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POINT SPECTRUM IN CONFLICT DYNAMICAL SYSTEMS WITH FRACTAL PARTITION

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Dedicated to outstanding Ukrainian mathematician Anatoly N. Kochubei to his 70th birthday anniversary

Abstract. We discuss the spectral problem for limit distributions of conflict dynamical systems on spaces subjected to fractal divisions. Conditions ensuring the existence of the point spectrum are established in two cases, the repulsive and the attractive interactions between the opponents. A key demand is the strategy of priority in a single region.

1. Introduction

In the present paper we continue the study, started in [24], of the problem connected with appearance of the point spectrum in the limiting singular distributions of dynamical systems on spaces subjected to fractal partitions. Such kind of distributions describe the limiting at time states of specific dynamical systems which we call conflict dynamical systems. They simulate processes of conflict interactions between alternative opponents in a wide sense. An essential feature of our constructions is a procedure of fractal partition of a conflict space which reflects the natural repeating of self-similar or similar structure elements (species) in the population dynamics [6, 7, 20].

Apparently, at first the spectral properties of limiting distributions associated with probability measures as states of the conflict dynamical systems have been studied by Albeverio, Koshmanenko, Prats'ovytyi, and Torbin in [2]–[4], see also [13, 18, 19]. Typically these distributions are pure singular and have a complicated fractal structure [4, 30, 31]. Precisely, it was shown by Koshmanenko in [16] that the class of pure singular continuous singular distributions represents a family of full measure in the space of all limit distributions for conflict dynamical systems. Significantly that pure point (discrete) distributions appears exotically under rather extremal conditions ensuring a rapidly strong local convergence of approximating measures (see condition (19)). This result has analogy with Jenssen-Wintner theorem for the infinite Bernoulli convolutions of probability values.

It should be noted that a notion of the conflict dynamical system, as a specific kind of dynamical system, was introduced and developed by Koshmanenko in the papers [5, 11, 1, 13, 14], (see also [12, 20]). A study of spectral properties for limit distributions takes its beginning in the paper [3]. A typical property of limiting measures to be pure singular continuous, apparently emerges the universal singularity phenomenon. It is well-known in functional analysis and linear operators theory (see, for example,

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[32, 26, 28, 31]). It follows that discrete and absolutely continuous functions and distributions, as well as, both, the point and the absolutely continuous spectra, arise very seldom exotically. Usually, typical objects of mathematical studding dealing with infinite-dimensional constructions (fractal partitions of underlying space) necessary have singular structure. Moreover, they are measured by the second Baire category [32]. Of course, practically, in reality, we use only physical spectral types, absolutely continuous and discrete or point. Our macro world pictures are far from singularities. It is the reason, why in this paper we are interesting in conditions which ensure the appearance of the point spectrum.

Certainly, we start with two sequences of piecewise uniformly distributed probability measures $\mu^t, \nu^t,\ t = 1, 2, \dots$ which approximate the evolution of conflict interaction between opponents at discrete time. We assume that the procedure of fractal partition of the conflict territory (space) into regions is performed simultaneously with the discrete time. It means that for given iterated functional system $\{K_i\}$, a number of contractive maps $K_{i_k},\ k = 1, 2, \dots$ acting on space at moment t exactly coincides with meaning of this time, $k = t$.

We prove weak convergence of μ^t, ν^t to the limit measures, $\mu^\infty = \lim_{t\to\infty} \mu^t, \quad \nu^\infty = \lim_{t\to\infty} \nu^t$, and seek conditions ensuring that at least one of the limit measures is a point. In our setting, if the point spectrum appears, it is concentrated nearly zero. The next important question is, how quickly the point spectrum condensed to zero. The answer have to establish the ranking for distribution of the conflict resource (space) depending of a number of contractive maps.

We note that the mixed types spectrums are impossible. This fact reflects else one analogy with theorems of Jenssen-Wintner type which assert existence of distributions with pure only types of spectrum, i.e., the pure absolutely continuous, the pure point, or the pure absolutely continuous. Nevertheless, there is the possibility to transform different types of spectra one to other in during of conflict redistributions of the space. So, the question on transformation of the singular continuous spectrum into point ones was studied in [17, 10, 18].

In the recent paper by Koshmanenko and Voloshyna, [24] it was established the criterion for appearance of the point spectrum at time limit for conflict dynamical systems on the space $\Omega = [0, 1]$. Our main result asserts that the condition of view (19) (see below) is fulfilled and the limit measure is pure point, if and only if, at least one of opponents picks the strategy of priority in a single region (see condition (17)).

In this paper, we generalize the above result for the case when the conflict space Ω is a compact from $\mathbb{R}^d$. Besides, here we consider two situations, with repulsive and attractive interactions.

2. Conflict dynamical systems in terms of probability measures

Here we shortly recall some basic notations and definitions relating to the theory of conflict dynamical systems in terms of probability measures. More details may be found in [20]–[23] and references given there.

Let Ω be a compact set from $\mathbb{R}^d, d \geq 1$. The Borel σ-algebra and Lebesgue measure are denoted by $\mathcal{B}$ and λ, respectively.

Everywhere in what follows a compact Ω is considered as a conflict space (space of living resource) for a couple of alternative sides A and B which we call by opponents. We suppose that distributions of opponents along Ω at an initial time moment, $t = 0$, may be represented by probability measures μ, ν from some space $\mathcal{M}(\Omega)$. We assume that intersection of their supports is nonempty,

$$\sup(\mu) \bigcap \sup(\nu) \neq \varnothing,$$

that is a reason for conflict between A and B. With aim to describe of conflict interactions between A and B we introduce some noncommutative binary map $\divideontimes$ in the space of probability measures $\mathcal{M}(\Omega)$

$$\mu \divideontimes \nu = \mu^1, \quad \nu \divideontimes \mu = \nu^1.$$

The sequential iteration of $\divideontimes$ with starting $\mu^0 = \mu,\ \nu^0 = \nu$ generates some trajectory at discrete time

$$\left\{ \begin{array}{c} \mu^t \\ \nu^t \end{array} \right\} \xrightarrow{\divideontimes} \left\{ \begin{array}{c} \mu^{t+1} \\ \nu^{t+1} \end{array} \right\}, \quad t = 0, 1, \ldots \tag{1}$$

All such kind trajectories with a fixed law of interaction $\divideontimes$ generate some dynamical system which describes the conflict redistributions of space Ω between opponents A and B. The constructed dynamical system we call the conflict dynamical system and denote by $\{X, \divideontimes\}$, where $X = \mathcal{M}(\Omega) \times \mathcal{M}(\Omega)$ and $\divideontimes$ is a binary map associated with a conflict interaction between opponents (for more details see [11, 13, 14]).

In what follows one can think, without lost of generality, that $\Omega = [0, 1]$ or $\Omega = [0, 1]^d, d \geq 1$. And $\mathcal{M}(\Omega)$ denotes some specific class of measures on Ω which are suitable for description of local priorities one of opponents over other.

The behavior of trajectories $\mu^t, \nu^t,\ t = 0, 1, \ldots$, the study its properties, constitutes the typical problems of the theory. In particular, one of main problem is the question on existence of the limit measures $\mu^\infty = \lim_{t\to\infty} \mu^t, \nu^\infty = \lim_{t\to\infty} \nu^t$. According to the already developed theory, [11, 20], it was proven the existence of final redistribution of space Ω between opponents under rather weak demands on a law of conflict interaction $\divideontimes$ in both, repulsive and attractive cases. However, for each concrete map $\divideontimes$ we have to prove the existence of limit measures $\mu^\infty,\ \ \nu^\infty$ again.

In the present paper we are interesting in the question, under what conditions at least one of the limit measures μ_∞, ν_∞ is pure point, i.e., it is concentrated at most on a countable set. We consider two cases for definition of $\divideontimes$: repulsive and attractive interactions (see below formulas (6)–(10) and (37)). In a role of $\mathcal{M}(\Omega)$ we use piecewise uniformly distributed measures or similar structure measures introduced in [15].

2.1. **Fractal regionalization. Self-similar and similar structure measures.** Below we develop the iterative construction of measures $\mu^k, \nu^k,\ k = 0, 1, \ldots$ connected with the fractal partition of space without effects of conflict interaction. Our construction directly depends of a fixed procedure of fractal partition (regionalization) of the conflict space Ω.

We suppose that Ω is subjected to iterative procedure of regionalization, which, in fact, coincides with typical construction of fractal divisions. It means that Ω admits a sequential separation into similar regions

$$\Omega = \bigcup_{i_1=1}^{n} \Omega_{i_1}, \quad 2 \leq n < \infty,$$

$$\Omega_{i_1} = \bigcup_{i_2=1}^{n} \Omega_{i_1 i_2}, \quad \Omega = \bigcup_{i_1, i_2=1}^{n} \Omega_{i_1 i_2}, \ldots$$

$$\Omega_{i_1 \ldots i_{k-1}} = \bigcup_{i_k=1}^{n} \Omega_{i_1 \ldots i_k}, \quad \Omega = \bigcup_{i_1, \ldots, i_k=1}^{n} \Omega_{i_1 \ldots i_k}, \ldots$$

with condition: $|\Omega_{i_1 \ldots i_k}| = \lambda(\Omega_{i_1 \ldots i_k}) \longrightarrow 0,\ k \to \infty$. Therefore, the subsets $\Omega_{i_1 \ldots i_k}$ generate a basic for the Borel σ-algebra $\mathcal{B}$.

More generally, the described procedure admits treating in terms of some iterated functional systems [6, 9, 29].

We recall that a family of contractive maps $K = \{K_i\}_{i=1}^n, 2 \le n < \infty$ in $\mathbb{R}^d$ is called (see Barnsley, Hutchinson, Triebel) an iterated functional system (IFS) on Ω, if the following conditions hold:

$$\forall i, \quad \Omega_i \subset \Omega, \quad \text{where} \quad \Omega_i := K_i\Omega,$$

and

$$\mathrm{int}(\Omega_i) \bigcap \mathrm{int}(\Omega_j) = \varnothing, \qquad i \neq j.$$

In what follows we additionally assume that

$$\Omega = \bigcup_{i=1}^n \Omega_i, \quad 0 < |\Omega_i| = q_i < 1, \quad 1 = \sum_{i=1}^n q_i. \tag{2}$$

Thus, every ITF with condition (2) generates some fractal division of Ω

$$\Omega = \bigcup_{i_1,\dots,i_k=1}^n \Omega_{i_1\dots i_k}, \quad \Omega_{i_1\dots i_k} := K_{i_1} \circ \cdots \circ K_{i_k}\Omega.$$

Let some ITF and a sequence of stochastic vectors $P = \{\mathbf{p}^k\}_{k=1}^\infty,\ \mathbf{p}^k \in \mathbb{R}^d$ are are given. Here, $\mathbf{p}^k = (p_i^k)_{i=1}^n,\ \ \sum_i p_i^k = 1$. With P we associate a sequence of piecewise uniformly distributed measures μ^k on Ω

$$\mu^k(\Omega_{i_1\dots i_k}) = p_{i_1}^1 p_{i_2}^2 \dots p_{i_k}^k, \quad k = 1, 2, \dots \tag{3}$$

According Hutchinson [9] and Triebel [29] (see also [30, 31, 15]) the sequence $\mu^k, k = 1, 2, \dots$ converges, at least in the weak sense, to some probability measure

$$\mu = \lim_{k\to\infty} \mu^k.$$

Moreover, there exists a one-to-one correspondence between above kind sequences P and a class of similar structure measures $\mu \in \mathcal{M}(\Omega)$ under fixed ITF on Ω. That is, μ is self-similar, if all vectors $\mathbf{p}^k$ are the same. In the general case, μ is a similar structure measure. It means that relations

$$\frac{\mu(\Omega_{i_1\dots i_k})}{\mu(\Omega_{i_1\dots i_{k-1}})} = p_{i_k}^k$$

are only dependent of a last index i_k (for details see ([15, 20]).

We recall, that usually μ is singular continuous, $\mu \in \mathcal{M}_{\mathrm{sc}}(\Omega)$. However, in some extremal cases, these measure may become pure point. It occurs under condition

$$\prod_{k=1}^\infty \max_{i_1\dots i_k} \mu^k(\Omega_{i_1\dots i_k}) > 0. \tag{4}$$

2.2. **Approximation of free states of dynamical systems.** In general, it is a non-trivial problem to construct the well-defined interaction map $\circledast$ in terms of probability measures (see constructions developed in [19], [21]–[23]). What is why further we will develop some approximating approach to description of the free states of dynamical systems in terms of simple measures for which the interaction map $\circledast$ has a rather transparent sense. We shall use the above described fractal regionalization of the conflict space Ω and take into account on each step of approximation more detailed structures of the initial measures.

A measure $\mu \in \mathcal{M}(\Omega)$ we call simple, if it has a piecewise uniformly distribution. So, in the case $\Omega = [0, 1]$ a density of simple measure is a simple function. Such class of probability measures we denote by $\mathcal{M}^{\mathrm{pud}}(\Omega)$ (pud – piecewise uniformly distributed).

Assume, some IFS, $K = \{K_k\}_{k=1}^\infty$, is fixed. Let opponents A and B are represented at a starting moment of time by a couple of measures $\mu, \nu \in \mathcal{M}(\Omega)$, which both have the similar structure. For setting of the conflict problem we put in correspondence to given measures μ, ν two approximating sequences of simple measures $\mu_k, \nu_k,\ k = 1, 2, \dots$

To avoid some pure mathematical problems we will assume some additional conditions.

We say that the iterated functional system K and a measure μ are *conformed*, if there exist a restriction K on a Borel subset $\Omega^\sharp \subseteq \Omega$ (not necessary closed), such that all properties of fractal regionalization described in the previous subsection are valid in terms $\Omega^\sharp_{i_1\dots i_k} = \Omega_{i_1\dots i_k} \bigcap \Omega^\sharp$, $\mu(\Omega^\sharp) = \mu(\Omega)$, and, additionally,

$$\mu(\Omega^\sharp_{i_1\dots i_k} \bigcap \Omega^\sharp_{j_1\dots j_k}) = 0, \quad k = 1, 2, \dots, \tag{5}$$

if $i_l \neq j_l$ at least for a one $1 \leq l \leq k$. Further we always assume that K and both initial measures μ, ν are conformed simultaneously and we can replace $\Omega_{i_1\dots i_k}$ by $\Omega^\sharp_{i_1\dots i_k}$ without changing of notations.

On the first step of approximation, $k = 1$, we replace starting measures $\mu, \nu \in \mathcal{M}(\Omega)$ by their rough versions $\mu_1, \ \nu_1$ from $\mathcal{M}^{\text{pud}}(\Omega)$ defined as follows:

$$\mu_1(\Omega_{i_1}) = \mu(\Omega_{i_1}) := p_{i_1}, \quad \nu_1(\Omega_{i_1}) = \nu(\Omega_{i_1}) := r_{i_1}, \quad i_1 = 1, \dots, n,$$

which are uniformly distributed along every set Ω_{i_1}. The last property distinguishes μ_1 from μ and ν_1 from ν, respectively. Two sets of values $p_{i_1}, \ r_{i_1}, \ i_i = 1, 2, \dots, n$ form two stochastic vectors, $\mathbf{p}_1 = (p_1, \dots, p_n)$ and $\mathbf{r}_1 = (r_1, \dots, r_n)$. Now one can develop some rough version of conflict redistribution of Ω between opponents A and B in terms of these vectors (see [11, 12]). However, we want to develop a more detailed conflict picture.

To this aim we produce the second approximation step and replace μ, ν by more exact their versions μ_2, ν_2 from $\mathcal{M}^{\text{pud}}(\Omega)$

$$\mu_2(\Omega_{i_1 i_2}) = \mu(\Omega_{i_1 i_2}) := p_{i_1 i_2}, \quad \nu_2(\Omega_{i_1 i_2}) = \nu(\Omega_{i_1 i_2}) := r_{i_1 i_2}.$$

With measures μ_2, ν_2 we associate a new couple of stochastic vectors $\mathbf{p}_2 = \{p_{i_2}\}_{i_2=1}^n, \ \mathbf{r}_2 = \{r_{i_2}\}_{i_2=1}^n$, with coordinates

$$p_{i_2} := \frac{p_{i_1 i_2}}{p_{i_1}}, \qquad r_{i_2} := \frac{r_{i_1 i_2}}{r_{i_1}}.$$

That is, these coordinates are well defined, i.e., they are independent of an index i_1 due to starting measures have the similar structure. We can repeat the above procedure for any natural k and define more and more close versions $\mu_k, \nu_k \in \mathcal{M}^{\text{pud}}(\Omega)$ of the starting measures μ, ν

$$\mu_k(\Omega_{i_1\dots i_k}) = \mu(\Omega_{i_1\dots i_k}) := p_{i_1\dots i_k}, \quad \nu_k(\Omega_{i_1\dots i_k}) = \nu(\Omega_{i_1\dots i_k}) := r_{i_1\dots i_k}.$$

With μ_k, ν_k one can associate a couple of stochastic vectors $\mathbf{p}_k = \{p_{i_k}\}_{i_k=1}^n, \quad \mathbf{r}_k = \{r_{i_k}\}_{i_k=1}^n$ with coordinates

$$p_{i_k} := \frac{p_{i_1\dots i_k}}{p_{i_1\dots i_{k-1}}}, \quad r_{i_k} := \frac{r_{i_1\dots i_k}}{r_{i_1\dots i_{k-1}}}.$$

Thus, having two sequences of stochastic vectors $P = (\mathbf{p}_k)_{k=1}^\infty, \quad R = (\mathbf{r}_k)_{k=1}^\infty$ we are able to solve the problem of conflict redistribution of the space Ω between opponents A and B with any accuracy in terms of vectors $\mathbf{p}_k, \mathbf{r}_k$ according to the theory developed in [11, 14].

Vice versa, if we have two sequences of stochastic vectors $P = (\mathbf{p}_k)_{k=1}^\infty, \ \ R = (\mathbf{r}_k)_{k=1}^\infty$,

$$\mathbf{p}_k = \{p_{i_k}\}_{i_k=1}^n, \quad \mathbf{r}_k = \{r_{i_k}\}_{i_k=1}^n, \quad \sum_{i=1}^n p_{i_k} = 1 = \sum_{i=1}^n r_{i_k}, \quad p_{i_k}, r_{i_k} \geq 0, \quad \forall k,$$

we may reconstruct two sequences of piecewise uniformly distributed probability measures $\mu_k, \nu_k \in \mathcal{M}^{\text{pud}}(\Omega)$ putting

$$\mu_k(\Omega_{i_1\dots i_k}) = p_{i_1} \cdots p_{i_k}, \quad \nu_k(\Omega_{i_1\dots i_k}) = r_{i_1} \cdots r_{i_k}, \quad k = 1, 2, \dots$$

According subsection 2.1, sequences μ_k, ν_k have at least the weak limits, μ, ν, which are the similar structure measures. Thus, we may state that the setting of conflict dynamical problem for opponents presented by similar structure measures $\mu, \nu \in \mathcal{M}(\Omega)$, admits consideration with any power of accuracy in terms of simple measures from $\mathcal{M}^{\text{pud}}(\Omega)$ or, equivalently, in terms of stochastic vectors.

We remark that in applications the values $\mu_k(\Omega_{i_1\dots i_k}), \nu_k(\Omega_{i_1\dots i_k})$ have interpretation of probabilities to observe some property of opponents in a region $\Omega_{i_1\dots i_k}$. Using these values for all $\Omega_{i_1\dots i_k},\ k \geq 1$ and due to conditions on IFS, one can calculate the probability to find this property in any Borel set $E \in \mathcal{B}$ from Ω. Moreover, the fractal structure of regional divisions Ω allows to get information about spectral properties of measures μ and ν under evolution produced by conflict interaction between opponents.

3. Conflict dynamical systems in terms of simple measures

In this section we describe in what way the measures μ_k, ν_k from previous section will transformed under action of the conflict interaction between opponents. In other words, we will describe the changes of states for dynamical system (1) in terms of simple measures.

So, we start with a couple of probability similar structure measures μ, ν on Ω which are conformed with some IFS.

At the first moment, $t = 1$, of the conflict interaction we put in correspondence to system the state with two changed (non-free) measures $\mu^{t=1}, \nu^{t=1} \in \mathcal{M}^{\mathrm{pud}}(\Omega)$ associated with stochastic vectors $\mathbf{p}^1 = (p^1_{i_1})$ and $\mathbf{r}^1 = (r^1_{i_1}),\ i_1 = 1, \dots, n$ whose coordinates are calculated as follows,

$$p^1_{i_1} = \frac{p_{i_1}(1 - r_{i_1})}{1 - \sum_{i_1} p_{i_1} r_{i_1}}, \quad r^1_{i_1} = \frac{r_{i_1}(1 - p_{i_1})}{1 - \sum_{i_1} p_{i_1} r_{i_1}}. \tag{6}$$

where

$$p_{i_1} = \mu(\Omega_{i_1}), \quad r_{i_1} = \nu(\Omega_{i_1}). \tag{7}$$

For the next moment of time, $t = 2$, we can define μ^2, ν^2 in two different way: to use formula of view (6) with values $p_{i_1 i_2},\ r_{i_1 i_2}$ from previous subsection instead $p_{i_1},\ r_{i_1}$, or another one, to calculate

$$p^2_{i_2} = \frac{p^1_{i_2}(1 - r^1_{i_2})}{1 - \sum_i p^1_i r^1_i}, \quad r^2_{i_2} = \frac{r^1_{i_2}(1 - p^1_{i_2})}{1 - \sum_i p^1_i r^1_i}, \tag{8}$$

and define

$$\mu^{t=2}(\Omega_{i_1 i_2}) = p_{i_1 i_2} := p^1_{i_1} \cdot p^2_{i_2}, \quad \nu^{t=2}(\Omega_{i_1 i_2}) = r_{i_1 i_2} := r^1_{i_1} \cdot r^2_{i_2}.$$

We prefer to take the second way. Thus, by iteration we find

$$\mu^{t=k}(\Omega_{i_1\dots i_k}) = p_{i_1\dots i_k} := p^1_{i_1} \cdots p^k_{i_k}, \quad \nu^{t=k}(\Omega_{i_1\dots i_k}) = r_{i_1\dots i_k} := r^1_{i_1} \cdots r^k_{i_k}, \quad k \geq 1, \tag{9}$$

where

$$p^k_{i_k} = \frac{p^{k-1}_{i_{k-1}}(1 - r^{k-1}_{i_{k-1}})}{1 - \sum_i p^{k-1}_i r^{k-1}_i}, \quad r^k_{i_k} = \frac{r^{k-1}_{i_{k-1}}(1 - p^{k-1}_{i_{k-1}})}{1 - \sum_i p^{k-1}_i r^{k-1}_i}. \tag{10}$$

Thus, formulas (6)–(10) describe the iterative construction of states for conflict dynamical system in terms of piecewise uniformly distributed measures $\mu^t, \nu^t,\ t = 1, 2, \dots$ Symbolically, we denote these constructions as follows

$$\mu^t \ast \nu^t = \mu^{t+1}, \quad \nu^t \ast \mu^t = \nu^{t+1},$$

where the map $\ast$ corresponds to the repulsive law of interaction between opponents. Below it will be shown that as a rule sequences $\mu^{t=k}(\Omega_{i_1\dots i_k}), \nu^{t=k}(\Omega_{i_1\dots i_k}), k = 1, 2, \dots$ converge to zero, with $t \to \infty$. According [30, 31], the limit measure μ^∞ is singular continuous if $\lim_{t=k\to\infty} \mu^t(\Omega_{i_i\dots i_k}) = 0$ for all sequences $i_1, \dots, i_k, \dots$ However, we are seeking the conditions which ensure the converging $\mu^{t=k}(\Omega_{i_1\dots i_k})$ or $\nu^{t=k}(\Omega_{i_1\dots i_k})$ to non-zero values at least for some sequences $i_1, \dots, i_k, \dots$

In what follows we will used the following notations:

$$\mu_{i_1\dots i_k\dots} = \prod_{k=1}^{\infty} p^k_{i_k} = \lim_{k\to\infty} p_{i_1\dots i_k}, \quad \nu_{i_1\dots i_k\dots} = \prod_{k=1}^{\infty} r^k_{i_k} = \lim_{k\to\infty} r_{i_1\dots i_k}, \tag{11}$$

where we recall that

$$p_{i_1\dots i_k} := p^1_{i_1}\cdots p^k_{i_k}, \quad r_{i_1\dots i_k} := r^1_{i_1}\cdots r^k_{i_k} \tag{12}$$

and $p^k_{i_k}, r^k_{i_k}$ are defined by (10). Besides, we recall that each point from Ω admits the representation

$$\Omega \ni \omega_{i_1\dots i_k\dots} = \bigcap_{k=1}^{\infty} \Omega_{i_1\dots i_k}, \tag{13}$$

i.e., it is associated with some directed sequence of indices $i_1,\dots,i_k,\dots$

3.1. **Weak convergence of interacted distributions.** Here we prove that both sequences of simple measures μ^t, ν^t defined by (9) have the weak limits with $t\to\infty$. This fact is not new, however our proof is transparent and differ from the way used in [30].

Below we deals only with μ^t, similar arguments are valid for ν^t.

Lemma 1. *The sequence of simple measures μ^t, $t=1,2,\dots$ defined by (9) converges in the weak sense to some probability measure $\mu^\infty \in \mathcal{M}(\Omega)$*

$$\mu^\infty = w - \lim_{t\to\infty} \mu^t.$$

Proof. Let us associate with measures $\mu^t, t\ge 1$ the linear functionals $\Phi_k(\varphi)$, $k\ge 1$

$$\Phi_k(\varphi) = \int_\Omega \varphi(x) d\mu^{t=k}(x), \quad k=1,2,\dots, \quad \varphi\in C(\mathbb{R}^d), \tag{14}$$

where $C(\mathbb{R}^d)$ denotes the space of continuous functions. Let a function φ is fixed, then without loss of generality we may assume that it is positive and bounded on Ω, i.e.,

$$0\le \varphi(x) \le M, \quad x\in\ \Omega.$$

Now one can estimate the values of functional $\Phi_k(\varphi)$ from above and from below by replacing the function $\varphi(x)$ in (14) by its minimal and maximal meanings on each region $\Omega_{i_1\dots i_k}$. To write such estimation explicitly we introduce notations

$$\varphi_{i_1\dots i_k,m} := \min_{x\in\Omega_{i_1\dots i_k}} \varphi(x), \quad \varphi_{i_1\dots.i_k,M} := \max_{x\in\Omega_{i_1\dots i_k}} \varphi(x).$$

Then, it is evident that

$$\Phi_{k,m}(\varphi)\ \le \Phi_k(\varphi)\ \le \Phi_{k,M}(\varphi)\ ,$$

where

$$\Phi_{k,\mathrm{ex}}(\varphi) := \sum_{i_1,\dots,i_k=1}^{n} \varphi_{i_1\dots i_k,\mathrm{ex}} \int_{\Omega_{i_1\dots i_k}} d\mu^{t=k}(x), \quad \mathrm{ex} = m \bigvee M.$$

Using that $\mu^{t=k}(\Omega_{i_1\dots i_k}) = p_{i_1\dots i_k} \equiv p^1_{i_1}\cdots p^k_{i_k}$, we can write

$$\Phi_{k,\mathrm{ex}}(\varphi) = \sum_{i_1,\dots,i_k=1}^{n} \varphi_{i_1\dots i_k,\mathrm{ex}}\cdot \mu^t(\Omega_{i_1\dots i_k}) = \sum_{i_1,\dots,i_k=1}^{n} \varphi_{i_1\dots i_k,\mathrm{ex}}\cdot p_{i_1\dots i_k}.$$

We state that differences $\Phi_{k,M}(\varphi)-\Phi_{k,m}(\varphi)$ monotonically decrease to zero as $k\to\infty$

$$\Phi_{k,M}(\varphi)-\Phi_{k,m}(\varphi) = \sum_{i_1,\dots,i_k=1}^{n} (\varphi_{i_1\dots i_k,M}-\varphi_{i_1\dots i_k,m}) p_{i_1\dots i_k} \le d_k \sum_{i_1,\dots,i_k=1}^{n} p_{i_1\dots i_k} = d_k \to 0.$$

Indeed, it is true since the sequence

$$d_k\ := \max_{i_1\dots i_k}\{\varphi_{i_1\dots i_k,M}-\varphi_{i_1\dots i_k,m}\}$$

decreases to zero because $\varphi(x)$ is continuous and bounded, and $\lambda(\Omega_{i_1\dots i_k})\to 0$, as $k\to\infty$.

Hence, there exists the limit functional

$$\Phi_\infty(\varphi) = \lim_{k\to 1} \Phi_{k,m}(\varphi) = \lim_{k\to 1} \Phi_{k,M}(\varphi) \ .$$

By construction,

$$\Phi_\infty(\varphi) = \lim_{k\to\infty} \sum_{i_1\ldots i_k=1}^{n} c_{i_1\ldots i_k} \int_{\Omega_{i_1\ldots i_k}} \varphi(x)dx, \quad c_{i_1\ldots i_k} \ := \ \frac{p_{i_1\ldots i_k}}{\lambda(\Omega_{i_1\ldots i_k})}.$$

This functional is positive and continuous on $C(\mathbb{R}^d)$. Therefore it is associated with some probability measure $\mu^\infty \in \mathcal{M}(\Omega)$, and we can write:

$$\Phi_\infty(\varphi) = \int_\Omega \varphi(x)d\mu^\infty(x).$$

□

We remark that the functional $\Phi_\infty(\varphi)$ and the measure μ^∞ are associated (see (9)) with infinite matrix

$$P = \{\mathbf{p}^k\}_{k=1}^{\infty} = \{p^k_{i_k}\}_{k=1,i_k=1}^{1,n}$$

and therefore μ^∞ is a similar structure measure.

In what follows we will use

Definition 1. A sequence of elements $p^k_{i_k}, k = 1, 2, \ldots$ from the matrix $P = (p^k_{i_k})_{k=1,i_k=1}^{\infty,n}$ is said to be 0-convergent, if

$$\sum_{k=1}^{\infty} p^k_{i_k} \ < \infty.$$

A sequence $p^k_{i_k}, k = 1, 2, \ldots$ is called 1-convergent, if

$$\prod_{k=1}^{\infty} p^k_{i_k} \ > 0. \tag{15}$$

In general, it is a non-trivial problem to select a 1-convergent sequence of elements from the matrix P since all vectors $\mathbf{p}^k$ are stochastic and $0 \le p^k_{i_k} \le 1$. We recall that condition (15) has an equivalent form

$$\sum_{k=1}^{\infty}(1 - p^k_{i_k}) < \infty.$$

A couple of sequences $p^k_{i_k}, p^k_{j_k}, \quad k = 1, 2, \ldots$ we call *equivalent*, if $p^k_{i_k} \neq p^k_{j_k}$ for at most finite many indices. However, if $p^k_{i_k} = p^k_{j_k}$ for at most finite many indices, then this couple of sequences we call *disjunctive*.

Proposition 1. *If the matrix P contains a 1-convergent sequence of its elements, then it is unique up to equivalence. All other disjoint sequences are 0-convergent.*

Proof. If a sequence $p^k_{i_k}, k = 1, 2, \ldots$ is 1-convergent, then with necessity $p^k_{i_k} \to 1, k \to \infty$. Since all $\mathbf{p}^k$ are stochastic, it is possible only for a unique sequence up to equivalence. Therefore all other sequences, disjoint to this 1-convergent sequence, are 0-convergent.

□

Assume a sequence $p_{ki_k}, k = 1, 2, \ldots$ is 1-convergent. Put in correspondence to each set of indices $i_1 \ldots i_k$ the linear functional

$$\Psi_k(\varphi) \ := \int_{\Omega_{i_1\ldots i_k}} \varphi(x)d\mu^k(x), \quad \varphi \in C(\mathbb{R}^d).$$

Lemma 2. *The sequence of functionals Ψ_k is convergent and the limit functional is generated by Dirac δ-function concentrated at point*

$$x_{\bar{i}_k} \equiv x_{i_1 \ldots i_k \ldots} = \bigcap_{k=1}^{\infty} \Omega_{i_1 \ldots i_k} : \tag{16}$$
$$\Psi_k(\varphi) \to \mu_{\bar{i}_k} \delta_{x=x_{\bar{i}_k}}(\varphi) = \mu_{\bar{i}_k} \varphi(x_{\bar{i}_k}),$$

where

$$\mu_{\bar{i}_k} = \prod_{k=1}^{\infty} p_{i_k}^k.$$

Proof. By construction,

$$\Psi_k(\varphi) = c_{i_1 \ldots i_k} \int_{\Omega_{i_1 \ldots i_k}} \varphi(x) d\lambda(x),$$

where

$$c_{i_1 \ldots i_k} = \frac{p_{i_1 \ldots i_k}}{\lambda(\Omega_{i_1 \ldots i_k})}.$$

Hence, we can use the estimation

$$\varphi_{i_1 \ldots i_k, m} c_{i_1 \ldots i_k} \lambda(\Omega_{i_1 \ldots i_k}) \le \Psi_k(\varphi) \le \varphi_{i_1 \ldots i_k, M} c_{i_1 \ldots i_k} \lambda(\Omega_{i_1 \ldots i_k}),$$

and get

$$\varphi_{i_1 \ldots i_k, m}\, p_{i_1 \ldots i_k} \le \Psi_k(\varphi) \le \varphi_{i_1 \ldots i_k, M} p_{i_1 \ldots i_k}.$$

Since the sequence of regions $\Omega_{i_1 \ldots i_k}$ shrinks to point $x_{\bar{i}_k}$ and the function φ is assumed continuous, two sequences $\varphi_{i_1 \ldots i_k, m}$ and $\varphi_{i_1 \ldots i_k, M}$ converge to the same value $\varphi(x_{\bar{i}_k})$. This proves (16). □

Using Lemma 2 we able to represent the meanings of limit functional Φ_∞ from Lemma 1 in terms of δ-functions.

Theorem 1. *Assume the matrix $P = (p_{ki_k})_{k=1, i_k=1}^{\infty, n}$ contains a $1-$convergent sequence. Then the limit functional $\Phi_\infty(\varphi) = \lim_{k\to\infty} \Phi_k(\varphi)$ (see Lemma 1) admits the following representation:*

$$\Phi_\infty(\varphi) = \sum_{\bar{x} \in \Gamma_\mu} \mu_{\bar{x}} \varphi(\bar{x}),$$

where Γ_μ denotes the countable set of points

$$\bar{x} \equiv x_{\bar{i}_k} = x_{i_1 \ldots i_k \ldots} = \bigcap_{k=1}^{\infty} \Omega_{i_1 \ldots i_k}$$

such that all corresponding sequences of matrix elements $\{p_{i_k}^k\}$ are $1-$convergent and mutually equivalent. That is, all constants $\mu_{\bar{x}}$ are given by products of matrix elements:

$$\mu_{\bar{x}} \equiv \mu_{\bar{i}_k} = \mu_{i_1 \ldots i_k \ldots} = \prod_{k=1}^{\infty} p_{i_k}^k.$$

4. Criterion for point spectra, repulsive case

Let us consider the conflict dynamical system $\{X, \divideontimes\}$ with trajectories (1), where the $\divideontimes$ denotes the repulsive interaction given by formulas (6)–(10).

Theorem 2. *For every couple of starting probability measures $\mu, \nu \in \mathcal{M}(\Omega)$, one of the limit measures μ^∞ or ν^∞, constructed in according with formulas (7)–(10) by the sequence of piecewise uniformly distributed probability measures $\mu^{t=k}, \nu^{t=k} \in \mathcal{M}(\Omega), t = 1, 2, \dots$ is pure point, $\mu^\infty \in \mathcal{M}_{\rm pp}$ or $\nu^\infty \in \mathcal{M}_{\rm pp}$, if and only if the condition*

$$\mu(\Omega_{\mathbf{i}}) > \nu(\Omega_{\mathbf{i}}) \tag{17}$$

or

$$\mu(\Omega_{\mathbf{i}}) < \nu(\Omega_{\mathbf{i}}) \tag{18}$$

fulfilled for a single fixed index $1 \le \mathbf{i} \le n$.

Proof. We will consider only the limit measure μ^∞. The proving for ν^∞ is the same. By the theory developed in [30, 31] (see also [3, 13, 4, 5, 20]) the limit measure μ^∞ is purely point if and only if

$$\prod_{k=1}^{\infty} \max_{i_1 \dots i_k} \mu^k(\Omega_{i_1 \dots i_k}) > 0. \tag{19}$$

This statement is some analog of Jenssen-Wintner theorem. We will prove that (19) follows from (17).

So, for $t = 1$, from (17), in accordance with (7), we obtain

$$\mu^1(\Omega_{\mathbf{i}}) = p^1_{\mathbf{i}} \ > \ \nu^1(\Omega_{\mathbf{i}}) = r^1_{\mathbf{i}}. \tag{20}$$

Moreover, the similar arguments show that it is true for any $k \ge 1$

$$\mu^k(\Omega_{i_1=\mathbf{i} \dots i_k=\mathbf{i}}) = p^k_{i_1=\mathbf{i} \dots i_k=\mathbf{i}} \ > \ \nu^k(\Omega_{i_1=\mathbf{i} \dots i_k=\mathbf{i}}) = r^k_{i_1=\mathbf{i} \dots i_k=\mathbf{i}}. \tag{21}$$

Since condition (17) is assumed only for a unique index $\mathbf{i}$, for all other indices we have

$$\mu^1(\Omega_i) < \nu^1(\Omega_i), \quad \forall i \ne \mathbf{i}, \tag{22}$$

and, by induction, for any k

$$\mu^k(\Omega_{i_1 \dots i_k}) < \nu^k(\Omega_{i_1 \dots i_k}), \quad i_1, \dots, i_k \ne \mathbf{i}. \tag{23}$$

Therefore, by Theorem of conflict [11] it yields

$$\mu^k(\Omega_{i_1 \dots i_k}) \to 0, \quad k \to \infty, \quad i_1, \dots, i_k \ne \mathbf{i}. \tag{24}$$

At the same time the last multiplier in $\mu^k(\Omega_{\mathbf{i} \dots \mathbf{i}}) = p^1_{\mathbf{i}} \cdots p^k_{\mathbf{i}}$ has a monotonic grow

$$p^k_{\mathbf{i}} \to 1, \quad k \to \infty,$$

because by (10), $p^k_{i_k} \to 0$ for all $i_k \ne \mathbf{i}$ due to condition (17). These facts shows that

$$\mu^k(\Omega_{\mathbf{i} \dots \mathbf{i}}) = \max_{i_1 \dots i_k} \mu^k(\Omega_{i_1 \dots i_k}). \tag{25}$$

Now we have to prove that

$$\prod_{k=1}^{\infty} \mu^k(\Omega_{\mathbf{i} \dots \mathbf{i}}) > 0. \tag{26}$$

Let us consider the helping constructions.

Assume we have a couple of stochastic vectors $\mathbf{p}^1, \ \mathbf{r}^1 \in \mathbb{R}^n, \ \ n \ge 2$ with different coordinates, $0 < p^1_i \ne r^1_i < 1, \ i = 1, \dots, n$. We will study the trajectory

$$\{\mathbf{p}^k, \mathbf{r}^k\} \xrightarrow{*} \{\mathbf{p}^{k+1}, \mathbf{r}^{k+1}\}, \quad k = 1, 2, \dots,$$

where the coordinates of $\{\mathbf{p}^k, \mathbf{r}^k\}$ are defined by the rule (10). Assume, as that

$$p^1_{\mathbf{i}} > r^1_{\mathbf{i}} \tag{27}$$

for a single only index $\mathbf{i}$. Now, we will prove that due to (27), the sequence $p_{\mathbf{i}}^k,\ k = 1, 2, \dots$ is 1-converging and therefore,

$$\prod_{k=1}^{\infty} p_{\mathbf{i}}^k > 0. \tag{28}$$

At first, consider the case $n = 2$. It easily seen that for any starting $\mathbf{p},\ \mathbf{r} \in \mathbb{R}^2_{+,1}$ already on the first step of conflict interaction, the coordinates of vectors $\mathbf{p}^1,\ \mathbf{r}^1$ become symmetric:

$$p_1^1 = r_2^1, \quad p_2^1 = r_1^1.$$

By this reason without lost of generality one can set

$$p_1^1 = a_1, \quad p_2^1 = 1 - a_1, \quad r_1^1 = 1 - a_1, \quad r_2^1 = a_1, \quad 0 < a_1 < 1.$$

Clearly that coordinates of vectors $\mathbf{p}^{t=k}$ and $\mathbf{r}^{t=k}$ for all $k \geq 1$ are also symmetric:

$$p_1^k = r_2^k = a_k, \quad p_2^k = r_1^k = 1 - a_k, \quad 0 < a_k < 1.$$

Let us fix one of indices from $\{1, 2\}$, for instance, put $\mathbf{i} = 1$. Then condition (27) means that $a_1 > 1/2$. Using this assumption we have to prove that the sequence $p_1^k = a_k$ increases to unite: $a_k \to 1$, so quickly that

$$\prod_{k=1}^{\infty} a_k > 0. \tag{29}$$

Recalling that coordinates p_i^k, r_i^k are calculated by formulas

$$p_i^k = p_i^{k-1} \cdot \frac{1 - r_i^{k-1}}{1 - \theta^{k-1}}, \quad r_i^k = r_i^{k-1} \cdot \frac{1 - p_i^{k-1}}{1 - \theta^k}, \tag{30}$$

we observe that iterated rule (30) in case $n = 2$ in really is the difference variant of the continuous one-dimensional maps

$$F_1(x) = \frac{x^2}{x^2 + (1-x)^2}, \quad F_2(x) = \frac{(1-x)^2}{(1-x)^2 + x^2}, \quad x \in \mathbb{R}^1.$$

It is easy to see that for $x = a_1 > 1/2$ the sequence

$$a_{k+1} = \frac{a_k^2}{a_k^2 + (1 - a_k)^2}, \quad k = 1, 2, \dots \tag{31}$$

converges to 1. We have to prove that condition $a > 1/2$ implies (42).

Instead (29) we will prove the convergence of series

$$\sum_{k=1}^{\infty} (1 - a_k) < \infty \tag{32}$$

that is equivalent. To prove (32) we can use the well-known Raabe criterion. Indeed, due to (31) it easy to see that value $k\left(\frac{1-a_k}{1-a_{k+1}} - 1\right)$ has the monotonic grows, if $a_1 > 1/2$. Moreover, due to $a_k \to 1$, it follows that $k\left(\frac{1-a_k}{1-a_{k+1}} - 1\right)$ goes to infinity with $k \to \infty$. By the Raabe criterion, it proves that series (32) is convergent. Therefore the infinite product (29) is strongly positive.

Let now $n > 2$. Since inequality (27) is fulfilled only for a single coordinate, it follows that inverse inequality is true for all other coordinates of vectors $\mathbf{p}^1, \mathbf{r}^1$, i.e., for coordinates with indices different of $\mathbf{i}$:

$$p_i^1 < r_i^1, \quad i \neq \mathbf{i}. \tag{33}$$

Let us show that due to (33) each sequence $p_i^k, i \neq \mathbf{i}, k = 1, 2, \dots$ is 0-converging, i.e.,

$$\sum_{k=1}^{\infty} p_i^k < \infty. \tag{34}$$

With this aim we remark that from (30) it follows:

$$p_i^k = p_i^{k-1} \cdot c_{ki} = p_i^1 \prod_{l=1}^{k} c_{li}, \quad c_{li} = \frac{1 - r_i^l}{1 - \theta_l}. \tag{35}$$

By the Theorem of conflict (see [12, 20])) and due to (33), all coordinates $r_i^k, i \neq \mathbf{i}$ convergent to strongly positive meanings, but $p_i^k,\ i \neq \mathbf{i}$ and θ_k tend to zero

$$r_i^k \to r_i^\infty > 0, \quad p_i^k \to 0, \quad \theta_k \to 0, \quad k \to \infty, \quad i \neq \mathbf{i}.$$

Therefore, there is some natural T, such that for $k = t > T$ all c_{ki} in (35) have the estimation independent of k

$$c_{ki} \leq c_i < 1.$$

If $T = 1$, then we may replace c_{ki} by k-power of c_i and get the estimation

$$p_i^k < p_i^1 c_i^k.$$

Therefore, series (34) is estimated by the sum of geometrical progression, i.e., it is finite and does not exceed $p_i^1/(1 - c_i)$. If $T > 1$, then the series (34) is convergent also by the same reason, because the sum of its first $T - 1$ terms is finite. Since the analogical argumentation is valued for all $i \neq \mathbf{i}$, we obtain

$$\sum_{i \neq \mathbf{i}} \Big(\sum_{k=1}^{\infty} p_i^k\Big) = \sum_{k=1}^{\infty} \Big(\sum_{i \neq \mathbf{i}} p_i^k\Big) = \sum_{k=1}^{\infty} (1 - p_{\mathbf{i}}^k) < \infty,$$

that is equivalent to (28). We proved that in the general case under condition (27), the sequence $p_{\mathbf{i}}^k, k = 1, 2, \dots$ is 1-converging.

Now we can come back to our problem in terms of measures.

Using that due to (9),

$$\mu^{t=k}(\Omega_{i_1 \dots i_k}) = p_{i_1 \dots i_k} := p_{i_1}^1 \cdots p_{i_k}^k$$

and inequalities (17), (20), and (27) are equivalent, we may conclude that the sequence $\mu^k(\Omega_{\underbrace{\mathbf{i} \dots \mathbf{i}}_{k}}),\ k \geq 1$ is 1-converging. Now, by virtue of (25), the inequalities (28) and (19) are equivalent. This completes the proof of the theorem. □

Thus we proved that the limit measure μ^∞ is pure point, $\mu^\infty \in \mathcal{M}_{\text{pp}}$, if and only if the opponent A chooses the strategy of a single priority, i.e., inequality $\mu(\Omega_{\mathbf{i}}) > \nu(\Omega_{\mathbf{i}})$ is fulfilled for an unique fixed index $1 \leq \mathbf{i} \leq n$.

5. The point spectrum in models with attractive interaction

Here we consider the situation with a map ⋇ corresponding to the attractive interaction. Now measures μ^t, ν^t are defined in the similar to previous constructions way

$$\mu^{t+1} = \mu^t \divideontimes \nu^t, \quad \nu^{t+1} = \nu^t \divideontimes \mu^t$$

with trajectories

$$\{\mu^t, \nu^t\} \xrightarrow{\divideontimes, t} \{\mu^{t+1}, \nu^{t+1}\}, \quad t = 1, 2, \dots \tag{36}$$

All measure μ^t, ν^t, $t \geq 1$ are uniformly distributed on regions $\Omega_{i_1 \dots i_k}$ (see the previous subsections), and are defined by beforehand given measures $\mu, \nu \in \mathcal{M}(\Omega)$ according to the iterative rule

$$\mu^t(\Omega_{i_1 \dots i_k}) = p_{i_1 \dots i_k} := p_{i_1}^1 \cdot \dots \cdot p_{i_k}^t,$$
$$\nu^t(\Omega_{i_1 \dots i_k}) = r_{i_1 \dots i_k} = r_{i_1}^1 \cdot \dots \cdot r_{i_k}^t, \quad t = k,$$

with

$$p_i^t = \frac{p_i^{t-1}(1+r_i^{t-1})}{1+\theta^{t-1}}, \quad r_i^t = \frac{r_i^{t-1}(1+p_i^{t-1})}{1+\theta^{t-1}}, \quad t = 1, \dots, \tag{37}$$

where $\theta^t = (\mathbf{p}^t, \mathbf{r}^t) = \sum_{i=1}^n p_i^t r_i^t$ and $p_i^0 \equiv p_i = \mu(\Omega_i)$, $r_i^0 \equiv r_i = \nu(\Omega_i)$.

In [23] (see also [27]) it was proven the existence of the limit invariant measures $\mu^\infty = \lim_{t\to\infty} \mu^t, \nu^\infty = \lim_{t\to\infty} \nu^t$. Here we find the sufficient conditions for $\mu^\infty, \nu^\infty \in \mathcal{M}_{\rm pp}$.

Let us introduce the values

$$\sigma_i^t := p_i^t + r_i^t, \quad \rho_i^t := p_i^t \cdot r_i^t, \quad t \geq 0.$$

Lemma 3. *If there exist a single region $\Omega_{\mathbf{i}}$, such that*

$$p_{\mathbf{i}} = \max_{i=1}^n \{p_i\}, \quad r_{\mathbf{i}} = \max_{i=1}^n \{r_i\}, \tag{38}$$

or

$$\sigma_{\mathbf{i}} = \sigma_{\max} = \max_{i=1}^n \{\sigma_i\}, \quad \rho_{\mathbf{i}} = \rho_{\max} = \max_{i=1}^n \{\rho_i\}, \tag{39}$$

then the sequences $p_{\mathbf{i}}^t, r_{\mathbf{i}}^t$, $t = 1, 2, \dots$ are 1-convergent, i.e.,

$$\prod_{t=1}^\infty p_{\mathbf{i}}^t > 0, \quad \prod_{t=1}^\infty r_{\mathbf{i}}^t > 0. \tag{40}$$

We note that (39) takes place, if (38) is true.

Proof. Assume condition (39) or (38) is fulfilled for a single $\mathbf{i}$. It means that for all $i \neq \mathbf{i}$

$$\sigma_i < \sigma_{\mathbf{i}}, \quad \rho_i < \rho_{\mathbf{i}}$$

or

$$p_i < p_{\mathbf{i}}, \quad r_i < r_{\mathbf{i}}.$$

Let us show that then each sequence $\{p_i^t\}_{t=1}^\infty, i \neq \mathbf{i}$, is 0-convergent, i.e.,

$$\sum_{t=1}^\infty p_i^t < \infty. \tag{41}$$

Due to results of Section 2.2 from [1]) under one of conditions (38), (39), all coordinates p_i^t, r_i^t, $i \neq \mathbf{i}$ converge to zero

$$p_i^t \to 0, \quad r_i^t \to 0, \quad \theta^t \to 1, \quad t \to \infty, \quad i \neq i_0.$$

From (37) we find that

$$p_i^{t+1} = p_i^t \cdot c_i^t = p_i \prod_{t=0}^\infty c_i^t, \quad c_i^t = \frac{1+r_i^t}{1+\theta^t}. \tag{42}$$

If $r_i^t < \theta^t$, then $c_i^{t+1} < c^t$. However, if $r_i^t > \theta^t$, then we have $c_i^{t+1} > c^t$. Nevertheless, since $r_i^t \to 0$ and $\theta^t \to 1$, all c_i^t, $t > T$ become less than unite for enough large T and moreover, they monotonically decreases. Thus we may write

$$0 < c_i^t \leq c < 1, \quad c := c_i^T.$$

Assume $T = 1$. Then series (41) is estimated by the series of geometrical progression $\sum_{l=1}^\infty p_i c^l$, since each terms of our series is less then the corresponding term of the later ones. This series is obviously converging due to its denominator $c < 1$. This proves that the sum (41) is finite. If $T > 1$, then series (41) is also converging because the partly sum of the first $T-1$ terms of this series is finite.

So, we prove that

$$\sum_{t=1}^\infty \Big(\sum_{i\neq\mathbf{i}} p_i^t\Big) = \sum_{t=1}^\infty (1 - p_{\mathbf{i}}^t) < \infty.$$

Therefore, the sequence $\{p_{\mathbf{i}}^t\}_{t=1}^\infty$ is 1-convergent, i.e., $\prod_{t=0}^\infty p_{\mathbf{i}}^t > 0$. □

We recall that by theorem of the Jessen-Wintner type, measures μ^∞, ν^∞ are pure point, if

$$\prod_{k=1}^{\infty} \max_{i_1\dots i_k} \mu^\infty(\Omega_{i_1\dots i_k}) > 0, \quad \prod_{k=1}^{\infty} \max_{i_1\dots i_k} \nu^\infty(\Omega_{i_1\dots i_k}) > 0, \tag{43}$$

where we have to take $\Omega^\sharp_{i_1\dots i_k}$ instead $\Omega_{i_1\dots i_k}$ since in general $\mu^t(\Omega_{i_1\dots i_k}) \neq \mu^\infty(\Omega_{i_1\dots i_k})$ and $\nu^t(\Omega_{i_1\dots i_k}) \neq \nu^\infty(\Omega_{i_1\dots i_k})$.

Theorem 3. *The limit measures μ^∞ and ν^∞, constructed in according with formulas (36)–(37) which describe the attractive interaction, are pure point, $\mu^\infty, \nu^\infty \in \mathcal{M}_{\rm pp}$, if there exists a single region $\Omega_{\mathbf{i}}$, such that one of condition* (38) *or* (39) *is fulfilled.*

Proof. We only remark that since conditions (40) and (43) are equivalent, the validity of the theorem follows from Lemma 2. □

6. Discussion

By Theorems 2 and 3, the limit measure μ^∞ is pure point and supported on the countable set of points $\bar{x} \in \Gamma_\mu$ corresponding to all 1-convergent sequences. Exactly the meanings of measure μ^∞ on these points one can calculate by the products formula for 1-convergent sequences of elements from matrix P (see Theorem 1)

$$\mu^\infty(\bar{x}) = \mu_{\bar{x}} = \prod_{k=1}^{\infty} p^k_{i_k}.$$

All these sequences are equivalent (see Definition 1) and have only finite numbers of multipliers different of $p^k_{\mathbf{i}}$, where $\mathbf{i}$ is fixed. Let $\bar{x}(l), l = 1, 2, \dots$ means that a point $\bar{x}$ corresponds to a sequence with l multipliers different of $p^k_{\mathbf{i}}$. Obviously

$$\sum_{l=0}^{L} \mu_{\bar{x}(l)} \to 1, \quad L \to \infty.$$

This fact allows to establish some ranking of values $\mu_{\bar{x}}$ depending of the step of fractal partition. In particular the main value of $\mu_{\bar{x}}$ corresponds to point $\bar{x}(l = 0)$ and admits calculation as follows:

$$\mu_{\bar{x}(0)} = \prod_{k=1}^{\infty} p^k_{\mathbf{i}}.$$

Finally we remark that Γ_μ is dense in Ω. Therefore the support of μ^∞ coincides with all Ω. Nevertheless the essential support of μ^∞ consists of only points from the set Γ_μ. That is, this set is self-similar and has zero topological and Hausdorff dimensions.

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