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CHARACTERIZATION OF SCHUR PARAMETER SEQUENCES OF POLYNOMIAL SCHUR FUNCTIONS

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Dedicated to Yu. M. Arlinskii on the occasion of his 70th birthday

Abstract. A function is called a Schur function if it is holomorphic in the open unit disk and bounded by one. In the paper, the Schur parameters of polynomial Schur functions are characterized.

0. Introduction. Formulation of the main theorem

0.1. **Schur algorithm, Schur parameters.** The main goal of this paper is to study the Schur parameter sequences of Schur functions of polynomial type. Let $\mathbb{D} := \{\zeta \in \mathbb{C} : |\zeta| < 1\}$. Then a function $\theta : \mathbb{D} \to \mathbb{C}$ is called a *Schur function* if θ is holomorphic in $\mathbb{D}$ and if for all $\zeta \in \mathbb{D}$ the inequality $|\theta(\zeta)| \leq 1$ is satisfied. The symbol $\mathcal{S}$ stands for the set of all Schur functions.

The classical paper [17] by I. Schur determined the starting point for a lot of developments touching several areas of mathematics. Let $\theta \in \mathcal{S}$. Following I. Schur, we set $\theta_0 := \theta$ and $\gamma_0 := \theta_0(0)$. Obviously, $|\gamma_0| \leq 1$. If $|\gamma_0| < 1$, then we consider the function $\theta_1 : \mathbb{D} \to \mathbb{C}$ defined by

$$\theta_1(\zeta) := \frac{1}{\zeta} \cdot \frac{\theta_0(\zeta) - \gamma_0}{1 - \overline{\gamma_0}\theta_0(\zeta)}. \tag{0.1}$$

In view of the Lemma of H. A. Schwarz, we have $\theta_1 \in \mathcal{S}$. As above, we set $\gamma_1 := \theta_1(0)$ and if $|\gamma_1| < 1$, we consider the function $\theta_2 : \mathbb{D} \to \mathbb{C}$ defined by

$$\theta_2(\zeta) := \frac{1}{\zeta} \cdot \frac{\theta_1(\zeta) - \gamma_1}{1 - \overline{\gamma_1}\theta_1(\zeta)}.$$

Further, we continue this procedure inductively. Namely, if in the j-th step a function occurs for which $|\gamma_j| < 1$ where $\gamma_j := \theta_j(0)$, we define $\theta_{j+1} : \mathbb{D} \to \mathbb{C}$ by

$$\theta_{j+1}(\zeta) := \frac{1}{\zeta} \cdot \frac{\theta_j(\zeta) - \gamma_j}{1 - \overline{\gamma_j}\theta_j(\zeta)}$$

and continue this procedure in the prescribed way. Then setting $\mathbb{N}_0 := \{0, 1, 2, \dots\}$ two cases are possible:

(1) The procedure can be carried out without end, i.e., $|\gamma_j| < 1$ for each $j \in \mathbb{N}_0$.
(2) There exists an $n \in \mathbb{N}_0$ such that $|\gamma_n| = 1$ and, if $n > 0$, then $|\gamma_j| < 1$ for each $j \in \{0, \dots, n-1\}$.

Thus, a sequence $(\gamma_j)_{j=0}^{w}$ is associated with each function $\theta \in \mathcal{S}$. Here we have $w = \infty$ (resp. $w = n$) in the first (resp. second) case. From I. Schur's paper [17] it is known that the second case occurs if and only if θ is a finite Blaschke product of degree n.

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Definition 0.1. The sequence $(\gamma_j)_{j=0}^{w}$ obtained by the above procedure is called the *sequence of Schur parameters* associated with the function $\theta \in \mathcal{S}$. The symbol Γ stands for the set of all corresponding sequences $(\gamma_j)_{j=0}^{w}$ of Schur parameters.

The following two properties established by I. Schur in [17] determine the particular role, which the Schur parameters play in the study of functions of class $\mathcal{S}$.

(a) Each sequence $(\gamma_j)_{j=0}^{w}$ of complex numbers, $0 \le w \le \infty$, which satisfies one of the conditions (1) or (2) is the sequence of Schur parameters of some function $\theta \in \mathcal{S}$.

(b) There is a one-to-one correspondence between the set of functions $\theta \in \mathcal{S}$ and the set Γ of corresponding sequences $(\gamma_j)_{j=0}^{w}$.

Thus, the Schur parameters are independent parameters, which determine the functions of class $\mathcal{S}$.

Two central aspects of these investigations can be roughly described as follows:

(1) Express distinguished properties of a Schur function in terms of its sequence of Schur parameters.

(2) Which properties of a Schur parameter sequence produce desired properties of its associated Schur function?

There is a rich literature dealing with these tasks (see, e.g., Simon [18], Khrushchev [15], Alpay/ Gohberg [1], [2], Arlinskii [3], [4] and the references cited therein). The investigations of this paper continue the authors' former work on this topic (see [7],[9]–[14] as well as [5], [6]).

The following result describes the interplay of the asymptotics of the sequence of Schur parameters of a function $\theta \in \mathcal{S}$ with the corresponding Szegö integral.

Theorem 0.2 (see Simon [18, Theorem 2.7.7])**.** *Let $\theta \in \mathcal{S}$ and let $(\gamma_j)_{j=0}^{w}$ be its sequence of Schur parameters. Denote $\underline{\theta}$ the boundary values of θ and let m be the normalized Lebesgue measure on $\mathbb{T}$. Then*

$$\prod_{j=0}^{w}(1-|\gamma_j|^2) = \exp\left\{\int_{\mathbb{T}} \log[1-|\underline{\theta}|^2]\,\mathrm{d}m\right\}.$$

In the case $w = \infty$, in particular,

$$\sum_{j=0}^{\infty}|\gamma_j|^2 < \infty \iff \int_{\mathbb{T}} \log[1-|\underline{\theta}|^2]\,\mathrm{d}m > -\infty.$$

0.2. **On some properties of the Schur parameters of pseudocontinuable Schur functions.** In this paper, we use ideas and methods from the paper [11], where the phenomenon of pseudocontinuability of functions of class $\mathcal{S}$ into the exterior of the unit disk was investigated in the language of Schur parameters. Against this background we introduce the following definitions and notations.

Let f be a function, which is meromorphic in $\mathbb{D}$ and which has nontangen-tial boundary limit values a.e. with respect to the Lebesgue measure on $\mathbb{T} := \{\zeta \in \mathbb{C} : |\zeta| = 1\}$. Denote by $\mathbb{D}_e := \{\zeta : |\zeta| > 1\}$ the exterior of the unit circle including the point infinity. The function f is said to admit a *pseudocontinuation of bounded type* into $\mathbb{D}_e$ if there exist functions α and $\beta \not\equiv 0$, which are bounded and holomorphic in $\mathbb{D}_e$ such that the nontangential boundary values of f and $\hat{f} := \frac{\alpha}{\beta}$ coincide a.e. on $\mathbb{T}$. From the Theorem of Luzin-Privalov (see, e.g., Koosis [16]) it follows that there exists at most one pseudocontinuation.

We denote by $\mathcal{S}\Pi$ the subset of all functions belonging to $\mathcal{S}$, which admit a pseudo-continuation of bounded type into $\mathbb{D}_e$. We note that the set J of all inner functions in $\mathbb{D}$ is a subset of $\mathcal{S}\Pi$. Indeed, if $\theta \in J$, then the function $\hat{\theta}(\zeta) := \overline{\theta^{-1}(\frac{1}{\overline{\zeta}})}$, $\zeta \in \mathbb{D}_e$, is the pseudocontinuation of θ.

The following statement shows the principal difference between the properties of Schur parameters of inner functions and the properties of Schur parameters of pseudocontinuable Schur functions, which are not inner.

Theorem 0.3 ([11, Theorem 4.3]). *Let $\theta \in \mathcal{S}\Pi$ and denote $(\gamma_j)_{j=0}^{w}$ the sequence of Schur parameters of θ. If θ is not inner, then $w = \infty$ and the product*

$$\prod_{j=0}^{\infty}(1 - |\gamma_j|^2) \tag{0.2}$$

converges. If θ is inner, then the product (0.2) *diverges.*

0.3. **First observations on Schur functions of polynomial type.** The main aim of this paper is the investigation of the Schur parameters of a particular subclass of the class $\mathcal{S}\Pi$.

A function $\theta \in \mathcal{S}$ is called *of polynomial type* if there exists a polynomial P with complex coefficients such that its restriction onto $\mathbb{D}$ coincides with θ. Clearly, P is then uniquely determined.

We denote by $\mathcal{SP}$ the set of all Schur functions of polynomial type. If $m \in \mathbb{N}_0$, we denote by $\mathcal{SP}_m$ the set of all functions $\theta \in \mathcal{SP}$ of the shape $\theta = \mathrm{Rstr.}_{\mathbb{D}}\, P$, where P is a polynomial of exact degree m.

Let $m \in \mathbb{N}_0$ and let $\theta \in \mathcal{SP}_m$. Then we are naturally led to the following two cases.

(1) $\theta \in \mathcal{SP}_m \cap J$
   Obviously, in this case, there exists a unique $u \in \mathbb{T}$ such that $\theta(\zeta) = u \cdot \zeta^m$. Hence, the Schur parameter sequence of θ is given by $\gamma_m = u$ and, if $m > 0$, additionally, by $\gamma_0 = \cdots = \gamma_{m-1} = 0$.

(2) $\theta \in \mathcal{SP}_m \setminus J$
   We start with a family of remarkable members of $\mathcal{SP}_m \setminus J$.

   Example 0.4. Let $m \in \mathbb{N}$ and $u \in \mathbb{D} \setminus \{0\}$. Define the function $\theta : \mathbb{D} \to \mathbb{C}$ by $\theta(\zeta) := u\zeta^m$. Obviously, then $\theta \in \mathcal{SP}_m \setminus J$ and the Schur parameter sequence $(\gamma_j)_{j=0}^{\infty}$ of θ is given by

$$\gamma_j = \begin{cases} 0, & \text{if } j \in \mathbb{N}_0 \setminus \{m\}, \\ u, & \text{if } j = m. \end{cases}$$

   We will see in Theorem 1.2 that the family of functions occuring in Example 0.4 plays an exceptional role within the class $\mathcal{SP}_m \setminus J$.

In view of $\mathcal{SP}_m \subseteq \mathcal{S}\Pi$, the application of Theorem 0.3 immediately yields:

Theorem 0.5. *Let $\theta \in \mathcal{SP}_m \setminus J$ with Schur parameters $(\gamma_j)_{j=0}^{m}$. Then $w = \infty$ and the product* (0.2) *converges.*

0.4. **Schur parameters of a rational Schur function.** From the preceding considerations it becomes clear that we are mainly interested in functions $\theta \in \mathcal{SP}_m \setminus J$. In this case we infer from Theorem 0.5 that the product (0.2) converges. In this context, we introduce the following notations. In the following, the symbol ℓ_2 stands for the space of all sequences $(z_j)_{j=0}^{\infty}$ of complex numbers such that $\sum_{j=0}^{\infty} |z_j|^2 < \infty$. Moreover,

$$\Gamma\ell_2 := \left\{\gamma = (\gamma_j)_{j=0}^{\infty} \in \ell_2 : \ \gamma_j \in \mathbb{D}, \ j \in \{0, 1, 2, \dots\}\right\}.$$

Thus, $\Gamma\ell_2$ is the subset of all sequences $\gamma = (\gamma_j)_{j=0}^{\infty}$ taken from the open unit disk $\mathbb{D}$ for which the product (0.2) converges. Hence, if $\theta \in \mathcal{SP} \setminus J$, then Theorem 0.5 implies that the Schur parameter sequence γ of θ belongs to $\Gamma\ell_2$. We define the coshift $W : \ell_2 \to \ell_2$ via

$$(\gamma_0, \gamma_1, \gamma_2, \dots) \mapsto (\gamma_1, \gamma_2, \gamma_3, \dots), \quad \gamma = (\gamma_j)_{j=0}^{\infty} \in \ell_2. \tag{0.3}$$

In the sequel, the system of functions $(L_n(\gamma))_{n=0}^{\infty}$, which was introduced for $\gamma \in \Gamma\ell_2$ in [10, formula (3.12)] will play an important role. We recall its construction. For $\gamma = (\gamma_0, \gamma_1, \gamma_2, \dots) \in \Gamma\ell_2$ we set

$$\begin{aligned} L_0(\gamma) &:= 1, \\ L_n(\gamma) &:= \sum_{r=1}^{n} (-1)^r \sum_{s_1+\ldots+s_r=n} \sum_{j_1=n-s_1}^{\infty} \sum_{j_2=s_1-s_2}^{\infty} \dots \\ &\quad \dots \sum_{j_r=s_{r-1}-s_r}^{\infty} \gamma_{j_1}\overline{\gamma_{j_1+s_1}}\gamma_{j_2}\overline{\gamma_{j_2+s_2}} \dots \gamma_{j_r}\overline{\gamma_{j_r+s_r}}. \end{aligned} \tag{0.4}$$

Here the summation runs over all ordered r-tuples $(s_1, \dots, s_r)$ of positive integers which satisfy $s_1 + s_2 + \dots + s_r = n$. For example,

$$L_1(\gamma) = -\sum_{j=0}^{\infty} \gamma_j\overline{\gamma_{j+1}}, \quad L_2(\gamma) = -\sum_{j=0}^{\infty} \gamma_j\overline{\gamma_{j+2}} + \sum_{j_1=1}^{\infty}\sum_{j_2=0}^{\infty} \gamma_{j_1}\overline{\gamma_{j_1+1}}\gamma_{j_2}\overline{\gamma_{j_2+1}}.$$

In view of $\gamma \in \Gamma l_2$ the series in (0.4) converges absolutely.

Let $n \in \mathbb{N}$ and let

$$\mathfrak{L}_n(\gamma) = \begin{pmatrix} \Pi_1 & 0 & 0 & \dots & 0 \\ \Pi_2 L_1(W\gamma) & \Pi_2 & 0 & \dots & 0 \\ \Pi_3 L_2(W\gamma) & \Pi_3 L_1(W^2\gamma) & \Pi_3 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \Pi_n L_{n-1}(W\gamma) & \Pi_n L_{n-2}(W^2\gamma) & \Pi_n L_{n-3}(W^3\gamma) & \dots & \Pi_n \end{pmatrix}, \tag{0.5}$$

where

$$\Pi_k := \prod_{j=k}^{\infty} \sqrt{1 - |\gamma_j|^2}, \quad k \in \{1, 2, 3, \dots\}. \tag{0.6}$$

We consider the matrices

$$\mathcal{A}_n(\gamma) := I_n - \mathfrak{L}_n(\gamma)\mathfrak{L}_n^*(\gamma), \quad n \in \mathbb{N}, \tag{0.7}$$

and their determinants

$$\sigma_n(\gamma) := \begin{cases} 1, & \text{if } n = 0, \\ \det \mathcal{A}_n(\gamma), & \text{if } n \in \mathbb{N}. \end{cases} \tag{0.8}$$

The important role, which the determinants introduced in (0.8) play for the investigation of Schur functions, becomes clear from the following statement.

Theorem 0.6 ([11, Theorem 5.9]). *Let $\theta \in \mathcal{S}$ with Schur parameter sequence $\gamma = (\gamma_j)_{j=0}^{w}$. Then the following statements are equivalent:*

(i) *θ is a rational noninner Schur function.*

(ii) *$w = \infty$, the sequence γ belongs to $\Gamma\ell_2$ and there exists an $n \in \mathbb{N}$ such that $\sigma_n(\gamma) = 0$.*

It is well-known that rational inner functions are finite Blaschke products and, as it was already mentioned above, I. Schur [17] proved that this case is characterized by $w < \infty$ and $|\gamma_w| = 1$.

0.5. **Formulation of the main theorem.** The main aim of this paper is the following result (see also Theorem 3.3 of this paper).

Theorem 0.7. *Let $\theta \in \mathcal{S}$ and let $\gamma = (\gamma_j)_{j=0}^{w}$ be the Schur parameter sequence of θ. Let $m \in \mathbb{N}$. Then $\theta \in \mathcal{SP}_m$ if and only if one of the following two conditions is satisfied:*

(1) *It hold $w = m$, $\gamma_0 = \gamma_1 = \cdots = \gamma_{m-1} = 0$, $\gamma_m = u \in \mathbb{T}$. In this case $\theta(\zeta) = u\zeta^m$, $\zeta \in \mathbb{D}$, and θ is obviously an inner function.*

(2) *It hold $w = \infty$, $\gamma \in \Gamma\ell_2$ and the relation*

$$\sigma_m > 0, \ \sigma_{m+1} = 0 \tag{0.9}$$

are satisfied, where the determinants σ_k, $k \in \{0, 1, 2, \dots\}$ are given via (0.8), whereas the matrix

$$\begin{pmatrix} -\overline{\gamma}_0\gamma_1 & -\overline{\gamma}_0 D_{\gamma_1}\gamma_2 & \dots & -\overline{\gamma}_0 \prod\limits_{j=1}^{m-2} D_{\gamma_j}\gamma_{m-1} & -\overline{\gamma}_0 \prod\limits_{j=1}^{m-1} D_{\gamma_j}\gamma_m - D_{\gamma_m}\alpha_1 \\ D_{\gamma_1} & -\overline{\gamma}_1\gamma_2 & \dots & -\overline{\gamma}_1 \prod\limits_{j=2}^{m-2} D_{\gamma_j}\gamma_{m-1} & -\overline{\gamma}_1 \prod\limits_{j=2}^{m-1} D_{\gamma_j}\gamma_m - D_{\gamma_m}\alpha_2 \\ 0 & D_{\gamma_2} & \dots & -\overline{\gamma}_2 \prod\limits_{j=3}^{m-2} D_{\gamma_j}\gamma_{m-1} & -\overline{\gamma}_2 \prod\limits_{j=3}^{m-1} D_{\gamma_j}\gamma_m - D_{\gamma_m}\alpha_3 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & D_{\gamma_{m-1}} & -\overline{\gamma}_{m-1}\gamma_m - D_{\gamma_m}\alpha_m \end{pmatrix}, \tag{0.10}$$

is nilpotent. Here

$$D_{\gamma_j} := \sqrt{1 - |\gamma_j|^2}, \quad j \in \{0, 1, 2, \dots\}, \tag{0.11}$$

and the vector

$$a := \operatorname{col}(\alpha_1, \alpha_2, \dots, \alpha_m, 1) \tag{0.12}$$

satisfies the homogeneous linear system

$$\mathcal{A}_{m+1}(\gamma) \cdot a = 0_{(m+1)\times 1}$$

and $\mathcal{A}_n(\gamma)$ is given by (0.7). In this case $\theta \in \mathcal{SP}_m \setminus J$.

As it follows from Theorem 0.6, in the second case the conditions (0.9) mean that the function θ is a rational Schur function and the nilpotency property of the matrix (0.10) implies that this rational function is a polynomial. From the shape of the matrix (0.10) it becomes immediately clear that its $(m-1)$-th power is the nonzero matrix (see also the proof of Theorem 3.2). Thus, the nilpotency index of this matrix equals m.

0.6. **About the content of the paragraphs.** In section 1, we start with a short summary on properties of the Taylor coefficient sequences of Schur functions. This leads us to the class of central Schur functions. At the end of Section 1, we characterize those polynomial Schur functions which are central Schur functions.

The central point of the strategy applied in this paper was to consider a Schur function as characteristic operator function of some unitary colligation. The main theme of Section 2 is to recall the main aspects of the operator-theoretic model associated with a Schur function, which was developed in [11]. This model turned out to be the key instrument to obtain the description of the Schur parameters of pseudocontinuable Schur functions. Since polynomial Schur functions are clearly pseudocontinuable, it is possible to get information about the Schur parameter sequences of polynomial Schur functions by speci-fying the more general considerations done in [11]. The specific feature of the polynomial situation is handled in Section 2, which contains an adaptation of a result due to Radu Theodorescu [21] to our concrete situation. The message of the main result (see Theorem 2.28) is that we consider a triangularization of a contraction T with respect to a particular orthogonal decomposition, which produces a nilpotent operator.

The final Section 3 is the central one of this paper. Here we obtain our main result (see Theorem 3.3 as well as Theorem 0.7). In [13] the authors described S-recurrence of Schur parameters of non-inner rational Schur functions. The additional property, which ensures that a rational Schur function θ is a polynomial of degree m, can be roughly described as follows: A certain matrix (see (0.10)) has to be nilpotent to the power m. At the end of this paper we illustrate Theorem 3.3 by the discussion of some examples of polynomial Schur functions under the view of this paper.

1. On Schur sequences

In this section, we summarize some important properties on the Taylor coefficients of Schur functions.

Let $n \in \mathbb{N}_0$ and let $(A_j)_{j=0}^n$ be a sequence from $\mathbb{C}$. Then we set

$$S_n := \begin{pmatrix} A_0 & 0 & \dots & 0 \\ A_1 & A_0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_n & A_{n-1} & \dots & A_0 \end{pmatrix}.$$

If the matrix $I_{n+1} - S_n S_n^*$ (or equivalently $I_{n+1} - S_n^* S_n$) is non-negative (resp. positive) Hermitian, then $(A_j)_{j=0}^n$ is called a *Schur sequence* (resp. *strict Schur sequence*).

Let $(A_j)_{j=0}^\infty$ be a sequence from $\mathbb{C}$. Then $(A_j)_{j=0}^\infty$ is called a Schur sequence (resp. strict Schur sequence) if for each $n \in \mathbb{N}_0$ the sequence $(A_j)_{j=0}^n$ is a Schur sequence (resp. strict Schur sequence).

The importance of Schur sequences is caused by the following result due to I. Schur [17].

Theorem 1.1. (a) *Let $\theta \in \mathcal{S}$ and let $(A_j)_{j=0}^\infty$ be the sequence of Taylor coefficients of θ. Then $(A_j)_{j=0}^\infty$ is a Schur sequence.*

(b) *Let $(A_j)_{j=0}^\infty$ be a Schur sequence. Then for $\zeta \in \mathbb{D}$ the sequence $(\sum_{j=0}^n A_j \zeta^j)_{j=0}^\infty$ converges and its limit function θ belongs to $\mathcal{S}$.*

Schur sequences have a remarkable inner structure. This will be described now.

Let $n \in \mathbb{N}_0$ and let $(A_j)_{j=0}^n$ be a sequence from $\mathbb{C}$. In the case $n \in \mathbb{N}$, for $j \in \{1, \dots, n\}$, we set

$$y_j := (A_1, A_2, \dots, A_j)^T, \quad z_j := (A_j A_{j-1}, \dots, A_1).$$

Further, we set

$$m_n := \begin{cases} 0, & \text{if } n = 0 \\ -z_{n-1} S_{n-2}^* (I_{n-1} - S_{n-2} S_{n-2}^*)^\dagger y_{n-1}, & \text{if } n \in \mathbb{N} \end{cases},$$

$$r_n := \begin{cases} 1 - |A_0|^2, & \text{if } n = 0 \\ 1 - |A_0|^2 - y_{n-1}^* (I_{n-1} - S_{n-2} S_{n-2}^*)^\dagger y_{n-1}, & \text{if } n \in \mathbb{N} \end{cases},$$

where $(I_{n-1} - S_{n-2} S_{n-2}^*)^\dagger$ stands for the Moore-Penrose inverse of the matrix $I_{n-1} - S_{n-2} S_{n-2}^*$. Let $(A_j)_{j=0}^\infty$ be a Schur sequence. Then for $s \in \mathbb{N}$ the number A_s belongs to the closed disk with center m_s and radius r_s (see [12, Section 3.5]). This observation leads us to an important subclass of infinite Schur sequences, namely a Schur sequence $(A_j)_{j=0}^\infty$ is called *central* if there exists an $s \in \mathbb{N}$ such that $A_k = m_k$ is satisfied for $k \in \{s, s+1, \dots\}$.

If $n \in \mathbb{N}_0$ and $(A_j)_{j=0}^n$ is a Schur sequence, then the sequence $(A_j)_{j=n+k}^\infty$, which is recursively defined by $A_{n+k} := m_{n+k}$ is a Schur sequence . It is called the *central Schur sequence* associated with $(A_j)_{j=0}^n$. The function $\theta \in \mathcal{S}$ with Taylor coefficients $(A_j)_{j=0}^\infty$ is then called the *central Schur function* associated with $(A_j)_{j=0}^n$.

The following result indicates the important role of the functions considered in Example 0.4 within the class $\mathcal{SP}_m \setminus J$. Namely, we see now that these functions are the only polynomials of degree m, which are non-inner central Schur functions.

Theorem 1.2. *Let $m \in \mathbb{N}$ and let $(A_j)_{j=0}^m$ be a strict Schur sequence such that the central Schur function f_c associated with $(A_j)_{j=0}^m$ belongs to $\mathcal{SP}_m \setminus J$. Then $A_m \in \mathbb{D} \setminus \{0\}$ and $A_j = 0$ for $j \in \{0, \dots, m-1\}$.*

Proof. We use the concrete expression for the function f_c, which was obtained in [14]. Namely, it was shown there that the function f_c is rational. We recall the corresponding quotient representation of f_c. Let $e_{m-1} : \mathbb{C} \to \mathbb{C}^{m+1}$ be defined by

$$e_{m-1}(\zeta) := (1, \zeta, \dots, \zeta^{m-1}).$$

Further, let $\pi_m : \mathbb{C} \to \mathbb{C}$ and $\rho_m : \mathbb{C} \to \mathbb{C}$ be defined by

$$\pi_m(\zeta) := A_0 + \zeta \cdot e_{m-1}(\zeta)(I_m - S_{m-1}S_{m-1}^*)^{-1} y_m$$

and

$$\rho_m(\zeta) := 1 + \zeta \cdot e_{m-1}(\zeta)(I_m - S_{m-1}^* S_{m-1})^{-1} S_{m-1}^* y_m.$$

Then, for $\zeta \in \mathbb{D}$, it holds $\rho_m(\zeta) \neq 0$ and

$$f_c(\zeta) = \frac{\pi_m(\zeta)}{\rho_m(\zeta)}. \tag{1.1}$$

Let P_c be the unique polynomial, which satisfies $\mathrm{Rstr.}_{\mathbb{D}} P_c = f_c$. Because of $f_c \in \mathcal{SP}_m \setminus J$, we have then

$$\deg P_c = m.$$

From (1.1) we get

$$\pi_m = \rho_m \cdot P_c,$$

which implies

$$\deg \pi_m = \deg \rho_m + \deg P_c = \deg \rho_m + m.$$

On the other side, from the construction of π_m it is clear that

$$\deg \pi_m \leq m.$$

Thus, $\deg \rho_m = 0$. Combining this with the construction of ρ_m, we see that

$$(I_m - S_{m-1}^* S_{m-1})^{-1} S_{m-1}^* y_m = 0_{(m-1)\times 1}$$

and, consequently, $S_{m-1}^* y_m = 0_{(m-1)\times 1}$. Thus,

$$\begin{pmatrix} \overline{A_0} & \overline{A_1} & \dots & \overline{A_{m-1}} \\ 0 & \overline{A_0} & \dots & \overline{A_{m-2}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \overline{A_0} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{1.2}$$

Because of $P_c(\zeta) = \sum_{j=0}^{m} A_j \zeta^j$ and $\deg P_c = m$, we have $A_m \neq 0$. Thus, we obtain successively from (1.2) that $A_j = 0$ for $j \in \{0, \dots, m-1\}$. Because $(A_j)_{j=0}^m$ is a strict Schur sequence, we even have $A_m \in \mathbb{D} \setminus \{0\}$. □

2. On an operator model of a Schur function in terms of its Schur parameters

2.1. **Shifts contained in contractions and unitary colligations.** Let T be a contraction acting in some Hilbert space $\mathfrak{H}$, i.e., $\|T\| \leq 1$ (in this paper all Hilbert spaces are assumed to be complex and separable, all operators are assumed to be linear). The operators

$$D_T := \sqrt{I_{\mathfrak{H}} - T^*T} \quad \text{and} \quad D_{T^*} := \sqrt{I_{\mathfrak{H}} - TT^*}$$

are called the *deficiency operators* of T. The closures of their ranges

$$\mathcal{D}_T := \overline{D_T(\mathfrak{H})} \quad \text{and} \quad \mathcal{D}_{T^*} := \overline{D_{T^*}(\mathfrak{H})}$$

are called the *deficiency spaces* of T. The dimensions of these spaces

$$\delta_T := \dim \mathcal{D}_T \quad \text{and} \quad \delta_{T^*} := \dim \mathcal{D}_{T^*}$$

are called the *deficiency numbers* of the contraction T. In this way, the condition $\delta_T = 0$ (resp. $\delta_{T^*} = 0$) characterizes isometric (resp. coisometric) operators, whereas the conditions $\delta_T = \delta_{T^*} = 0$ characterize unitary operators. Note that an operator is called coisometric if its adjoint is isometric. Clearly, $TD_T^2 = D_{T^*}^2 T$. From here (see, e.g., Sz.-Nagy/ Foias [19, Chapter I]) it follows that $TD_T = D_{T^*}T$. Passing to the adjoint operators we obtain

$$T^* D_{T^*} = D_T T^*. \tag{2.1}$$

Starting from the contraction T we can always find Hilbert spaces $\mathfrak{F}$ and $\mathfrak{G}$ and operators $F : \mathfrak{F} \to \mathfrak{H}$, $G : \mathfrak{H} \to \mathfrak{G}$, and $S : \mathfrak{F} \to \mathfrak{G}$ such that the operator matrix

$$U = \begin{pmatrix} T & F \\ G & S \end{pmatrix} : \mathfrak{H} \oplus \mathfrak{F} \to \mathfrak{H} \oplus \mathfrak{G} \tag{2.2}$$

is unitary, i.e., the conditions $U^*U = I_{\mathfrak{H}\oplus\mathfrak{F}}$ and $UU^* = I_{\mathfrak{H}\oplus\mathfrak{G}}$ are satisfied. Obviously, these identities can be rewritten in the form

$$\begin{aligned} T^*T + G^*G = I_{\mathfrak{H}}, \quad F^*F + S^*S = I_{\mathfrak{F}}, \quad T^*F + G^*S = 0, \\ TT^* + FF^* = I_{\mathfrak{H}}, \quad GG^* + SS^* = I_{\mathfrak{G}}, \quad TG^* + FS^* = 0. \end{aligned} \tag{2.3}$$

As an example for such a construction one can consider the spaces $\mathfrak{F} := \mathcal{D}_{T^*}$, $\mathfrak{G} := \mathcal{D}_T$, and the operators

$$F := \mathrm{Rstr.}_{\mathcal{D}_{T^*}} D_{T^*} : \mathfrak{F} \to \mathfrak{H}, \quad G := D_T : \mathfrak{H} \to \mathfrak{G}, \quad S := \mathrm{Rstr.}_{\mathcal{D}_{T^*}} (-T^*) : \mathfrak{F} \to \mathfrak{G}.$$

Using (2.1), it is easily checked that the conditions (2.3) are fulfilled in this case. Note that in the general situation from (2.3) it follows $G^*G = D_T^2,\ FF^* = D_{T^*}^2$. Hence,

$$\overline{G^*(\mathfrak{G})} = \mathcal{D}_T, \quad \overline{F(\mathfrak{F})} = \mathcal{D}_{T^*}. \tag{2.4}$$

Definition 2.1. The ordered tuple

$$\Delta = (\mathfrak{H}, \mathfrak{F}, \mathfrak{G}; T, F, G, S) \tag{2.5}$$

consisting of three Hilbert spaces $\mathfrak{H}$, $\mathfrak{F}$, $\mathfrak{G}$ and four operators $T,\ F,\ G,\ S$, where

$$T : \mathfrak{H} \to \mathfrak{H}, \quad F : \mathfrak{F} \to \mathfrak{H}, \quad G : \mathfrak{H} \to \mathfrak{G}, \quad S : \mathfrak{F} \to \mathfrak{G}$$

is called a unitary colligation (or more short colligation) if the operator matrix U given via (2.2) is unitary.

The operator T is called the *fundamental operator* of the colligation Δ. Clearly, the fundamental operator of a colligation is a contraction. The operation of representing a contraction T as fundamental operator of a unitary colligation is called embedding T in a colligation. The space $\mathfrak{H}$ of the colligation Δ is called *inner*, whereas the spaces $\mathfrak{F}$ and $\mathfrak{G}$ are called *outer*. This embedding permits us to use the spectral theory of unitary operators for the study of contractions (see, e.g., Sz.-Nagy/ Foias [19]).

The spaces $\mathfrak{H}_{\mathfrak{F}} := \bigvee_{n=0}^{\infty} T^n F(\mathfrak{F})$, $\mathfrak{H}_{\mathfrak{G}} := \bigvee_{n=0}^{\infty} T^{*n} G^*(\mathfrak{G})$ and their orthogonal complements $\mathfrak{H}_{\mathfrak{F}}^{\perp} := \mathfrak{H} \ominus \mathfrak{H}_{\mathfrak{F}}$, $\mathfrak{H}_{\mathfrak{G}}^{\perp} := \mathfrak{H} \ominus \mathfrak{H}_{\mathfrak{G}}$ play an important role in the theory of colligations. Clearly,

$$\mathfrak{H} = \mathfrak{H}_{\mathfrak{F}} \oplus \mathfrak{H}_{\mathfrak{F}}^{\perp}, \tag{2.6}$$

$$\mathfrak{H} = \mathfrak{H}_{\mathfrak{G}}^{\perp} \oplus \mathfrak{H}_{\mathfrak{G}}. \tag{2.7}$$

The spaces $\mathfrak{H}_{\mathfrak{F}}$ and $\mathfrak{H}_{\mathfrak{G}}$ are called the spaces of *controllability* and *observability*, respectively. From (2.4) it follows that these spaces can also be defined in an alternate way, namely

$$\mathfrak{H}_{\mathfrak{F}} := \bigvee_{n=0}^{\infty} T^n \mathcal{D}_{T^*}, \quad \mathfrak{H}_{\mathfrak{G}} := \bigvee_{n=0}^{\infty} T^{*n} \mathcal{D}_T. \tag{2.8}$$

Consequently, the spaces $\mathfrak{H}_{\mathfrak{F}}$ and $\mathfrak{H}_{\mathfrak{G}}$ do not depend on the concrete way of embedding T in a colligation. Note that $\mathfrak{H}_{\mathfrak{F}}$ is invariant with respect to T, whereas $\mathfrak{H}_{\mathfrak{G}}$ is invariant with respect to T^*. This means that $\mathfrak{H}_{\mathfrak{F}}^{\perp}$ and $\mathfrak{H}_{\mathfrak{G}}^{\perp}$ are invariant with respect to T^* and T, respectively. Switching over to the kernel of the adjoint operators in the identities (2.8), we obtain

$$\mathfrak{H}_{\mathfrak{F}}^{\perp} = \bigcap_{n=0}^{\infty} \ker(D_{T^*} T^{*n}), \quad \mathfrak{H}_{\mathfrak{G}}^{\perp} = \bigcap_{n=0}^{\infty} \ker(D_T T^n). \tag{2.9}$$

Theorem 2.2. *The identities* $\mathfrak{H}_{\mathfrak{G}}^{\perp} = \{h \in \mathfrak{H} : \ \|T^n h\| = \|h\|, \ n = 1, 2, 3, \ldots\}$ *and* $\mathfrak{H}_{\mathfrak{F}}^{\perp} = \{h \in \mathfrak{H} : \ \|T^{*n} h\| = \|h\|, \ n = 1, 2, 3, \ldots\}$ *hold true.*

Proof. For $n = 1, 2, 3, \ldots$, clearly, $\|T^n h\|^2 = (T^{*n} T^n h, h) = (T^{*n-1} T^{n-1} h, h) - (T^{*n-1} D_T^2 T^{n-1} h, h)$. Now the first assertion follows from (2.9) and the identity $\|T^{n-1} h\|^2 - \|T^n h\|^2 = \|D_T T^{n-1} h\|^2$, $n = 1, 2, 3, \ldots$. Analogously, the second assertion can be proved. □

Corollary 2.3. *The space* $\mathfrak{H}_{\mathfrak{G}}^{\perp}$ *(resp.* $\mathfrak{H}_{\mathfrak{F}}^{\perp}$*) is characterized by the following properties:*

(a) $\mathfrak{H}_{\mathfrak{G}}^{\perp}$ *(resp.* $\mathfrak{H}_{\mathfrak{F}}^{\perp}$*) is invariant with respect to* T *(resp.* T^**).*
(b) $\mathrm{Rstr.}_{\mathfrak{H}_{\mathfrak{G}}^{\perp}} T$ *(resp.* $\mathrm{Rstr.}_{\mathfrak{H}_{\mathfrak{F}}^{\perp}} T^*$*) is an isometric operator.*
(c) $\mathfrak{H}_{\mathfrak{G}}^{\perp}$ *(resp.* $\mathfrak{H}_{\mathfrak{F}}^{\perp}$*) is the maximal subspace of* $\mathfrak{H}$ *having the properties (a) and (b).*

From the preceding consideration we immediately obtain the following result.

Theorem 2.4. *The identity* $\mathfrak{H}_{\mathfrak{G}}^{\perp} \cap \mathfrak{H}_{\mathfrak{F}}^{\perp} = \{h :\in \mathfrak{H} : \ \|T^{*n} h\| = \|h\| = \|T^n h\|, \ n = 1, 2, 3, \ldots\}$ *holds true.*

Corollary 2.5. *The subspace* $\mathfrak{H}_{\mathfrak{G}}^{\perp} \cap \mathfrak{H}_{\mathfrak{F}}^{\perp}$ *is the maximal among all subspaces* $\mathfrak{H}'$ *of* $\mathfrak{H}$ *having the following properties:* $\mathfrak{H}'$ *reduces* T *and* $\mathrm{Rstr.}_{\mathfrak{H}'} T$ *is a unitary operator.*

A contraction T on $\mathfrak{H}$ is called *completely nonunitary* if there is no nontrivial reducing subspace $\mathfrak{L}$ of $\mathfrak{H}$ for which the operator $\mathrm{Rstr.}_{\mathfrak{L}} T$ is unitary. Consequently, a contraction is completely nonunitary if and only if $\mathfrak{H}_{\mathfrak{G}}^{\perp} \cap \mathfrak{H}_{\mathfrak{F}}^{\perp} = \{0\}$. The colligation Δ given in (2.5) is called *simple* if $\mathfrak{H} = \mathfrak{H}_{\mathfrak{F}} \vee \mathfrak{H}_{\mathfrak{G}}$. Hence, the colligation Δ is simple if and only if its fundamental operator T is a completely nonunitary contraction. Taking into account the Wold decomposition for isometric operators (see, e.g., Sz-Nagy/ Foias [19, Chapter I]) from Corollary 2.3 we infer the following result:

Theorem 2.6. *Let* T *be a completely nonunitary contraction in* $\mathfrak{H}$*. Then the subspace* $\mathfrak{H}_{\mathfrak{G}}^{\perp}$ *(resp.* $\mathfrak{H}_{\mathfrak{F}}^{\perp}$*) is characterized by the following properties:*

(a) $\mathfrak{H}_{\mathfrak{G}}^{\perp}$ *(resp.* $\mathfrak{H}_{\mathfrak{F}}^{\perp}$*) is invariant with respect to* T *(resp.* T^**).*
(b) *The operator* $\mathrm{Rstr.}_{\mathfrak{H}_{\mathfrak{G}}^{\perp}} T$ *(resp.* $\mathrm{Rstr.}_{\mathfrak{H}_{\mathfrak{F}}^{\perp}} T^*$*) is unilateral shift.*

(c) $\mathfrak{H}^{\perp}_{\mathfrak{G}}$ *(resp.* $\mathfrak{H}^{\perp}_{\mathfrak{F}}$*) is the maximal subspace of* $\mathfrak{H}$ *having the properties (a) and (b).*

We say that a unilateral shift $V : \mathfrak{L} \to \mathfrak{L}$ is *contained* in the contraction T if $\mathfrak{L}$ is a subspace of $\mathfrak{H}$, which is invariant with respect to T and $\mathrm{Rstr.}_{\mathfrak{L}}\, T = V$ is satisfied.

Definition 2.7. Let T be a completely nonunitary contraction in $\mathfrak{H}$. Then the shift $V_T := \mathrm{Rstr.}_{\mathfrak{H}^{\perp}_{\mathfrak{G}}}\, T$ is called the maximal shift contained in T.

By a *coshift* we mean an operator the adjoint of which is a unilateral shift. We say that a coshift $\tilde{V} : \tilde{\mathfrak{L}} \to \tilde{\mathfrak{L}}$ is *contained* in T if the unilateral shift $\tilde{V}^*$ is contained in T^*. Then from Theorem 2.6 it follows that the operator $V_{T^*} = \mathrm{Rstr.}_{\mathfrak{H}^{\perp}_{\mathfrak{F}}}\, T^*$ is the maximal shift contained in T^*. If $\mathfrak{H}^{\perp}_{\mathfrak{G}} = \{0\}$ (resp. $\mathfrak{H}^{\perp}_{\mathfrak{F}} = \{0\}$), we will say that the shift V_T (resp. V_{T^*}) has multiplicity zero.

Definition 2.8. Let T be a completely nonunitary contraction in $\mathfrak{H}$. Then the coshift $\tilde{V}_T := (V_{T^*})^*$ is called the maximal coshift contained in T.

Let T be a completely nonunitary contraction in $\mathfrak{H}$. From the above considerations it follows that the decomposition (2.6) corresponds to the block representation

$$T = \begin{pmatrix} T_{\mathfrak{F}} & * \\ 0 & \tilde{V}_T \end{pmatrix}. \tag{2.10}$$

Analogously, the decomposition (2.7) corresponds to the block representation

$$T = \begin{pmatrix} V_T & * \\ 0 & T_{\mathfrak{G}} \end{pmatrix}. \tag{2.11}$$

The properties of a completely nonunitary contraction T are determined in many aspects by the mutual position of the subspaces $\mathfrak{H}^{\perp}_{\mathfrak{F}}$ and $\mathfrak{H}^{\perp}_{\mathfrak{G}}$. According to this we note that in [11, Subsection 3.2] (see also Subsection 2.4 of this paper) a description of the mutual position of these subspaces was given.

Theorem 2.9. *Let* T *be a completely nonunitary contraction in* $\mathfrak{H}$*. Then the multiplicities of the maximal shifts* V_T *and* V_{T^*} *are not greater than* δ_{T^*} *and* δ_T*, respectively.*

Proof. See [11, Theorem 1.9]. □

Corollary 2.10. *Let* Δ *be a simple unitary colligation of type* (2.5)*. Denote* $\mathfrak{L}_0$ *and* $\tilde{\mathfrak{L}}_0$ *the generating wandering subspaces for the maximal shifts* V_T *and* V_{T^*}*, respectively. Then* $\overline{P_{\mathfrak{L}_0} F(\mathfrak{F})} = \mathfrak{L}_0$*,* $\overline{P_{\tilde{\mathfrak{L}}_0} G^*(\mathfrak{G})} = \tilde{L}_0$*, where* $P_{\mathfrak{L}_0}$ *and* $P_{\tilde{\mathfrak{L}}_0}$ *are the orthogonal projections from* $\mathfrak{H}$ *onto* $\mathfrak{L}_0$ *and* $\tilde{\mathfrak{L}}_0$*, respectively.*

Proof. See [11, p. 181]. □

Remark 2.11. In [9, part III] it was shown that the multiplicity of the shift V_T coincides with δ_{T^*} if and only if the multiplicity of the shift V_{T^*} coincides with δ_T. Moreover, all remaining cases connected with the inequalities

$$0 \leq \dim \mathfrak{L}_0 < \delta_{T^*}, \quad 0 \leq \dim \tilde{\mathfrak{L}}_0 < \delta_T$$

are possible.

2.2. **Characteristic operator functions.** Let $\mathfrak{F}$ and $\mathfrak{G}$ be Hilbert spaces.

Definition 2.12. The symbol $\mathcal{S}(\mathbb{D}; \mathfrak{F}, \mathfrak{G})$ denotes the set of all operator-valued functions which are defined and holomorphic in $\mathbb{D}$ and the values of which are contractive operators acting between $\mathfrak{F}$ and $\mathfrak{G}$.

Definition 2.13. Let Δ be the unitary colligation given in (2.5). The operator function

$$\theta_\Delta(\zeta) := S + \zeta G(I_{\mathfrak{H}} - \zeta T)^{-1} F, \quad \zeta \in \mathbb{D},$$

is called the characteristic operator function (c.o.f) of the colligation Δ.

The next result is very important (see, e.g., Brodskii [8]):

Theorem 2.14. *The characteristic operator function θ_Δ of the unitary colligation Δ belongs to the class $\mathcal{S}(\mathbb{D};\mathfrak{F},\mathfrak{G})$. Conversely, suppose that θ is an operator function belonging to the class $\mathcal{S}(\mathbb{D};\mathfrak{F},\mathfrak{G})$. Then there exists a simple unitary colligation Δ of the form* (2.5) *such that θ is the characteristic operator function of Δ.*

Definition 2.15. Let $\Delta_k = (\mathfrak{H}_k, \mathfrak{F}, \mathfrak{G}; T_k, F_k, G_k, S_k)$, $k = 1, 2$, be unitary colligations. Then Δ_1 and Δ_2 are called unitarily equivalent if $S_1 = S_2$ and if there exists a unitary operator $Z : \mathfrak{H}_1 \to \mathfrak{H}_2$ which satisfies $ZT_1 = T_2Z$, $ZF_1 = F_2$, $G_2Z = G_1$.

It can be easily seen that the characteristic operator functions of unitarily equivalent colligations coincide. In this connection it turns out to be important that the converse statement is also true (see, e.g., Brodskii [8]):

Theorem 2.16. *If the characteristic operator functions of two simple colligations coincide then the colligations are unitarily equivalent.*

2.3. **Description of the model of a unitary colligation if the infinite product $\prod_{j=0}^{\infty}(1-|\gamma_j|^2)$ converges.** Let $\theta \in \mathcal{S}$ be such that the sequence $\gamma = (\gamma_j)_{j=0}^{\infty}$ of its Schur parameters fulfills convergence of the product (0.2), i.e. $\gamma \in \Gamma\ell_2$. Further, let

$$\Delta = (\mathfrak{H}, \mathfrak{F}, \mathfrak{G}; T, F, G, S) \tag{2.12}$$

be a simple unitary colligation, which satisfies $\theta_\Delta = \theta$. In the considered case is $\mathfrak{F} = \mathfrak{G} = \mathbb{C}$. we take 1 as basis vector of the one-dimensional vector space $\mathbb{C}$. Set

$$\phi_1' := F(1), \quad \widetilde{\phi}_1' := G^*(1). \tag{2.13}$$

Then using (2.8) we have $\mathfrak{H}_{\mathfrak{F}} = \bigvee_{n=0}^{\infty} T^n\phi_1'$, $\mathfrak{H}_{\mathfrak{G}} = \bigvee_{n=0}^{\infty} T^{*n}\widetilde{\phi}_1'$. If $(f_\alpha)_{\alpha\in\mathcal{A}}$ is some family of vectors from $\mathcal{H}$, the symbol $\bigvee_{\alpha\in\mathcal{A}} f_\alpha$ denotes the smallest (closed) linear subspace, which contains all vectors, which belong to this family. Hence, the Gram-Schmidt orthogonalization procedure determines a unique orthonormal basis $(\phi_k)_{k=0}^{\infty}$ of the subspace $\mathfrak{H}_{\mathfrak{F}}$ such that for $n \in \mathbb{N}$ the conditions

$$\bigvee_{k=1}^{n} \phi_k = \bigvee_{k=0}^{n-1} T^k\phi_1', \quad (T^{n-1}\phi_1', \phi_n) > 0 \tag{2.14}$$

are satisfied. In view of the convergence of the infinite product (0.2), we have $\mathfrak{H}_{\mathfrak{F}}^{\perp} = \mathfrak{H}\ominus\mathfrak{H}_{\mathfrak{F}} \neq \{0\}$ (see [11, Corollary 2.10]). Denote by $\widetilde{L}_0$ the generating wandering subspace for the shift V_{T^*}. Then

$$\mathfrak{H}_{\mathfrak{F}}^{\perp} = \bigoplus_{n=0}^{\infty}(T^*)^n\widetilde{L}_0, \tag{2.15}$$

where, in view of Theorem 2.9, we have $\dim \widetilde{L}_0 = 1$. In view of Corollary 2.10, there exists a unique unit vector $\psi_1 \in \widetilde{L}_0$ such that

$$(\widetilde{\phi}_1', \psi_1) > 0, \tag{2.16}$$

where $\widetilde{\phi}_1'$ is defined in (2.13). In view of (2.15) and (2.16), the sequence $(\psi_k)_{k\in\mathbb{N}}$, where $\psi_k := (T^*)^{k-1}\psi_q$, $k \in \{1, 2, \ldots, \}$ is the unique orthonormal basis in $\mathfrak{H}_{\mathfrak{F}}^{\perp}$ satisfying the conditions

$$(\widetilde{\phi}_1', \psi_1) > 0, \quad \psi_{k+1} = T^*\psi_k, \quad k \in \{1, 2, \ldots\}. \tag{2.17}$$

Definition 2.17. The constructed orthonormal basis

$$\phi_1, \phi_2, \dots ; \psi_1, \psi_2, \dots \tag{2.18}$$

of $\mathfrak{H}$, which satisfies the conditions (2.14) and (2.17) is called *canonical*.

From the form of its construction it becomes clear that the canonical basis of $\mathfrak{H}$ is uniquely defined by conditions (2.14) and (2.17). This allows us to identify in the following considerations operators and their matrix representations with respect to this basis. We note that we suppose in the sequel that the vectors of the canonical basis are ordered as in (2.18). From the above considerations it follows that the matrix of the operator T with respect to the canonical basis of $\mathfrak{H}$ has block triangular form

$$T = \begin{pmatrix} T_{\mathfrak{F}} & \widetilde{R} \\ 0 & \widetilde{V}_T \end{pmatrix}, \quad \widetilde{V}_T := (V_{T^*})^*.$$

The following result is taken from [11, Theorem 2.13].

Theorem 2.18. *Let $\theta \in \mathcal{S}$ be such that its Schur parameters $(\gamma_j)_{j=0}^{\infty}$ yields a convergent product* (0.2)*. Further, let Δ be a simple unitary colligation of the form* (2.12)*, which satisfies $\theta_\Delta = \theta$. Let the canonical basis of the space $\mathfrak{H}$ be given by* (2.18)*. Then the operators T, F, G, and S have the following matrix representations with respect to this basis:*

$$T = \begin{pmatrix} T_{\mathfrak{F}} & \widetilde{R} \\ 0 & \widetilde{V}_T \end{pmatrix}, \tag{2.19}$$

where the operators in (2.19) *are given by*

$$T_{\mathfrak{F}} = \begin{pmatrix} -\overline{\gamma}_0\gamma_1 & -\overline{\gamma}_0 D_{\gamma_1}\gamma_2 & \dots & -\overline{\gamma}_0 \prod\limits_{j=1}^{n-1} D_{\gamma_j}\gamma_n & \dots \\ D_{\gamma_1} & -\overline{\gamma}_1\gamma_2 & \dots & -\overline{\gamma}_1 \prod\limits_{j=2}^{n-1} D_{\gamma_j}\gamma_n & \dots \\ 0 & D_{\gamma_2} & \dots & -\overline{\gamma}_2 \prod\limits_{j=3}^{n-1} D_{\gamma_j}\gamma_n & \dots \\ \vdots & \vdots & & \vdots & \\ 0 & 0 & \dots & -\overline{\gamma}_{n-1}\gamma_n & \dots \\ 0 & 0 & \dots & D_{\gamma_n} & \dots \\ \vdots & \vdots & & \vdots & \end{pmatrix}, \tag{2.20}$$

$$\widetilde{R} = \begin{pmatrix} -\overline{\gamma}_0 \prod\limits_{j=1}^{\infty} D_{\gamma_j} & 0 & 0 & \dots \\ -\overline{\gamma}_1 \prod\limits_{j=2}^{\infty} D_{\gamma_j} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \\ -\overline{\gamma}_n \prod\limits_{j=n+1}^{\infty} D_{\gamma_j} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix}, \quad \widetilde{V}_T = \begin{pmatrix} 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

$$F = \operatorname{col}(D_{\gamma_0}, 0, 0, \dots ; 0, 0, 0, \dots)\ , \tag{2.21}$$

$$G = (\gamma_1 D_{\gamma_0}, \gamma_2 \prod_{j=0}^{1} D_{\gamma_j}, \dots, \gamma_n \prod_{j=0}^{n-1} D_{\gamma_j}, \dots ; \prod_{j=0}^{\infty} D_{\gamma_j}, 0, 0, \dots), \tag{2.22}$$

$$S = \gamma_0, \tag{2.23}$$

where $\{D_{\gamma_j}\}_{j=0}^{\infty}$ *are defined in* (0.11).

Remark 2.19. According to a short description of the history connected with the model representation (2.19)–(2.22), we refer the reader to [11, 2.8. Comments].

2.4. **On some connections between the maximal shifts V_T and V_{T^*} and the pseudocontinuability of the corresponding c. o. f. θ.** Let $\theta \in \mathcal{S}$ be such that the product (0.2) formed from its Schur parameter sequence $(\gamma_j)_{j=0}^{\infty}$ converges. Further, let Δ be a simple unitary colligation of the form (2.12) which satisfies $\theta_\Delta = \theta$. Then from Theorem 2.18 and Remark 2.11 it follows that the contraction T (resp. T^*) contains a nontrivial maximal shift V_T (resp. V_{T^*}). Here, the multiplicities of the shifts V_T and V_{T^*} coincide and are equal to one. We consider the decompositions (2.6). Let

$$\mathfrak{N}_{\mathfrak{G}\mathfrak{F}} := \mathfrak{H}_{\mathfrak{G}} \cap \mathfrak{H}_{\mathfrak{F}}^{\perp}, \quad \mathfrak{N}_{\mathfrak{F}\mathfrak{G}} := \mathfrak{H}_{\mathfrak{F}} \cap \mathfrak{H}_{\mathfrak{G}}^{\perp}, \tag{2.24}$$

$$\mathfrak{H}_{\mathfrak{G}\mathfrak{F}} := \mathfrak{H}_{\mathfrak{G}} \ominus \mathfrak{N}_{\mathfrak{G}\mathfrak{F}}, \quad \mathfrak{H}_{\mathfrak{F}\mathfrak{G}} := \mathfrak{H}_{\mathfrak{F}} \ominus \mathfrak{N}_{\mathfrak{F}\mathfrak{G}}. \tag{2.25}$$

Then we have the following decompositions of $\mathfrak{H}$:

$$\mathfrak{H} = \mathfrak{H}_{\mathfrak{G}}^{\perp} \oplus \mathfrak{H}_{\mathfrak{G}\mathfrak{F}} \oplus \mathfrak{N}_{\mathfrak{G}\mathfrak{F}}, \tag{2.26}$$

$$\mathfrak{H} = \mathfrak{N}_{\mathfrak{F}\mathfrak{G}} \oplus \mathfrak{H}_{\mathfrak{F}\mathfrak{G}} \oplus \mathfrak{H}_{\mathfrak{F}}^{\perp}. \tag{2.27}$$

We note that with respect to the orthogonal decomposition (2.26) the contraction T admits the triangulation

$$T = \begin{pmatrix} V_T & * & * \\ 0 & T_{\mathfrak{G}\mathfrak{F}} & * \\ 0 & 0 & \widetilde{V}_{T_{\mathfrak{G}}} \end{pmatrix}, \tag{2.28}$$

where V_T is the maximal shift contained in T and $\widetilde{V}_{T_{\mathfrak{G}}}$ is the maximal coshift contained in $T_{\mathfrak{G}}$ (see representation (2.11)). Hence, it follows from (2.7), (2.11), (2.26), and (2.28) that with respect to the orthogonal decomposition

$$\mathfrak{H}_{\mathfrak{G}} = \mathfrak{H}_{\mathfrak{G}\mathfrak{F}} \oplus \mathfrak{N}_{\mathfrak{G}\mathfrak{F}}$$

the operator $T_{\mathfrak{G}}$ admits the block representation

$$T_{\mathfrak{G}} = \begin{pmatrix} T_{\mathfrak{G}\mathfrak{F}} & * \\ 0 & \tilde{V}_{T_{\mathfrak{G}}} \end{pmatrix}. \tag{2.29}$$

Analogously, the orthogonal decomposition (2.27) corresponds to the triangulation

$$T = \begin{pmatrix} V_{T_{\mathfrak{F}}} & * & * \\ 0 & T_{\mathfrak{F}\mathfrak{G}} & * \\ 0 & 0 & \widetilde{V}_T \end{pmatrix}, \tag{2.30}$$

and it follows from (2.6), (2.10), and (2.27) that

$$T_{\mathfrak{F}} = \begin{pmatrix} V_{T_{\mathfrak{F}}} & * \\ 0 & T_{\mathfrak{F}\mathfrak{G}} \end{pmatrix}.$$

From (2.24) and (2.25) it follows that

$$\mathfrak{H}_{\mathfrak{G}\mathfrak{F}} = \overline{P_{\mathfrak{H}_{\mathfrak{G}}} \mathfrak{H}_{\mathfrak{F}}}, \quad \mathfrak{H}_{\mathfrak{F}\mathfrak{G}} = \overline{P_{\mathfrak{H}_{\mathfrak{F}}} \mathfrak{H}_{\mathfrak{G}}}. \tag{2.31}$$

Thus,

$$\dim \mathfrak{H}_{\mathfrak{G}\mathfrak{F}} = \dim \mathfrak{H}_{\mathfrak{F}\mathfrak{G}}. \tag{2.32}$$

The following criterion of pseudocontinuability of a noninner Schur function (see, e.g., [11, Theorem 4.5]) plays an important role in our subsequent investigation.

Theorem 2.20. *Let $\theta \in \mathcal{S}$ and let Δ be a simple unitary colligation of the form* (2.12) *which satisfies $\theta_\Delta = \theta$. Then the conditions $\mathfrak{N}_{\mathfrak{G}\mathfrak{F}} \neq \{0\}$ and $\mathfrak{N}_{\mathfrak{F}\mathfrak{G}} \neq \{0\}$ are equivalent. They are satisfied if and only if $\theta \in \mathcal{S}\Pi \setminus J$.*

Corollary 2.21. *If $\theta \in \mathcal{SP} \setminus J$, then $\mathfrak{N}_{\mathfrak{G}\mathfrak{F}} \neq \{0\}$ and $\mathfrak{N}_{\mathfrak{F}\mathfrak{G}} \neq \{0\}$.*

We consider the sequence of vectors

$$\widetilde{\psi}_j = \sum_{k=j}^{\infty} \Pi_k L_{k-j}(W^j\gamma)\phi_k + \sum_{k=1}^{\infty} Q(W^{k+j-1}\gamma)\psi_k, \quad j \in \{1, 2, 3, \dots\}, \tag{2.33}$$

where

$$Q(\gamma) = -\sum_{j=0}^{\infty} \gamma_j L_j(\gamma)$$

and $L_k(\gamma), \Pi_k, W$ are defined in (0.4), (0.6), and (0.3), respectively. The following result indicates the key role of the vector sequence $(\widetilde{\psi}_j)_{j=1}^{\infty}$. It follows from Section 3 in [11] (see, in particular, Theorem 3.6 and formula (3.4) in [11]).

Theorem 2.22. *The vector sequence $(\widetilde{\psi}_j)_{j=1}^{\infty}$ is an orthonormal basis of the subspace $\mathfrak{H}_{\mathfrak{G}}^{\perp}$, which satisfies*

$$V_T\widetilde{\psi}_j = \widetilde{\psi}_{j+1}, \quad j \in \{1, 2, \dots\}.$$

Remark 2.23. Thus, the one-dimensional subspace generated by the vector $\widetilde{\psi}_1$ is a generating wandering subspace for the shift V_T.

2.5. **Construction of a countable total vector system in $\mathfrak{H}_{\mathfrak{G}\mathfrak{F}}$ and computation of the corresponding Gram determinants.** Let $\theta \in \mathcal{S}$ be such that its Schur parameter sequence $(\gamma_j)_{j=0}^{\infty}$ belongs to $\Gamma\ell_2$. Further, let Δ be a simple unitary colligation of the form (2.12), which satisfies $\theta_\Delta = \theta$.

Theorem 2.24 ([11, Theorem 5.1])**.** *The linear span of vectors*

$$h_n := \phi_n - \Pi_n \sum_{j=1}^{n} \overline{L_{n-j}(W^j\gamma)}\widetilde{\psi}_j, \quad n \in \mathbb{N} \tag{2.34}$$

is dense in $\mathfrak{H}_{\mathfrak{G}\mathfrak{F}}$. Here $(\phi_k)_{k=1}^{\infty}$ denotes the orthonormal system taken from the canonical basis (2.18) *and $(\widetilde{\psi}_k)_{k=1}^{\infty}$ is the orthonormal system defined in* (2.33)*, whereas W, $(L_k(\gamma))_{k=1}^{\infty}$, and $(\Pi_k)_{k=1}^{\infty}$ are defined in* (0.3)*,* (0.4)*, and* (0.6)*, respectively.*

Corollary 2.25. *It holds*

$$\begin{pmatrix} (h_1,h_1) & (h_2,h_1) & \dots & (h_n,h_1) \\ (h_1,h_2) & (h_2,h_2) & \dots & (h_n,h_2) \\ \vdots & \vdots & & \vdots \\ (h_1,h_n) & (h_2,h_n) & \dots & (h_n,h_n) \end{pmatrix} = I - \mathfrak{L}_n(\gamma)\mathfrak{L}_n^*(\gamma), \quad n \in \mathbb{N}. \tag{2.35}$$

Proof. The assertion is an immediate consequence of (2.34). □

Note that the determinants $\sigma_n(\gamma)$, $n \in \mathbb{N}$, which were defined in (0.8) have a lot of remarkable properties (see [11, Theorem 5.5]).

(1) For $r \in \mathbb{N}$, it holds $0 \le \sigma_r(\gamma) < 1$ and $\sigma_r(\gamma) \ge \sigma_{r+1}(\gamma)$. Moreover, $\lim_{n\to\infty} \sigma_n(\gamma) = 0$.

(2) If there exists some $n_0 \in \{0, 1, 2, \dots\}$, which satisfies $\sigma_{n_0}(\gamma) > 0$ and $\sigma_{n_0+1}(\gamma) = 0$, then $\operatorname{rank} \mathcal{A}_n(\gamma) = n_0$ holds true for $n \ge n_0$. Hereby, $n_0 = \dim \mathfrak{H}_{\mathfrak{G}\mathfrak{F}} (= \dim \mathfrak{H}_{\mathfrak{F}\mathfrak{G}})$, where $\mathfrak{H}_{\mathfrak{G}\mathfrak{F}}$ and $\mathfrak{H}_{\mathfrak{F}\mathfrak{G}}$ are given via (2.25).

Conversely, if $\dim \mathfrak{H}_{\mathfrak{G}\mathfrak{F}} (= \dim \mathfrak{H}_{\mathfrak{F}\mathfrak{G}})$ is a finite number n_0, then $\sigma_{n_0}(\gamma) > 0$ and $\sigma_{n_0+1}(\gamma) = 0$.

Remark 2.26. From Theorem 2.24, relation (2.35) and the concrete forms (0.7) of $\mathcal{A}_n(\gamma)$ and (0.8) of $\sigma_n(\gamma)$, respectively, it follows that $\dim \mathfrak{H}_{\mathfrak{G}\mathfrak{F}} = m$ if and only if the first m vectors of the sequence $(h_k)_{k=1}^{\infty}$ are linear independent (which means they form a basis of the space $\mathfrak{H}_{\mathfrak{G}\mathfrak{F}}$), whereas all vectors of the remaining sequence $(h_k)_{k=m+1}^{\infty}$ are linear combinations of the sequence $(h_k)_{k=1}^{m}$.

2.6. **On a theorem of R. I. Teodorescu.** The investigations by R. I. Teodorescu (see [20], [21]) on contractive operators in Hilbert spaces the c. o. f. of which is of polynomial type played an important role in our considerations. In particular, the following result influenced our approach.

Theorem 2.27 ([21]). *Let T be a contraction on a Hilbert space such that the characteristic operator function of T is a polynomial of degree m. Then $\mathfrak{H}$ admits a decomposition into three orthogonal subspaces such that the corres-ponding matrix of T has the form*

$$\begin{pmatrix} T_1 & * & * \\ 0 & T_2 & * \\ 0 & 0 & T_3 \end{pmatrix},$$

where $T_1,\ T_3^*$ are isometries and $T_2^m = 0$.

We formulate a modified version of Theorem 2.27, which is connected with the above constructed operator model.

Theorem 2.28. *Let $\theta \in \mathcal{S}$ with Schur parameter sequence $\gamma = (\gamma_j)_{j=0}^{w}$. Further, let Δ be a simple unitary colligation of type* (2.12) *which satisfies $\theta_\Delta = \theta$. Then the following statements are equivalent:*

(i) $\theta \in \mathcal{SP} \setminus J$.
(ii) *$w = \infty$, the sequence γ belongs to $\Gamma\ell_2$ and there exists an $n \in \mathbb{N}$ such that the operator $T_{\mathfrak{G}\mathfrak{F}}$ taken from the triangulation* (2.28) *is nilpotent.*

If one of the equivalent conditions (i) and (ii) is satisfied, then

$$\operatorname{degree} \theta = \min\{n \in \mathbb{N} :\ (T_{\mathfrak{G}\mathfrak{F}})^n = 0\}. \tag{2.36}$$

Remark 2.29. Note that the analogous assertion is true also for the triangu-lation (2.30).

Proof. Suppose that the Taylor series representation of θ is given by

$$\theta(\zeta) = \sum_{n=0}^{\infty} c_n \zeta^n. \tag{2.37}$$

From the form (2.12) of Δ and Definition 2.13 we obtain

$$\theta_\Delta(\zeta) = S + \sum_{n=1}^{\infty} \zeta^n G T^{n-1} F. \tag{2.38}$$

In view of $\theta_\Delta = \theta$ the identity theorem for holomorphic functions gives us

$$c_0 = S, \quad c_n = GT^{n-1}F, \quad n \in \mathbb{N}. \tag{2.39}$$

(i)→(ii). Theorem 0.5 implies that the product (0.2) converges. Conse-quently, $\gamma \in \Gamma\ell_2$. Let m be the degree of the polynomial θ. From (2.39) we get

$$F^*(T^*)^r G^* = \overline{c_{r+1}} = 0, \quad r \in \{m, m+1, \dots\}.$$

From this it follows

$$F^*(T^*)^k (T^*)^r (T^*)^\ell G^* = 0 \tag{2.40}$$

for $k, \ell \in \{0, 1, 2, \dots\}$ and $r \in \{m, m+1, \dots\}$. If we fix k and r in (2.40) and vary ℓ, then from (2.8) and the second equality (2.8) we obtain

$$F^*(T^*)^k (T^*)^r h = 0 \tag{2.41}$$

for each $h \in \mathfrak{H}_{\mathfrak{G}}$, $k \in \{0, 1, 2, \dots\}$ and $r \in \{m, m+1, \dots\}$. In (2.41) we fix now r and vary k. Then, from (2.8) and the first equality (2.9), we obtain

$$(T^*)^r h \in \mathfrak{H}_{\mathfrak{F}}^{\perp}, \quad r \in \{m, m+1, \dots\} \tag{2.42}$$

for each $h \in \mathfrak{H}_{\mathfrak{G}}$. The subspace $\mathfrak{H}_{\mathfrak{G}}$ is invariant with respect to T^* and $T_{\mathfrak{G}}^* = \mathrm{Rstr.}_{\mathfrak{H}_{\mathfrak{G}}} T^*$. Thus, for each $h \in \mathfrak{H}_{\mathfrak{G}}$ and $r \in \{m, m+1, \dots\}$, the equality

$$(T_{\mathfrak{G}}^*)^r h \in \mathfrak{H}_{\mathfrak{G}} \cap \mathfrak{H}_{\mathfrak{F}}^{\perp} = \mathfrak{N}_{\mathfrak{G}\mathfrak{F}} \tag{2.43}$$

follows from (2.42) and the first equality (2.24).

We note that the contraction $T_{\mathfrak{G}}$ admits the triangulation (2.29). Thus, the condition (2.45) is equivalent to

$$(T_{\mathfrak{G}\mathfrak{F}})^m = 0.$$

Hence, if for $r \in \mathbb{N}$ it holds $(T_{\mathfrak{G}\mathfrak{F}})^r = 0$, then

$$r \leq \operatorname{degree} \theta. \tag{2.44}$$

Thus, condition (ii) is satisfied.

(ii)→(i). The operators V_t and $\widetilde{V}_{T_{\mathfrak{G}}}$ occuring in the block representation (2.28) are a unilateral shift and a unilateral coshift, respectively. Hence,

$$V_T^* V_T = I_{\mathfrak{G}^{\perp}}, \quad \widetilde{V}_{T_{\mathfrak{G}}} \widetilde{V}_{T_{\mathfrak{G}}}^* = I_{\mathfrak{N}_{\mathfrak{G}\mathfrak{F}}}.$$

This implies that with respect to the orthogonal decomposition (2.26) we have the block representations

$$I - T^*T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}, \quad I - TT^* = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

From this relation and the conditions of unitarity of a colligation (see (2.3))

$$T^*T + G^*G = I_{\mathfrak{H}}, \quad TT^* + FF^* = I_{\mathfrak{H}}$$

we conclude that with respect to the orthogonal decomposition (2.26) the mappings G and F have the block representations

$$G = (0, *, *), \quad F = \begin{pmatrix} * \\ * \\ 0 \end{pmatrix}.$$

From these representations and triangulation (2.28) we obtain

$$c_n = GT^{n-1}F = GP_{\mathfrak{G}\mathfrak{F}} T_{\mathfrak{G}\mathfrak{F}}^{n-1} P_{\mathfrak{G}\mathfrak{F}} F, \quad n \in \mathbb{N}. \tag{2.45}$$

where $P_{\mathfrak{G}\mathfrak{F}}$ is the orthogonal projection from $\mathfrak{H}$ onto $\mathfrak{H}_{\mathfrak{G}\mathfrak{F}}$. Because of (ii) there exists an $n \in \mathbb{N}$ such that $T_{\mathfrak{G}\mathfrak{F}}^n = 0$. Then, in view of (2.45),

$$c_l = 0, \quad l > n.$$

Thus, $\theta \in \mathcal{SP}$ and

$$\operatorname{degree} \theta \leq \min\{n \in \mathbb{N} : \ (T_{\mathfrak{G}\mathfrak{F}})^n = 0\}. \tag{2.46}$$

Taking into account $\gamma \in \Gamma\ell_2$, we infer from Theorem 0.3 that $\theta \notin J$. Hence, (i) is satisfied.

From (2.44) and (2.46) we infer

$$\operatorname{degree}\theta = \min\{n \in \mathbb{N} : (T_{\mathfrak{G}\mathfrak{F}})^n = 0\}.$$

□

3. The main result

3.1. The matrix of the operator $T_{\mathfrak{G}\mathfrak{F}}$ with respect to the basis $(h_k)_{k=1}^m$ in the case $\dim \mathfrak{H}_{\mathfrak{G}\mathfrak{F}} = m < +\infty$. The following assertion plays an important role in our approach.

Theorem 3.1. *Let $\theta \in \mathcal{S}$ be such that its Schur parameter sequence $\gamma = (\gamma_j)_{j=0}^\infty$ satisfies $\gamma \in \Gamma\ell_2$. Further, let Δ be a simple unitary colligation of the form* (2.12) *which satisfies $\theta_\Delta = \theta$. Consider the orthogonal decomposition* (2.26)*) of $\mathfrak{H}$ and let $\dim \mathfrak{H}_{\mathfrak{G}\mathfrak{F}} = m < +\infty$. Then the matrix of the operator $T_{\mathfrak{G}\mathfrak{F}}$ (see* (2.28)*) with respect to the basis $(h_k)_{k=1}^m$ (see* (2.34)*) is given by*

$$(3.1)\qquad \begin{pmatrix} -\overline{\gamma}_0\gamma_1 & -\overline{\gamma}_0 D_{\gamma_1}\gamma_2 & \dots & -\overline{\gamma}_0 \prod\limits_{j=1}^{m-2} D_{\gamma_j}\gamma_{m-1} & -\overline{\gamma}_0 \prod\limits_{j=1}^{m-1} D_{\gamma_j}\gamma_m - D_{\gamma_m}\alpha_1 \\ D_{\gamma_1} & -\overline{\gamma}_1\gamma_2 & \dots & -\overline{\gamma}_1 \prod\limits_{j=2}^{m-2} D_{\gamma_j}\gamma_{m-1} & -\overline{\gamma}_1 \prod\limits_{j=2}^{m-1} D_{\gamma_j}\gamma_m - D_{\gamma_m}\alpha_2 \\ 0 & D_{\gamma_2} & \dots & -\overline{\gamma}_2 \prod\limits_{j=3}^{m-2} D_{\gamma_j}\gamma_{m-1} & -\overline{\gamma}_2 \prod\limits_{j=3}^{m-1} D_{\gamma_j}\gamma_m - D_{\gamma_m}\alpha_3 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & D_{\gamma_{m-1}} & -\overline{\gamma}_{m-1}\gamma_m - D_{\gamma_m}\alpha_m \end{pmatrix},$$

where the vector

$$(3.2)\qquad a := \operatorname{col}(\alpha_1, \alpha_2, \dots, \alpha_m, 1)$$

satisfies the homogeneous linear system

$$(3.3)\qquad \mathcal{A}_{m+1}(\gamma) \cdot a = 0_{(m+1)\times 1}$$

and $\mathcal{A}_n(\gamma)$ is given by (0.7).

Proof. The assumption $\dim \mathfrak{H}_{\mathfrak{G}\mathfrak{F}} = m < +\infty$ implies (see Remark 2.26) that the vectors $(h_k)_{k=1}^m$ form a basis of the space $\mathfrak{H}_{\mathfrak{G}\mathfrak{F}}$, whereas the vectors $(h_k)_{k=m+1}^\infty$ are linear combinations of them. In the language of determinants $(\sigma_n(\gamma))_{n\in\mathbb{N}}$ (see (0.8)) this means that

$$\sigma_m(\gamma) > 0, \quad \sigma_{m+1}(\gamma) = 0.$$

Thus, there exist complex numbers $\alpha_1, \dots, \alpha_n$ such that

$$(3.4)\qquad h_{m+1} = -\alpha_1 h_1 - \alpha_2 h_2 - \dots - \alpha_m h_m.$$

It can be seen from (2.35) that the vector

$$a := \operatorname{col}(\alpha_1, \alpha_2, \dots, \alpha_m, 1)$$

satisfies the equation

$$\mathcal{A}_{m+1}(\gamma) \cdot a = 0_{(m+1)\times 1}.$$

In order to obtain the shape (3.1) of the matrix $T_{\mathfrak{G}\mathfrak{F}}$ we apply the operator T on both sides of identity (2.34). This gives us

$$(3.5)\qquad Th_n = T\phi_n - \Pi_n \sum_{j=1}^n \overline{L_{n-j}(W^j\gamma)} T\widetilde{\psi}_j, \quad n \in \mathbb{N}.$$

From the model representation (2.20) we get

$$T\phi_1 = T_{\mathfrak{F}}\phi_1 = -\overline{\gamma_0}\gamma_1\phi_1 + D_{\gamma_1}\phi_2,$$

$$T\phi_n = T_{\mathfrak{F}}\phi_n = -\sum_{k=1}^{n-1}(\overline{\gamma_{k-1}}\prod_{j=k}^{n-1}D_{\gamma_j}\gamma_n)\phi_k - \overline{\gamma_{n-1}}\gamma_n\phi_n + D_{\gamma_n}\phi_{n+1}, \quad n \geq 2.$$

Inserting these expressions into (3.5) and taking into account Remark 2.23, we obtain

$$Th_1 = -\overline{\gamma_0}\gamma_1\phi_1 + D_{\gamma_1}\phi_2 - \Pi_1\widetilde{\psi}_2, \tag{3.6}$$

$$\begin{aligned} Th_n = &-\sum_{k=1}^{n-1}(\overline{\gamma_{k-1}}\prod_{j=k}^{n-1}D_{\gamma_j}\gamma_n)\phi_k - \overline{\gamma_{n-1}}\gamma_n\phi_n + D_{\gamma_n}\phi_{n+1} \\ &- \Pi_n\sum_{j=1}^{n}\overline{L_{n-j}(W^j\gamma)}\widetilde{\psi}_{j+1}, \quad n \geq 2. \end{aligned} \tag{3.7}$$

Taking into account the orthogonality relations $\widetilde{\psi}_j \perp \mathfrak{H}_{\mathfrak{G}}$, $j \in \mathbb{N}$, from (2.34) we infer

$$h_n = P_{\mathfrak{G}\mathfrak{F}}h_n = P_{\mathfrak{G}\mathfrak{F}}\phi_n, \quad n \in \mathbb{N},$$

where $P_{\mathfrak{G}\mathfrak{F}}$ is the orthogonal projection from $\mathfrak{H}$ onto $\mathfrak{H}_{\mathfrak{G}\mathfrak{F}}$. Further from the triangulation (2.28) we get

$$P_{\mathfrak{G}\mathfrak{F}}Th = T_{\mathfrak{G}\mathfrak{F}}h, \quad h \in \mathfrak{H}_{\mathfrak{G}\mathfrak{F}}.$$

For this reason, applying the orthoprojection $P_{\mathfrak{G}\mathfrak{F}}$ on both sides of the identities (3.6) and (3.7) , for $1 \leq n \leq m-1$, we get

$$T_{\mathfrak{G}\mathfrak{F}}h_1 = -\overline{\gamma_0}\gamma_1 h_1 + D_{\gamma_1}h_2,$$

$$T_{\mathfrak{G}\mathfrak{F}}h_n = -\sum_{k=1}^{n-1}(\overline{\gamma_{k-1}}\prod_{j=k}^{n-1}D_{\gamma_j}\gamma_n)h_k - \overline{\gamma_{n-1}}\gamma_n h_n + D_{\gamma_n}h_{n+1}, \quad 2 \leq n \leq (m-1).$$

Taking into account (3.4), from (3.7) it follows, for $n = m$, then

$$\begin{aligned} T_{\mathfrak{G}\mathfrak{F}}h_m &= -\sum_{k=1}^{m-1}(\overline{\gamma_{k-1}}\prod_{j=k}^{m-1}D_{\gamma_j}\gamma_m)h_k - \overline{\gamma_{m-1}}\gamma_m h_m + D_{\gamma_m}h_{m+1} \\ &= -\sum_{k=1}^{m-1}(\overline{\gamma_{k-1}}\prod_{j=k}^{m-1}D_{\gamma_j}\gamma_m + D_{\gamma_m}\alpha_k))h_k - (\overline{\gamma_{m-1}}\gamma_m + D_{\gamma_m}\alpha_m)h_m. \end{aligned}$$

□

The following assertion specifies Theorem 2.28.

Theorem 3.2. *Let $\theta \in \mathcal{S}$ with Schur parameter sequence $\gamma = (\gamma_j)_{j=0}^{w}$. Let Δ be a simple unitary colligation of type (2.12) which satisfies $\theta_\Delta = \theta$. Further, let $m \in \mathbb{N}$. Then the following statements are equivalent:*

(i) *$\theta \in \mathcal{SP}_m \setminus J$.*

(ii) *$w = \infty$, the sequence γ belongs to $\Gamma\ell_2$, the subspace $\mathfrak{H}_{\mathfrak{G}\mathfrak{F}}$ taken from the decomposition (2.26) satisfies $\dim \mathfrak{H}_{\mathfrak{G}\mathfrak{F}} = m$ and the operator $T_{\mathfrak{G}\mathfrak{F}}$ taken from the triangulation (2.28) is nilpotent.*

If one of the equivalent conditions (i) and (ii) is satisfied, then the nilpotency index of $T_{\mathfrak{G}\mathfrak{F}}$ equals m.

Proof. (i)→(ii). Let $\theta \in \mathcal{SP}_m \setminus J$. Then, from Theorem 2.28 we get $w = \infty, \gamma \in \Gamma\ell_2$ and $T_{\mathfrak{G}\mathfrak{F}}$ is nilpotent. Let $r := \dim \mathfrak{H}_{\mathfrak{G}\mathfrak{F}}$. Then $(T_{\mathfrak{G}\mathfrak{F}})^r = 0$. Thus, taking into account (2.36), it follows

$$r \geq m. \tag{3.8}$$

From the shape (3.1) of the matrix $T_{\mathfrak{G}\mathfrak{F}}$ with respect to the basis $(h_k)_{k=1}^r$ it follows

$$T_{\mathfrak{G}\mathfrak{F}}^{(r-1)} h_1 = \prod_{k=0}^{r-1} D_{\gamma_k} h_r + \sum_{k=1}^{r-1} c_k h_k \neq 0. \tag{3.9}$$

Thus, $r \leq m$. Combining this with (3.8) we get $r = m$.

(ii)→(i). In this case, it follows from Theorem 2.28 that $\theta \in \mathcal{SP}_m \setminus J$. Moreover, for $r = m$ we see from (3.9) that in the given case $T_{\mathfrak{G}\mathfrak{F}}^{(m-1)} \neq 0$. However, since $T_{\mathfrak{G}\mathfrak{F}}$ is a nilpotent operator and $\dim \mathfrak{H}_{\mathfrak{G}\mathfrak{F}} = m$, then $T_{\mathfrak{G}\mathfrak{F}}^m = 0$. Thus, it follows from (2.36) that $\operatorname{degree} \theta = m$. □

3.2. **Main theorem and examples.** The main aim of this paper is to prove the following result.

Theorem 3.3. *Let $\theta \in \mathcal{S}$ and let Δ be a simple unitary colligation of the form* (2.12) *which satisfies $\theta_\Delta = \theta$. Let $\gamma = (\gamma_j)_{j=0}^w$ be the Schur parameter sequence of θ. Let $m \in \mathbb{N}$. Then $\theta \in \mathcal{SP}_m$ if and only if one of the following two conditions is satisfied:*

(1) *It hold $w = m$, $\gamma_0 = \gamma_1 = \cdots = \gamma_{m-1} = 0$, $\gamma_m = u \in \mathbb{T}$. In this case $\theta(\zeta) = u\zeta^m$, $\zeta \in \mathbb{D}$, and θ is obviously an inner function.*
(2) *It hold $w = \infty$, $\gamma \in \Gamma\ell_2$ and the relation*

$$\sigma_m > 0, \quad \sigma_{m+1} = 0 \tag{3.10}$$

are satisfied, where the determinants σ_k, $k \in \{0, 1, 2, \dots\}$ are given via (0.8), *whereas matrix* (3.1) *is nilpotent. In this case $\theta \in \mathcal{SP}_m \setminus J$ and the nilpotency index of matrix* (3.1) *equals m.*

Proof. Obviously, condition (1) provides a complete characterization of polynomials, which are inner functions.

Let $\theta \in \mathcal{SP}_m \setminus J$. Then Theorem 3.2 and Theorem 3.1 imply that $\dim \mathfrak{H}_{\mathfrak{G}\mathfrak{F}} = m$, $T_{\mathfrak{G}\mathfrak{F}}$ is nilpotent and the nilpotency index of matrix (3.1) equals m. Further, in view of Remark 2.26, we have

$$\sigma_m > 0, \quad \sigma_{m+1} = 0.$$

Hence, the necessity of the conditions of the Theorem is proved.

Conversely, the conditions (3.10) mean that $\dim \mathfrak{H}_{\mathfrak{G}\mathfrak{F}} = m$. Thus, we infer from Theorem 3.2 and Theorem 3.1 that $\theta \in \mathcal{SP}_m \setminus J$. □

The following result (see [13, Section 2]) is important in connection with the consideration of subsequent examples.

Theorem 3.4. *Let $m \in \mathbb{N}$, let $\theta \in \mathcal{S} \backslash J$, and let $\gamma = (\gamma_j)_{j=0}^\infty$ be the Schur parameter sequence of θ. Assume that the conditions* (3.10) *are satisfied and let the vector*

$$a := \operatorname{col}(\alpha_1, \alpha_2, \dots, \alpha_m, 1) \tag{3.11}$$

satisfies the linear equation (3.11). *Further, let*

$$\lambda := -\left[\prod_{k=1}^m D_{\gamma_k}\right] p, \quad p := \operatorname{col}(\alpha_1, \alpha_2, \dots, \alpha_m). \tag{3.12}$$

If $n \in \{m, m+1, \dots\}$*, then*

$$\gamma_{n+1} = \left[\prod_{k=1}^{n} D_{\gamma_k}^{-1}\right] \left(\left[\overleftarrow{\prod_{j=0}^{n-m+1}} \mathfrak{M}_m^{-1}(W^j\gamma) \right] \lambda, \left[\prod_{k=n-m+1}^{n} D_{\gamma_k}^{-1} \right] \eta_m^{-1}(W^{n-m}\gamma) \right), \tag{3.13}$$

where D_{γ_j} *and* W *are defined via* (0.11) *and* (0.3)*, respectively,*

$$\mathfrak{M}_m(\gamma) := \begin{pmatrix} D_{\gamma_1} & 0 & 0 & \dots & 0 \\ -\gamma_1\overline{\gamma}_2 & D_{\gamma_2} & 0 & \dots & 0 \\ -\gamma_1 D_{\gamma_2}\gamma_3 & -\gamma_2\overline{\gamma}_3 & D_{\gamma_3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\gamma_1 \prod_{j=2}^{m-1} D_{\gamma_j}\overline{\gamma}_m & -\gamma_2 \prod_{j=3}^{m-1} D_{\gamma_j}\overline{\gamma}_m & -\gamma_3 \prod_{j=4}^{m-1} D_{\gamma_j}\overline{\gamma}_m & \dots & D_{\gamma_m} \end{pmatrix} \tag{3.14}$$

$$\eta_m(\gamma) := \left(\overline{\gamma_1},\ \overline{\gamma_2} D_{\gamma_1},\ \dots,\ \overline{\gamma_m} \left[\prod_{j=1}^{m-1} D_{\gamma_j} \right] \right)^T. \tag{3.15}$$

Remark 3.5. We note that in the case $m = 1$ the assertion of Theorem 3.4 was already obtained in [11, Theorem 5.22].

Example 3.6. I. Schur considered in his fundamental paper [17, Part II] as an example the function θ given by

$$\theta(\zeta) := \frac{1+\zeta}{2}, \quad \zeta \in \mathbb{D} \tag{3.16}$$

and showed that its Schur parameter sequence is given by

$$\gamma_0 = \frac{1}{2}, \quad \gamma_1 = \frac{2}{3}, \quad \gamma_2 = \frac{2}{5}, \quad \gamma_3 = \frac{2}{7}, \cdots \tag{3.17}$$

This example is of interest for us because the function θ belongs to $\mathcal{SP}_1 \setminus J$. We verify that the conditions of Theorem 3.3 are satisfied for this function. First of all, we note that as it was proved in [13, p. 247] the elements of the sequence (3.17) satisfy the conditions

$$\gamma_{n+1} = \lambda \frac{\gamma_n}{\prod_{j=1}^{n} [1 - |\gamma_j|^2]}, \quad n \geq 1, \tag{3.18}$$

where $\lambda = \frac{1}{3}$. Observe that relation (3.18) is just the condition (3.13) for $m = 1$.

Since the sequence (3.17) belongs to $\Gamma\ell_2$ and consists of nonzero elements, we have in this case

$$\sigma_1(\gamma) = 1 - \prod_{j=1}^{\infty} (1 - |\gamma_j|^2) > 0.$$

In [11] or [13] it was shown that each sequence for which (3.18) is fulfilled, satisfies $\sigma_2(\gamma) = 0$. Thus, for the sequence (3.17) we have $\sigma_1(\gamma) > 0$ and $\sigma_2(\gamma) = 0$ what completely agrees with the fact that the function (3.16) is a polynomial of first degree.

It remains to prove that in the case $m = 1$ the matrix (3.1) formed from the sequence (3.17) is nilpotent, which means that the number $-\overline{\gamma_0}\gamma_1 - D_{\gamma_1}\alpha_1$ has to be zero. In the given case $\gamma_0 = \frac{1}{2}$, $\gamma_1 = \frac{2}{3}$ and, in view of (3.12), we have $-D_{\gamma_1}\alpha_1 = \lambda$. But we have observed above that $\lambda = \frac{1}{3}$. Consequently,

$$-\overline{\gamma_0}\gamma_1 - D_{\gamma_1}\alpha_1 = 0.$$

Hence, the data of the polynomial (3.16) are in harmony with Theorem 3.3.

Example 3.7. We consider the function θ given by

$$\theta(\zeta) := \frac{1+\zeta^2}{2}, \quad \zeta \in \mathbb{D} \tag{3.19}$$

In this case the function θ belongs to $\mathcal{SP}_2 \setminus J$. Taking into account Example 3.6 and the shape of the function (3.19), we obtain

$$\gamma_0 = \frac{1}{2}, \quad \gamma_1 = 0, \quad \gamma_2 = \frac{2}{3}, \quad \gamma_3 = 0, \quad \gamma_4 = \frac{2}{5}, \quad \gamma_5 = 0, \quad \gamma_6 = \frac{2}{7}, \cdots \tag{3.20}$$

From Theorem 3.3 and Theorem 3.4 it follows that the Schur parameter sequence $(\gamma_j)_{j=0}^{\infty}$ of θ fulfills the conditions (3.13) for $m = 2$. This can be verify directly.

Writing the equations (3.13) for $n = 2$ and $n = 3$, (i.e., for γ_3 and γ_4,) and solving the corresponding system of linear equations of $\lambda = (\lambda_1, \lambda_2)^T$, we obtain

$$\lambda_1 = \frac{1}{3}, \quad \lambda_2 = 0. \tag{3.21}$$

Furthermore, we see that the conditions (3.13) are satisfied for $m = 2$, $n \geq 4$ and these $\lambda_1,\ \lambda_2$.

In this example as well as in the preceding, we have $\sigma_1(\gamma) > 0$. If it would be $\sigma_2(\gamma) = 0$, then as it was shown in [11], [13] the Schur parameter sequence (3.20) would satisfy the condition (3.18). However, it can be easily checked that this is not true. Hence, $\sigma_2(\gamma) > 0$. In [13] it was shown that if for $m = 2$ the condition (3.13) is fulfilled, then the corresponding sequence satisfies $\sigma_3(\gamma) = 0$. This agrees completely with the fact that the function (3.19) is a polynomial of second degree.

It remains to prove that in the case $m = 2$ the nilpotency index of the matrix (3.1) formed from the sequence (3.20) equals 2. This matrix is given by

$$\begin{pmatrix} -\overline{\gamma_0}\gamma_1 & -\overline{\gamma_0}D_{\gamma_1}\gamma_2 - D_{\gamma_2}\alpha_1 \\ D_{\gamma_1} & -\overline{\gamma_1}\gamma_2 - D_{\gamma_2}\alpha_2 \end{pmatrix}. \tag{3.22}$$

Taking into account that the vector $a = (\alpha_1, \alpha_2)^T$ satisfies (3.12) and the concrete form (3.21) of the coordinates of the vector $\lambda = (\lambda_1, \lambda_2)^T$, we obtain

$$\frac{1}{3} = -D_{\gamma_1}D_{\gamma_2}\alpha_1, \quad \alpha_2 = 0.$$

From this and formulas (3.20) for the Schur parameters of the function (3.19) we obtain that the matrix (3.21) has shape

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Thus,

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, all conditions of Theorem 3.3 are satisfied for the polynomial θ.

References

1. D. Alpay, I. Gohberg, *Discrete analogs of canonical systems with pseudo-exponential potential. Definitions and formulas for the spectral matrix functions.* In: The State Space Method. Generalizations and Applications (Eds.: D. Alpay, I. Gohberg), Oper. Theory: Adv. Appl., vol. 161, Birkhäuser, Basel, 2006, pp. 1–47.
2. D. Alpay, I. Gohberg, *Inverse problems for discrete analogs of canonical systems.* In: Interpolation, Schur Functions and Moment Problems (Eds.: D. Alpay, I. Gohberg), Oper. Theory: Adv. Appl., vol. 165, Birkhäuser, Basel, 2006, pp. 31–65.
3. Yu. M. Arlinskii, *Schur parameters, Toeplitz matrices, and Krein shorted operators*, Integr. Equ. Oper. Theory **71** (2011), 417–453.

4. Yu. M. Arlinskii, *The Schur problem and block operator CMV matrices*, Complex Anal. Oper. Theory **8** (2014), 875–923.
5. S. Bogner, B. Fritzsche, B. Kirstein, *The Schur-Potapov algorithm for sequences of complex $p \times q$ matrices.* I, Complex Anal. Oper. Theory **1** (2007), 55–95.
6. S. Bogner, B. Fritzsche, B. Kirstein, *The Schur-Potapov algorithm for sequencecs of complex $p \times q$ matrices.* II, Complex Anal. Oper. Theory **1** (2007), 235–278.
7. S. S. Boiko, V. K. Dubovoy, B. Fritzsche, B. Kirstein, *Shift operators contained in contractions and pseudocontinuable matrix-valued Schur functions*, Math. Nachr. **278** (2005), no. 7–8, 784–807.
8. M. S. Brodskii, *Unitary operator colligations and their characteristic functions* (Russian), Uspekhi Math. Nauk **33** (1978), no. 4(202), 141–168; English transl.: Russian Math. Surveys **33** (1978), no. 4, 159–191.
9. V. K. Dubovoy, *Indefinite metric in Schur's interpolation problem for analytic functions* (Russian), Teor. Funkcii, Funkcional. Anal. i Prilozen. (Kharkov), part I: **37** (1982), 14–26; part II: **38** (1982), 32–39; part III: **41** (1984), 55–64; part IV: **42** (1984), 46–57; part V: **45** (1986), 16–26; part VI: **47** (1987), 112–119; English transl.: part I: II. Ser., Amer. Math. Soc. **144** (1989), 47–60; part II: II. Ser. Amer. Math. Soc. **144** (1989), 61–70; part IV: Oper. Theory: Adv. Appl. **95** (1997), 93–104; part V: J. Sov. Math. **48** (1990), no. 4, 376–386; part VI: J. Sov. Math. **48** (1990), no. 6, 701–706.
10. V. K. Dubovoy, *Schur's parameters and pseudocontinuation of contractive holomorphic functions in the unit disk* (Russian), Dopovidi NAN Ukr. **2** (1998), 24–29.
11. V. K. Dubovoy, *Shift operators contained in contractions, Schur parameters and pseudocontinuable Schur functions.* In: Interpolation, Schur Functions and Moment Problems (Eds.: D. Alpay, I. Gohberg), Oper. Theory: Adv. Appl., vol. 165, Birkhäuser, Basel, 2006, 175–250.
12. V. K. Dubovoy, B. Fritzsche, B. Kirstein, *Matricial Version of the Classical Schur Problem*, Teubner-Texte zur Mathematik, Band 129, B. G. Teubner, Stuttgart—Leipzig, 1992.
13. V. K. Dubovoy, B. Fritzsche, B. Kirstein, *The $\mathcal{S}$-recurrence of Schur parameters of non-inner rational Schur functions.* In: Topics in Operator Theory. Volume 1. Operators, Matrices and Analytic Functions, Oper. Theory: Adv. Appl. **202** (2010), 151–194.
14. B. Fritzsche, B. Kirstein, *Representations of central matricial Schur functions in both nondegenerate and degenerate cases*, Analysis **24** (2004), 41–62.
15. S. V. Khrushchev, *Schur's algorithm, orthogonal polynomials, and convergence of Wall's continued fractions in $L^2(\mathbb{T})$*, J. Approx. Theory **108** (2001), 161–248.
16. P. Koosis, *Introduction to H^p Spaces*, Cambridge Univ. Press, Cambridge etc., 1998.
17. I. Schur, *Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind*, J. Reine und Angew. Math., part I: **147** (1917), 205–232; part II: **148** (1918), 122–145.
18. B. Simon, *Orthogonal Polynomials on the Unit Circle*, part 1: *Classical Theory*, Amer. Math. Soc. Colloq. Publ., Providence, RI, vol. 54, 2004.
19. B. Sz.-Nagy, C. Foias, *Harmonic analysis of operators in Hilbert space*, North Holland Publishing Co., Amsterdam—Budapest, 1970.
20. R. Teodorescu, *Fonctions caractéristiques constantes*, Acta Sci. Math. (Szeged) **38** (1976), no. 1–2, 183–185.
21. R. Teodorescu, *Sur la fonction caractéristique d'une contraction*, Rev. Roumaine Math. Pures Appl. **23** (1978), no. 10, 1583–1585.

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