely defined operator $T : D(T) \subset L^p(K) \to L^p(K)$ is said to be *commute with translation* if the domain $D(T)$ of T is invariant under translations, i.e., $L_x(D(T)) \subset D(T)$ for all $x \in K$ and T commutes with every translations, i.e., $(Tf)_x = T(f_x)$ for $x \in K$ and $f \in D(T)$.

(ii) An operator T is called *translation invariant* if T is closable and commute with translations.

Denote the set of all translation invariant operator on $L^p(K)$ by $M(L^p(K))$. A subset of $M(L^p(K))$ consisting of all closed translation invariant operator is denoted by $M_c(L^p(K))$.

Throughout this paper, K will denote a commutative hypergroup with a fixed Haar measure λ. Also $\widehat{K}$ will denote the dual of K and let $\mathcal{S}$ will be the support of the Plancherel measure π_K. For an operator T, we will write $D(T)$ for its domain.

3. Wendel's theorem for closed linear operators on $L^1(K)$

In this section, we prove Wendel's theorem for densely defined closed linear operators on $L^1(K)$. We begin this section with a Lemma which gives conditions under which the domain $D(T)$ of a closed linear operator T on $L^1(K)$ to be a right Banach $L^1(K)$-module.

Lemma 3.1. *Let $T : D(T) \subset L^1(K) \to L^1(K)$ be a closed linear operator. Suppose that, for $f \in D(T)$ and $g \in L^1(K)$, we have*

(i) $f * g \in D(T)$,
(ii) $(Tf) * g = T(f * g)$.

Then $D(T)$ is right Banach $L^1(K)$-module under the usual convolution of functions and the norm $\|\cdot\|_T$ defined by $\|f\|_T := \|Tf\| + \|f\|$, $f \in D(T)$. Moreover, we have $D(T) * L^1(K) = D(T)$.

Proof. Since $L^1(K)$ is a Banach algebra with respect to convolution, it is clear from (i) that $D(T) * L^1(K) \subset D(T)$, i.e., $D(T)$ is a right ideal of $L^1(K)$. Now, it is easy to check that $D(T)$ becomes a right $L^1(K)$-module with respect to the norm $\|\cdot\|_T$. Since $L^1(K)$ has a bounded right approximate identity and $D(T)*L^1(K)$ is a closed subspace of $D(T)$ [9, Theorem 32.22], we have $D(T) * L^1(K) = D(T)$. □

Here is the promised characterization of a closed linear operator (need not be densely defined) to be a translation invariant operator.

Theorem 3.2. *Let $T : D(T) \subset L^1(K) \to L^1(K)$ be a closed linear operator. Then the following statements are equivalent:*

(i) $(Tf)_x = T(f_x)$, *for all* $f \in D(T)$, $x \in K$.
(ii) $(Tf) * \nu = T(f * \nu)$, *for all* $f \in D(T)$, $\nu \in M(K)$.
(iii) $(Tf) * g = T(f * g)$, *for all* $f \in D(T)$, $g \in L^1(K)$.

Proof. **(i) ⇒ (ii).** Let $\mu \in M(K)$. Then, by [6, Theorem 3.3.2], there exists a net $\{T_\alpha\} \subset \operatorname{span}\{L_x : x \in K\}$ such that

$$\lim_\alpha \|T_\alpha f - f * \mu\| = 0, \quad \forall f \in L^1(K).$$

Now, by assumption, T is a translation invariant operator and therefore $D(T)$ is translation invariant and $T_\alpha(Tf) = T(T_\alpha f)$ for all α. Thus, we get

$$\lim_\alpha \|T(T_\alpha f) - (Tf) * \mu\| = \lim_\alpha \|T_\alpha(Tf) - (Tf) * \mu\| = 0.$$

As T is a closed operator, we get $f * \mu \in D(T)$ and $T(f * \mu) = (Tf) * \mu$.

(ii) ⇒ (iii). This is clear from the fact $L^1(K)$ can be embedded inside $M(K)$ via $f \mapsto f\lambda$.

(iii) ⇒, (i). By Lemma 3.1, we get $D(T) * L^1(K) = D(T)$. Therefore, for every $f \in D(T)$ there exist $g \in D(T)$ and $h \in L^1(K)$ such that $f = g * h$. Now, since L_x, $x \in K$, is a bounded translation invariant operator and it is a multiplier. Therefore, by Weldel's theorem for bounded operators [17] we have

$$(Tf)_x = (T(g * h))_x = ((Tg) * h)_x = (Tg) * h_x = T(g * h_x) = T((g * h)_x) = T(f_x).$$

This establishes the implication. □

In [17], the authors prove Wendel's theorem for bounded linear operators. Our next corollary is an analogue of the Weldel's theorem for densely defined closed linear operators. Here, w_F denotes the weak topology $\sigma(L^1(K), F)$.

Corollary 3.3. *Let $T : L^1(K) \to L^1(K)$ be a densely defined closed linear operator. Then the following statements are equivalent:*

(i) $(Tf)_x = T(f_x)$, *for all* $f \in D(T)$, $x \in K$.
(ii) $(Tf) * \nu = T(f * \nu)$, *for all* $f \in D(T)$, $\nu \in M(K)$.
(iii) $(Tf) * g = T(f * g)$, *for all* $f \in D(T)$, $g \in L^1(K)$.

(iv) *There exist a translation invariant separating subset $F \in L^\infty(K)$ and a net $\{\mu_\alpha\}_\alpha \subset M(K)$ such that*

$$w_F - \lim_\alpha \mu_\alpha * f = T(f)$$

defines T, i.e., the above equality holds for all $f \in D(T)$ and

$$\{f : \exists g \in L^1(K)\, s.t.\, w_F - \lim_\alpha \mu_\alpha * f = g\} = D(T).$$

Proof. In view of Theorem 3.2, it is enough to show that the statements (i) and (iv) are equivalent. Suppose that (i) holds and hence (ii) and (iii) also hold. Let $\{u_\alpha\}_\alpha \subset D(T)$ be a bounded left approximate identity in $L^1(K)$ bounded by $K > 0$. Note that the domain $D(T^*)$ of the adjoint T^* of T is separating in $L^1(K)$ as $D(T^*)$ is weak*-dense in $L^\infty(K)$ [12, Proposition 5.2]. Also, for $\varphi \in D(T^*)$, φT is a continuous linear functional on $D(T)$. Thus, for $f \in D(T)$,

$$\lim_\alpha \langle (Tu_\alpha) * f, \varphi\rangle = \lim_\alpha \langle T(u_\alpha * f), \varphi\rangle = \langle Tf, \varphi\rangle, \quad \varphi \in D(T^*).$$

Therefore, if $\mu_\alpha = (Tu_\alpha)\lambda \in M(K)$, then by setting $F := D(T^*)$, we have

$$w_F - \lim_\alpha \mu_\alpha * f = Tf. \tag{1}$$

Now, suppose that there exist g and h in $L^1(K)$ such that

$$\lim_\alpha \langle \mu_\alpha * h, \varphi\rangle = \langle g, \varphi\rangle, \quad \forall \varphi \in D(T^*).$$

Then, we have

$$\langle g, \varphi\rangle = \langle h, \overline{\varphi T}\rangle = \langle h, T^*\varphi\rangle, \quad \forall \varphi \in D(T^*),$$

where $\overline{\varphi T}$ is the unique continuous extension of φT to $L^1(K)$. Therefore, we get $h \in D(T^{**})$ and $T^{**}h = g$. As we know that $T^{**} = T$ [12, Proposition 5.2], it follows that $h \in D(T)$ and $Th = g$. Thus, T is completely determined by (1).

Now, it remains to show that $F := D(T^*)$ is invariant under composition with right translations. Let $x \in K$ and $\varphi \in D(T^*)$. Then

$$|\langle (Tf)_x, \varphi\rangle| = |\langle Tf_x, \varphi\rangle| \leq \|\varphi T\| \|L_x\| \|f\| \leq \|\varphi T\|\, \|f\|, \quad \forall f \in D(T).$$

Therefore, we have $\langle (\cdot)_x, \varphi\rangle \in D(T^*)$, which is the required condition. Hence, (i) $\Rightarrow$ (iv) is established.

We now prove the other implication. Let $f \in D(T)$. Then, for $h \in F$ and $x \in K$,

$$\lim_\alpha \langle \mu_\alpha * f - Tf, h\rangle = 0$$

and

$$\lim_\alpha \langle \mu_\alpha * f_x - (Tf)_x, h\rangle = \lim_\alpha \langle (\mu_\alpha * f - Tf)_x, h\rangle = 0,$$

i.e., $w_F - \lim_\alpha \mu_\alpha * f_x = (Tf)_x$. Hence, by assumption we have $f_x \in D(T)$ and $Tf_x = (Tf)_x$. □

Remark. We remark here that the statements given in this section holds true with same proof even for a general hypergroup (not necessarily commutative) possessing a Haar measure.

4. Characterization of unbounded multipliers on $L^1(K)$ and $L^2(K)$

We begin this section with the following important lemma. As the proof follows similar lines as in [2, Lemma 1] by using the Young's inequality and [10, Theorem 5.1 D], we omit the proof of it.

Lemma 4.1. *Let $1 \le p, q, r \le \infty$ satisfy $\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$. If $A \subset L^p(K)$ and $B \subset L^q(K)$ are dense subsets, then the linear span span$\{A*B\}$ of A and B is weak-$*$ dense in $L^r(K)$. Further, if $r < \infty$, then span$\{A * B\}$ is norm dense in $L^r(K)$.*

The following theorem presents an unbounded version of [7, Proposition 3], which characterizes translation invariant operators. Also, it says that the domain of an unbounded translation invariant operator on $L^1(K)$ is a dense ideal.

Theorem 4.2. *Let $1 \le p < \infty$ and let T be a translation invariant operator on $L^p(K)$. Then we have $L^1(K) * D(T) \subset D(\overline{T})$ and $\overline{T}(f * g) = f * Tg$ for all $f \in L^1(K)$ and $g \in D(T)$.*

Proof. Since T is translation invariant, for $h \in D(T^*)$ and $g \in D(T)$ we get

$$\begin{aligned}
(h * \kappa Tg)(x) &= \int_K h(x * y)\,(\kappa Tg)(\check{y})\,d\lambda(y) \\
&= \int_K h(x * y)\,(Tg)(y)\,d\lambda(y) \\
&= \int_K h(y)\,(Tg)(\check{x} * y)\,d\lambda(y), \quad \text{by [10, Theorem 5.1D]}, \\
&= \int_K h(y)\,(Tg)_{\check{x}}(y)\,d\lambda(y) \\
&= \int_K h(y)\,(Tg_{\check{x}})(y)\,d\lambda(y) \\
&= \langle h,\, Tg_{\check{x}}\rangle = \langle T^*h,\, g_{\check{x}}\rangle, \quad \forall\, x \in K.
\end{aligned}$$

So we have

$$|(h * \kappa Tg)(x)| = |\langle T^*h,\, g_{\check{x}}\rangle| \le \|T^*h\|_{p'}\,\|g_{\check{x}}\|_p \le \|T^*h\|_{p'}\,\|g\|_p$$

and hence $\|h * \kappa Tg\|_\infty \le \|T^*h\|_{p'}\,\|g\|_p$.

Let $f \in L^1(K)$. Then, Young's inequality gives

$$|\langle f * h, Tg\rangle| = |(f * h * \kappa Tg)(0)| \le \|f * h * \kappa Tg\|_\infty \le \|f\|_1\|T^*h\|_{p'}\,\|g\|_p,$$

hence, by definition of the adjoint, $f * h \in D(T^*)$.

Now, note that

$$\begin{aligned}
(h * \kappa Tg)(x) &= \langle T^*h, g_{\check{x}}\rangle = \int_K (T^*h)(y)\,g_{\check{x}}(y)\,d\lambda(y) \\
&= \int_K (T^*h)(x * y)\,(\kappa g)(\check{y})\,d\lambda(y) = (T^*h * \kappa g)(x),
\end{aligned}$$

and thus, for all $g \in D(T)$,

$$\langle T^*(f * h), g\rangle = \langle f * h, T^*(g)\rangle = (f * h * \kappa T^*g)(0) = (f * T^*h * \kappa g)(0) = \langle f * T^*h, g\rangle.$$

By the density of $D(T)$, we get $T^*(f * h) = f * T^*h$. This gives

$$\begin{aligned}
\langle T^*h, f * g\rangle &= (T^*h * \kappa g * \kappa f)(0) = (h * (\kappa Tg) * \kappa f)(0) \\
&= (h * \kappa(f * Tg))(0) = \langle h, f * Tg\rangle.
\end{aligned}$$

Since $\overline{T}$ is the $\sigma(L^{p'}, L^p)$-dual of T^*, where $\sigma(L^{p'}, L^p)$ denotes the weak-topology (see [18, Corollary IV.7.1]), we have $f * g \in D(\overline{T})$. Hence $L^1(K) * D(T) \subset D(\overline{T})$.

Further,

$$\langle h, f * Tg\rangle = \langle T^* h, f * g\rangle = \langle h, \overline{T}(f * g)\rangle.$$

By using $\sigma(L^{p'}, L^p)$- density of $D(T^*)$, we get $\overline{T}(f * g) = f * Tg$ for all $f \in L^1(K)$ and $g \in D(T)$. ☐

The following corollary gives a characterization of translation invariant operator on $L^1(K)$ in terms of the Fourier transform. A version of the following corollary for bounded translation invariant operators can be found in [7, Theorem 1].

Corollary 4.3. *Let T be a densely defined linear operator on $L^1(K)$ with the domain $D(T)$. Then the following statements are equivalent:*

(i) *T is translation invariant.*
(ii) *T is closable, $f*g \in D(\overline{T})$, and $\overline{T}(f*g) = f*Tg$ for all $f \in L^1(K)$ and $g \in D(T)$.*
(iii) *There exists a unique $\varphi_T \in C(\widehat{K})$ such that $\widehat{Tf} = \varphi_T \widehat{f}$ for all $f \in D(T)$.*

Proof. **(i)⇒(ii).** This follows from Theorem 4.2.

(ii) ⇒ (iii). Since the maximal ideal space of $L^1(K)$ is $\widehat{K}$, it becomes a particular case of [19, Theorem 1].

(iii) ⇒ (i). We know that for $x \in K$ and $\chi \in \widehat{K}$,

$$\widehat{f_x}(\chi) = \chi(x)\widehat{f}(\chi) = \widehat{\delta_{\check{x}}}(\xi)\widehat{f}(\xi).$$

Therefore, we get

$$\widehat{(Tf)_x} = \widehat{\delta_{\check{x}}}(\xi)\widehat{Tf}(\xi) = \widehat{\delta_{\check{x}}}(\xi)\varphi_T(\xi)\widehat{f}(\xi) = \varphi_T(\xi)\widehat{\delta_{\check{x}}}(\xi)\widehat{f}(\xi) = \varphi_T(\xi)\widehat{f_x}(\xi) = \widehat{Tf_x}(\xi).$$

The fact that the map $f \mapsto \widehat{f}$ from $L^1(K)$ to $C_0(\widehat{K})$ is a norm decreasing homomorphism ensures that T commutes with translations. Hence (i) holds. ☐

The function $\varphi_T : \widehat{K} \to \mathbb{C}$ in the above Corollary is called as *the symbol* associated with the translation invariant operator T.

In general, we do not know if any continuous function on $\widehat{K}$ is a symbol for a translation invariant operator. But if K is compact then we have the following result which shows that any complex valued function is a symbol for some translation invariant operator defined on $L^1(K)$.

Corollary 4.4. *Let K be a commutative compact hypergroup and let $\varphi : \widehat{K} \to \mathbb{C}$ be any function. Then there exists a translation invariant operator T on $L^1(K)$ such that $\varphi_T = \varphi$.*

Proof. Since K is compact it follows that its dual $\widehat{K}$ is discrete and therefore φ is continuous. Note that the set $D := \mathcal{F}^{-1}(C_c(\widehat{K}))$ is a dense subset of $L^2(K)$ because the Fourier transform $\mathcal{F}$ is an isometry from $L^2(K)$ onto $L^2(\widehat{K})$ and $C_c(\widehat{K})$ is dense in $L^2(\widehat{K})$. Since $L^2(K)$ is dense in $L^1(K)$ this implies that D is also dense in $L^1(K)$. Now, note that $\varphi\widehat{f} \in C_c(\widehat{K}) \cap \mathcal{F}(L^1(K))$. Therefore, we can define an operator $T : D \to L^1(K)$ by $Tf = \mathcal{F}^{-1}(\varphi\widehat{f})$. Now, by Corollary 4.3, T is a translation invariant operator on $L^1(K)$ and φ is symbol for T. ☐

The following theorem presents some basic properties of translation invariant operators.

Theorem 4.5. *Let T be a translation invariant operator. Then*

(i) *Let $1 \leq p < \infty$. If C is a core for T and D is a dense subset $L^1(K)$ then span$[D * C]$ is a core for T.*

(ii) *Let* $1 \leq p \leq 2$. *If* $D(\overline{T}) \cap L^1(K)$ *is a core for* $\overline{T}$, *then the dual operator* T^* *on* $L^{p'}(K)$ *is an extension of the translation invariant operator* $\kappa T \kappa$ *defined from* $\kappa(span[C_c(K) * (D(\overline{T}) \cap L^1(K))]) \subset L^{p'}(K)$ *to* $L^{p'}(K)$.

Proof. **(i)** By Lemma 4.1 span$[D * C]$ is dense in $L^p(K)$. Therefore, by Theorem 4.2, $\overline{T}|_{\mathrm{span}[D*C]}$ is a densely defined closable operator on $L^p(K)$. Let $f \in D(T)$. Then, there exists a sequence $\{f_n\} \subset C$ such that $\lim \|f_n - f\|_p = 0$ and $\lim \|Tf_n - \overline{T}f\| = 0$. Let $\{h_n\} \subset L^1(K)$ be a bounded approximate identity for $L^p(K)$ as in [13, Lemma 3.3] and hence $\lim \|h_n * f_n - f_n\|_p = 0$ and $\lim \|h_n * Tf_n - Tf_n\|_p = 0$. Since D is dense in $L^1(K)$ we can choose $\{h_n\}$ from D. Therefore, by Theorem 4.2, it follows that span$[D * C]$ is a core of T.

(ii) Again, by Lemma 4.1, the space span$[C_c(K) * (D(\overline{T}) \cap L^1(K))] = D$ (say) is a dense subspace of $D(\overline{T})$. Since $p' \geq p$, D is weak*-dense in $L^{p'}(K)$ (norm dense if $p > 1$). Now, for $f \in \kappa D$ and $g \in D(\overline{T}) \cap L^1(K)$, Theorem 4.2 yields that

$$\langle f, \overline{T}g\rangle = (f * \kappa(\overline{T}f))(0) = \overline{T}(\kappa f * g)(0) = (\overline{T}\kappa f * g)(0) = \langle \kappa\overline{T}\kappa f, g\rangle.$$

Now, $\kappa\overline{T}\kappa f \in L^{p'}(K)$ give $f \in D(T^*)$ because $D(\overline{T}) \cap L^1(K)$ is a core. Thus, $\langle T^* f, g\rangle = \langle \kappa\overline{T}\kappa f, g\rangle$ and this shows $T^* f = \kappa\overline{T}\kappa f$. □

Notation: For $1 \leq p \leq \infty$ and φ a measurable function on $\mathcal{S}$ with respect to π_K, set

$$D^p(\varphi) = \{f \in L^p(\mathcal{S}) : \varphi f \in L^p(\mathcal{S})\}.$$

In [7], Sina characterized the bounded multiplies on $L^2(K)$ in terms multiplication operators of L^∞- functions on $\mathcal{S}$. Our next theorem characterizes the unbounded multipliers on $L^2(K)$. It says that there is a one to one correspondence between the set of translation invariant operators on $L^2(K)$ and set of all complex-valued measurable functions on $\mathcal{S}$.

Theorem 4.6.

(i) *Let* T *be a translation invariant operator on* $L^2(K)$. *Then there exists a measurable function* $\varphi_T : \mathcal{S} \to \mathbb{C}$ *such that* $\widehat{Tf} = \varphi_T \widehat{f}$ *for all* $f \in D(T)$. *Further,* φ_T *is uniquely determined locally a.e..*

(ii) *Let* $\varphi : \mathcal{S} \to \mathbb{C}$ *be a measurable function. Then the operator*

$$T : \mathcal{F}^{-1}(D^2(\varphi)) \subset L^2(K) \to L^2(K)$$

given by $Tf = \mathcal{F}^{-1}(\varphi\widehat{f})$ *is a translation invariant operator, where* $\mathcal{F}^{-1}$ *is the inverse of the Fourier transform* $\mathcal{F}$ *on* $L^2(K)$.

Proof. **(i)** First note that, by [11, Theorem 3.24, p. 275], the operator $S := T^*\overline{T}$ is a densely defined, self-adjoint, positive operator and the core of $\overline{T}$ is $D(T^*\overline{T})$. Therefore, $(I + S)^{-1}$ is a translation invariant operator on $L^2(K)$. Hence, by [7, Theorem 2], there exists a measurable function $\varphi \in L^\infty(\mathcal{S})$ such that

$$\mathcal{F}((I + S)^{-1} f) = \varphi\widehat{f}, \quad \forall f \in L^2(K).$$

By applying [6, Corollary 3.2.10] to our setting, we get a measurable set $E \subset \mathcal{S}$ such that

$$\overline{\mathcal{F}(D(S))} = \overline{\mathcal{F}(\mathrm{Range}(I + S)^{-1})} = \chi_E \cdot L^2(\mathcal{S}).$$

Note that $\varphi(\chi) \neq 0$ locally a.e. because $\mathcal{S}\backslash E$ is locally null w.r.t. π_K. Set

$$\varphi_S(\xi) = \begin{cases} \frac{1}{\varphi(\xi)} - 1 & \text{if } \varphi(\xi) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\|\varphi\|_\infty \leq \|(I+S)^{-1}\|_{\mathcal{B}(L^2(K))}$, φ_S is measurable and $\varphi_S \geq 0$ a. e. locally. Now, for $f \in D(S) = \text{Range}(I+S)^{-1}$, there exists $g \in L^2(K)$ such that $f = (I+S)^{-1}g$ and hence $\widehat{f} = \mathcal{F}((I+S)^{-1}g) = \varphi\widehat{f}$. This also shows that $(I+S)f = g$ and hence

$$\widehat{Sf} = \widehat{g} - \widehat{f} = \frac{1}{\varphi}\varphi\widehat{g} - \widehat{f} = \left(\frac{1}{\varphi} - 1\right)\widehat{f} = \varphi_S\widehat{f} \quad \text{a.e..}$$

Since $\text{Range}((I+S)^{-1}) = D(T^*\overline{T}) \subset D(\overline{T})$, the operator $\overline{T}(I+T^*\overline{T})^{-1}$ is well-defined on $D(\overline{T}(I+S)^{-1}) = L^2(K)$. It follows from closed graph theorem that the operator $\overline{T}(I+S)^{-1}$ is bounded on $L^2(K)$. Further, $\overline{T}(I+S)^{-1}$ commutes with translations and hence there exists a symbol $\psi \in L^\infty(S)$ for it.

Set $\varphi_T = \psi \cdot (1+\varphi_S)$. Then φ_T is measurable function and we have, for $f \in D(T^*\overline{T})$

$$\widehat{\overline{T}f} = \mathcal{F}(\overline{T}(I+S)^{-1}(I+S)f) = \psi\mathcal{F}((I+S)f) = \psi(\widehat{f} + \varphi_S\widehat{f}) = \varphi_T\widehat{f}.$$

Since $D(T^*\overline{T})$ is a core for $\overline{T}$, we get $\widehat{Tf} = \varphi_T\widehat{f}$.

Next, we prove the uniqueness of φ_T. Let φ_1 and φ_2 be two symbols for the operator T. Set

$$F = \{\xi \in \mathcal{S} : \varphi_1(\xi) \neq \varphi_2(\xi)\}.$$

Then, for all $f \in D(T)$, $\chi_F \cdot (\varphi_1 - \varphi_2)\widehat{f} = \chi_F \cdot \mathcal{F}(Tf - Tf) = 0$ a.e.. Therefore, $\chi_F \cdot \widehat{f} = 0$ and

$$L^2(\mathcal{S}) = \overline{\mathcal{F}(D(T))} \subset \chi_{\mathcal{S}\setminus F} \cdot L^2(\mathcal{S})$$

implies that F is locally null with respect to π_K. Hence, $\varphi_1 = \varphi_2$ a.e. locally.

(ii) The multiplication operator M_φ from $D^2(\varphi) \subset L^2(\mathcal{S})$ to $L^2(\mathcal{S})$ given by $M_\varphi(f) = \varphi f$ is a densely defined closed operator. Now, the proof follows by using the fact that the Fourier transform is an isometry on $L^2(K)$. □

5. Unbounded translation invariant operators on $L^p(K)$

In this section, we deal with translation invariant operators on L^p- spaces on commutative hypergroups. We present results related to interpolation and duality for translation invariant operators and uniqueness of closed extension of a translation invariant operator. At last, we prove that the set of all closed translation invariant operators on $L^1(K)$ or $L^2(K)$ bears a commutative algebra structure.

We begin this section with the following important lemma.

Lemma 5.1. *Let $1 \leq p_1 \leq r \leq p_2 < \infty$ and let $T \in M(L^{p_1}(K)) \cap M(L^{p_2}(K))$. Then we have $T \in M(L^r(K))$.*

Proof. Let $T \in M(L^{p_1}(K)) \cap M(L^{p_2}(K))$. It is clear that it is enough to show that T is closable on $L^r(K)$ for proving that $T \in M(L^r(K))$. Throughout this proof, for the sake of clarity, we write the domain of T as $D(T_r)$ if the operator T is acting on $L^r(K)$.

We prove the closability of T on $L^r(K)$ by proving that $D((T_r)^*)$ is dense in $L^{r'}(K)$. Since $L^{p_2}(K)$ is a reflexive space it follows that $D((T_{p_2})^*)$ is dense in $L^{p_2'}(K)$ and therefore, by Theorem 4.5 (i), $\text{span}[C_c(K) * D((T_{p_2'})^*)]$ is a core for $T^*_{p_2}$ (which is the adjoint of T acting as an operator on $L^{p_2}(K)$). For $f \in C_c(K) * D(T^*_{p_2})$ and $g \in D(T_r)$ we get, by Lemma 4.1,

$$T^*_{p_2}f \in C_c(K) * L^{p_2'}(K) \subset L^{r'}(K)$$

and

$$\langle T^*_{p_2}f, g\rangle = \langle f, T_{p_2}g\rangle = \langle f, T_r g\rangle = \langle T^*_r f, g\rangle,$$

which gives that $\text{span}[C_c(K) * D((T_{p_2})^*)] \subset D((T_r)^*)$. Now, $\text{span}[C_c(K) * D((T_{p_2})^*)]$, and hence $D((T_r)^*)$ is dense in $L^{r'}(K)$ by Lemma 4.1, which proves our assertion. Therefore, T is closable on $L^r(K)$. □

For bounded translation invariant operator the following is true [6, p. 32]:

$$M(L^1(K)) \subset M(L^p(K)) \subset M(L^q(K)) \subset M(L^2(K)),$$

where $1 \leq p \leq 2$ and $p \leq q \leq p'$. We prove these inclusions for the unbounded translation invariant operators in the next two results. The following theorem is well-known for bounded translation invariant operators (see [6, Chapter 3]). It can be seen as an interpolation theorem.

Theorem 5.2. *Let $1 \leq p \leq 2$, $p \leq q \leq p', q < \infty$ and let T be a translation invariant operator on $L^p(K)$ such that $D(\overline{T}) \cap L^1(K)$ is dense in $L^p(K)$. Then $\overline{T}|_D$ is a translation invariant operator on $L^q(K)$, where $D := span[C_c(K) * (D(\overline{T}) \cap L^1(K))]$.*

Proof. Note that, Lemma 5.1 together with Theorem 4.6 and corollary 4.3 gives that $\overline{T}|_D \in M(L^r(K))$ for $1 \leq r \leq 2$ and hence the case $p = 1$ follows. Now, let $1 < p \leq 2$. Therefore $1 < p \leq p' < \infty$ and so $D = \text{span}[C_c(K) * (D(\overline{T}) \cap L^1(K))]$ is dense in $L^{p'}(K)$. With the help of Theorem 4.5, one can see that $\overline{T}|_D \subset \kappa T^* \kappa$. Hence $\overline{T}|_D$ is closable on $L^{p'}(K)$. By combining this with the assumption, we get that $\overline{T}|_D \in M(L^p(K)) \cap M(L^{p'}(K))$. Hence, by Lemma 5.1, we get that $\overline{T}|_D \in M(L^q(K))$. □

Our next corollary says that $M(L^1(K)) \subset M(L^p(K))$ for $1 \leq p < \infty$ and will be useful for application purposes.

Corollary 5.3. *Let T be a translation invariant operator on $L^1(K)$. Assume that $D(T)$ is dense in $L^1(K) \cap L^p(K)$, for $1 \leq p < \infty$. If T maps $D(T)$ into $L^1(K) \cap L^p(K)$ then T is a translation invariant operator on $L^p(K)$.*

Proof. The Theorem 5.2 above shows that $\overline{T}|_D \in M(L^p(K))$ where $D := \text{span}[C_c(K) * D(\overline{T})]$. Therefore, for proving this corollary, it is enough to prove that the closure of $\overline{T}|_D$ on $L^p(K)$ extends T. Suppose that $\{g_n\} \in C_c(K)$ is a approximate identity for $L^p(K)$ (see [13, Lemma 3.3]) such that

$$g_n * f \to f \quad \text{and} \quad \overline{T}|_D(g_n * f) = g_n * Tf = Tf$$

in p-norm. This shows that $T \subset \overline{\overline{T}|_D}$. □

The following lemma will be used in proving the next theorem. Here, we assume that K is a strong hypergroup.

Lemma 5.4. *Let $1 \leq p < \infty, T \in M_c(L^p(K))$, $D_1 := \mathcal{F}(D(T))$ and let h be a measurable function on $\mathcal{S}$ with respect to π_K such that $hf \in L^1(\mathcal{S})$ for all $f \in D_1$. Then $\int_{\mathcal{S}} hf d\pi_K = 0$ for all $f \in D_1$ implies that $h = 0$ a.e. π_K-locally.*

Proof. Let U and V be subsets of $\mathcal{S}$ such that $\pi_K(U) < \infty$ and $\pi_K(V) < \infty$. Then $\chi_U * \chi_V \in \mathcal{F}(L^1(K))$ (see [1, Lemma 2.1]). Therefore, by Theorem 4.2, we have $(\chi_U * \chi_V) \cdot D_1 \subset D_1$. Choose neighborhoods U_n of e such that

$$\lim_n \|\chi_U * \mu_n - \chi_U\|_2 = 0,$$

where $\mu_n = \frac{\chi_{U_n}}{\pi_K(U_n)}$ (see [8, Theorem 25.15]). We can suppose (possibly by taking subsequence) that the convergence $\chi_U * \mu_n \to \chi_U$ is π_K-a.e. and also, we have

$$\lim_n \|(\chi_U * \mu_n)hf - hf\chi_U\|_1 = 0.$$

Thus, by using the estimate $\|\chi_U * \mu\|_\infty \leq \|\chi_U\|_\infty \frac{1}{\pi_K(U_n)} \|\chi_{U_n}\|_1 = 1$, we get $\int_{\mathcal{S}} h\chi_U f \, d\pi_K = 0$ for all $f \in D_1$ and for all U such that $\pi_K(U) < \infty$. By approximating any measurable subset by finite π_K- measure sets we get $\int_{\mathcal{S}} h\chi_U f \, d\pi_K = 0$ for all π_K-measurable subset of $\mathcal{S}$ and $f \in D_1$. Further, by the weak*-density of simple function in $L^\infty(\mathcal{S})$, we get $\int_K h \, g \, f \, d\pi_K = 0$ for all $g \in L^\infty(\mathcal{S})$ and $f \in D_1$.

Finally, if $\pi_K(U) < \infty$, take $g := (1+|h|)^{-1}\chi_U$, so that $g \in L^2(\mathcal{S}) \cap L^\infty(\mathcal{S})$. In particular, $hg \in L^2(\mathcal{S})$. Whenever $\pi_K(U) < \infty$, the density of D_1 gives that $\chi_U \cdot h = 0$. Hence $h = 0$ a.e. π_K-locally. □

Our next result is about the uniqueness of the closed extension of a translation invariant operator on $L^p(K)$. Here also we assume that K is a strong hypergroup.

Theorem 5.5. *Let $1 \leq p < \infty$ and let T_1 and T_2 be closed translation invariant operator on $L^p(K)$, i.e., $T_1, T_2 \in M_c(L^p(K))$ such that $T_1 \subset T_2$. If $D(T_i) \cap L^1(K)$ is a core for $T_i\, (i = 1, 2)$ or $p = 2$ then $T_1 = T_2$.*

Proof. First assume that $p \neq 2$. Then, by Theorem 4.5 (i),

$$D = \text{span}[(D(T_1) \cap L^1(K)) * (D(T_2) \cap L^2(K))] \subset D(T_1) \cap D(T_2)$$

is a core for T_2. Then $T_2 = \overline{T_2|_D}$. But we also have $T_2|_D \subset T_1 \subset T_2$. Therefore, $T_1 = T_2$.

Now, assume that $p = 2$. Then $D_1 = \mathcal{F}(D(T_1)) \subset \mathcal{F}(D(T_2)) \subset D^2(\varphi_{T_2})$, where φ_{T_2} is the symbol associated with T_2. Therefore, it is enough to show that D_1 is a core for the multiplication operator $M_{\varphi_{T_2}} : L^2(\mathcal{S}) \to L^2(\mathcal{S})$ given by $M_\varphi(f) = \varphi_{T_2} f$ with the domain $D^2(\varphi)$.

First, we prove that $(M_{\varphi_{T_2}}|_{D_1})^* = M^*_{\varphi_{T_2}}$. To see this, note that $M_{\varphi_{T_2}}|_{D_1} \subset M_{\varphi_{T_2}}$ and therefore $(M_{\varphi_{T_2}})^* \subset (M_{\varphi_{T_2}}|_{D_1})^*$. Now, for the other side inclusion, let $g \in (D(M_{\varphi_{T_2}}|_{D_1})^*)$. Then we have, for all $f \in D_1$,

$$\int_{\mathcal{S}} (g\varphi_{T_2} - (M_{\varphi_{T_2}}|_{D_1})^* g)\, f\, d\pi_K = \langle g, (M_{\varphi_{T_2}}|_{D_1} f) f\rangle - \langle (M_{\varphi_{T_2}}|_{D_1})^* g, f\rangle = 0.$$

Therefore, by Lemma 5.4, we get $g\varphi_{T_2} - (M_{\varphi_{T_2}}|_{D_1})^* g = 0$ a.e. π_K-locally and so $g \in D^2(\varphi_{T_2}) = D(M^*_{\varphi_{T_2}})$. Hence, $(M_{\varphi_{T_2}}|_{D_1})^* = M^*_{\varphi_{T_2}}$. This implies that $M_{\varphi_{T_2}} = M^{**}_{\varphi_{T_2}} = (M_{\varphi_{T_2}}|_{D_1})^{**} = \overline{M_{\varphi_{T_2}}|_{D_1}}$. Therefore, D_1 is core for $M_{\varphi_{T_2}}$. Hence the proof. □

The following theorem says that the space of all closed translation invariant operators on $L^p(K)$ for $p = 1$ or 2 is a commutative algebra over the field of complex numbers.

Theorem 5.6. *Let $p = 1$ or 2. Then $M_c(L^p(K))$, the set of all closed translation invariant operators on $L^p(K)$, forms a commutative algebra over $\mathbb{C}$ with respect to the following operations: For $T, S \in M_c(L^p(K))$ and $\alpha \in \mathbb{C}$*

$$T +_{M_c} S = \overline{T + S}, \quad \alpha \cdot_{M_c} T = \overline{\alpha T} \quad \textit{and} \quad T \circ_{M_c} S = \overline{T \circ S}.$$

Proof. For $p = 1$, the operation $+_{M_c}$ and $\circ_{M_c}$ is defined on a common core of for S and T, namely, $D(S) * D(T)$ which is contained in $D(T + S) \cap D(T \circ T)$. Therefore, all the three operations are well-defined. Verifying all algebraic operations is a routine check. By Corollary 4.3, $T + S$ and $T \circ S$ are given by symbols $\varphi_T + \varphi_S$ and $\varphi_T \varphi_S$, where φ_T and φ_S are symbols of T and S respectively. By Corollary 4.3, $T + S$ and $T \circ S$ are closable with a unique closed extension.

For $p = 2$, the proof is similar to $p = 1$ case by noting that the common core for T and S, namely, $\mathcal{F}^{-1}(D^2((1 + |\varphi_T|)(1 + |\varphi_S|))) \subset D(T + S) \cap D(T \circ S)$ as $|\varphi_T|, |\varphi_S|, |\varphi_T + \varphi_S|$ and $|\varphi_S \varphi_T|$ are less or equal to $(1 + |\varphi_T|)(1 + |\varphi_S|)$. □

Acknowledgments. The first author thanks the Council of Scientific and Industrial Research, India, for the research grant. He would also like to express his deep gratitude to Prof. Ajit Iqbal Singh for suggesting this problem.

References

1. Ajit Kaur Chilana and K. A. Ross, *Spectral synthesis in hypergroups*, Pacific J. Math. **76** (1978), no. 2, 313–328.
2. D. Alboth, *Unbounded translation invariant operators on locally compact abelian groups*, J. Math. Anal. Appl. **193** (1995), no. 2, 431–446.
3. D. Alboth, *Unbounded translation invariant operators and the derivation property*, J. London Math. Soc. **56** (1997), no. 1, 1–15.
4. V. A. Babalola, *Characterization of unbounded multipliers maps*, J. Math. Anal. Appl. **88** (1982), 133–142.
5. W. R. Bloom and Herbert Heyer, *Harmonic Analysis on Probability Measures on Hypergroups*, De Gruyter, Berlin, 1995. (Reprint: 2011).
6. S. Degenfeld-Schonburg, *Multipliers for Hypergroups: Concrete Examples, Application to Time Series*, Doctoral Dissertation, 2012.
7. S. Degenfeld-Schonburg and R. Lasser, *Multipliers on L^p-spaces for hypergroups*, Rocky Mountain J. Math. **43** (2013), no. 4, 1115–1139.
8. E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis.* Vol. I*: Structure of Topological Groups, Integration Theory, Group Representations.* 2nd ed. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 115. Springer-Verlag, Berlin—New York, 1979.
9. E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis.* Vol. II*: Structure and Analysis for Compact Groups. Analysis on Locally Compact Abelian Groups*, Die Grundlehren der Mathematischen Wissenschaften, vol. 152. Springer-Verlag, New York—Berlin, 1970.
10. R. I. Jewett, *Spaces with an abstract convolution of measures*, Adv. Math. **18** (1975), 1–101.
11. T. Kato, *Perturbation Theory of Linear Operators*, Springer-Verlag, Berlin, 1976.
12. H. Komatsu, *Semigroups of operators in locally convex spaces*, J. Math. Soc. Japan **16** (1964), 230–262.
13. V. Kumar, N. Shravan Kumar, and R. Sarma, *Characterization of multipliers on hypergroups.* (Communicated).
14. R. Lasser, *Fourier-Stieltjes transforms on hypergroups*, Analysis **2** (1982), 281–303.
15. A. Olubummo, *Unbounded multiplier operators*, J. Math. Anal. Appl. **71** (1979), 359–365.
16. L. Pavel, *Multipliers for the L^p-spaces of a hypergroup*, Rocky Mountain J. Math. **37** (2007), no. 3, 987–1000.
17. R. Sarma, N. Shravan Kumar and Vishvesh Kumar, *Multipliers on vector-valued L^1-spaces for hypergroups, accepted in*, Acta. Math. Sinica (Eng. Ser.). DOI: 10.1007/s10114-018-7303-7.
18. H. H. Schaefer, *Topological Vector Spaces*, MacMillan Co., New York, 1962.
19. J. Wood, *Unbounded multipliers on commutative Banach algebra*, Pacific J. Math. **85** (1979), no. 2, 479–484.

Department of Mathematics, Indian Institute of Technology, Delhi, Delhi - 110 016, India
E-mail address: `vishveshmishra@gmail.com`

Department of Mathematics, Indian Institute of Technology, Delhi, Delhi - 110 016, India
E-mail address: `shravankumar@maths.iitd.ac.in`

Department of Mathematics, Indian Institute of Technology, Delhi, Delhi - 110 016, India
E-mail address: `ritumoni@gmail.com`

Received 03/10/2018