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ANALYTICITY AND OTHER PROPERTIES OF FUNCTIONALS $I(f,p)=\int_A |f(t)|^p dt$ AND $n(f,p)=\left(\frac{1}{\mu(A)}\int_A |f(t)|^p dt\right)^{\frac{1}{p}}$ AS FUNCTIONS OF VARIABLE p

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Abstract. We showed that for each function $f(t)$, which is not equal to zero almost everywhere in the Lebesgue measurable set, functionals $I(f,z)=\int_A |f(t)|^z dt$ as functions of a complex variable $z=p+iy$ are continuous on the domain and analytic on a set of all inner points of this domain. The functions $I(f,p)$ as functions of a real variable p are strictly convex downward and log-convex on the domain.

We proved that functionals $n(f,p)$ as functions of a real variable p are analytic at all inner points of the interval, in which the function $n(f,p)\neq 0$ except the point $p=0$, continuous and strictly increasing on this interval.

Let us consider the functional $n(f,p)=\left(\frac{1}{\mu(A)}\int_A |f(t)|^p dt\right)^{\frac{1}{p}}$, where A is an arbitrary Lebesgue measurable set, $\mu(A)$ is its Lebesgue measure and p is an arbitrary nonzero real number. These functionals define the spaces of functions, which are normalized if $p\geq 1$ with the norm $\|f\|_{L_{p(A)}}=n(f,p)$, are Euclidean if $p=2$, and for arbitrary $p>0$ contain the spaces of the continuous functions defined on a closed bounded set A.

We always use the functional $n(f,p)$ in the definitions of the main approximative characteristics of functions and functional classes in these spaces. For example, $n(f-A(f),p)$ is the approximation of the function f by the given operator $A: f\to A(f)$ or $\inf\limits_{u\in U_n} n(f-u,p)$ is the best approximation of the function f by the finite-dimensional subspace U_n.

The functional $n(f,p)$ can be regarded as the main approximative characteristic of the function f, the value of which is equal to the deviation of f from the zero function. This functional is used not only in the theory of approximation of functions, but also in other areas of mathematics.

If $p=2$, then according to Parseval's equality for finding the value of the functional $n(f,2)$ we only need to find a sum of a numerical series with the terms that are the squares of the Fourier coefficients of the function f with respect to the complete orthonormal sequence of functions on the set A. For all other fixed $p\neq 2$ the value of the main approximative characteristic is estimated by means of inequalities, which are called upper and lower estimates. So approximative characteristics as functions of the variable p are estimated using the tabular procedure of setting of the function. But the tabular procedure of setting of the function makes it impossible to establish even the simplest its properties such as monotonicity, continuity and other.

If $p<1$, then the space of functions defined by a functional $n(f,p)$ is not normalizable, which creates some inconvenience not only in approximation theory. Therefore, if $p<1$ and especially when $p<0$ in many cases properties of the approximative characteristics are not considered.

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Remark. If $0 < \mu B_0 = \mu\{t \in A : f(t) = 0\} < \mu(A)$, then for an arbitrary $p > 0$, using the additive property of the integral, we obtain

$$\begin{aligned}(1)\qquad &\left(n(f,p) = \left(1 - \frac{\mu(B_0)}{\mu(A)}\right)^{\frac{1}{p}} \left(\frac{1}{\mu(A \setminus B_0)} \int_{A\setminus B_0} |f(t)|^p dt\right)^{\frac{1}{p}}\right) \\ &\wedge \left(0 < 1 - \frac{\mu(B_0)}{\mu(A)} = a < 1\right) \wedge \left(\left(a^{\frac{1}{p}}\right)' = -\frac{1}{p^2}(\ln a) a^{\frac{1}{p}} > 0\right) \\ &\wedge \left(\lim_{p\to 0+0} a^{\frac{1}{p}} = 0\right) \wedge \left(\lim_{p\to 0-0} a^{\frac{1}{p}} = +\infty\right) \wedge \left(\lim_{p\to+\infty} a^{\frac{1}{p}} = 1\right) \wedge \left(\lim_{p\to-\infty} a^{\frac{1}{p}} = 1\right).\end{aligned}$$

From (1) follows that the function $a^{\frac{1}{p}}$ is analytic in the intervals $(-\infty, 0)$, $(0, +\infty)$ and strictly increasing on these intervals. Therefore, the intervals of analyticity and strict increase of the function $n(f,p)$ coincide with intervals of analyticity and strict increase of the function $\left(\frac{1}{\mu(A\setminus B_0)} \int_{A\setminus B_0} |f(t)|^p dt\right)^{\frac{1}{p}}$.

So, we can consider measurable functions $f(t)$, which are not equal to zero nearly everywhere in the Lebesgue measurable set A, that is, for which $\mu B_0 = 0$. Then the functional $n(f,p)$ can be defined also for $p < 0$, and the functional $I(f,z) = \int_A |f(t)|^z dt$, $z = p + iy$, – for complex values of the variable z.

If $|f(t)|$ is a constant k almost everywhere on the set A, then for any $p \neq 0$ equality $n(f,p) = k$ holds and this trivial case we do not consider.

Let's denote by

$$\begin{aligned} L_{p(A)} = \Bigg\{ &\left(f(t) : n(f,p) = \left(\frac{1}{\mu(A)} \int_A |f(t)|^p dt\right)^{\frac{1}{p}} < \infty\right) \wedge (p \in \mathbb{R} \setminus \{0\}) \\ &\wedge \left(n(f, 0+0) = \lim_{p\to 0+0} n(f,p)\right) \wedge \left(n(f, 0-0) = \lim_{p\to 0-0} n(f,p)\right) \\ &\wedge \left(n(f, +\infty) = \lim_{p\to+\infty} n(f,p)\right)\Bigg\}, \\ L_{\infty(A)} = \Bigg\{ &f(t) : \|f\|_{L_{\infty(A)}} \stackrel{\text{def}}{=} \inf_{E:\mu E = 0} \sup_{t\in A\setminus E} \{|f(t)|\} < \infty\Bigg\}\end{aligned}$$

the spaces of measurable essentially bounded functions.

It is known that (see [1, p. 143])

$$(2)\qquad \lim_{p\to+\infty} n(f,p) = \|f\|_{L_{\infty(A)}}, \quad \lim_{p\to 0+0} n(f,p) = e^{\frac{1}{\mu(A)}\int_A \ln|f(t)| dt} = n(f, 0+0).$$

For any $p \neq 0$ the following equality holds

$$(3)\qquad n(f,p) = \frac{1}{n\left(\frac{1}{f}, -p\right)}.$$

From equalities (2) and (3) we obtain

$$\begin{aligned}(4)\qquad & n(f, -\infty) = \lim_{p\to-\infty} n(f,p) = \frac{1}{\|1/f\|_{L_{\infty(A)}}}, \\ & n(f, 0-0) = \frac{1}{\lim\limits_{-p\to 0+0} n\left(\frac{1}{f}, -p\right)} = \frac{1}{e^{\frac{1}{\mu(A)}\int_A \ln\left|\frac{1}{f(t)}\right| dt}} = e^{\frac{1}{\mu(A)}\int_A \ln|f(t)| dt}.\end{aligned}$$

From formulas (2), (4) we get that if the function $n(f,p)$ is defined in the neighborhood of 0, then it is continuous at this point.

If the integral $\int_A |f(t)|^p dt$ is divergent for $p > 0$, then $f \notin L_{p(A)}$ and we will assume that $n(f,p) = +\infty$. If $f \notin L_{p(A)}$ and $\frac{1}{f} \notin L_{p(A)}$ for arbitrary $p > 0$, then we will assume that $n(f,0) = e^{\frac{1}{\mu(A)} \int_A \ln|f(t)|dt}$. If $\int_A \ln|f(t)|dt = -\infty$, then according to the equalities (2), (4) we have $n(f,0) = 0$. If $p < 0$ then $\frac{1}{\mu(A)} \int_A |f(t)|^{-p} dt = k$ or $\int_A |f(t)|^{-p} dt = +\infty$. Therefore, according to the formula (2) we obtain

$$n(f,p) = \frac{1}{\left(\frac{1}{\mu(A)} \int_A |f(t)|^{-p} dt\right)^{-\frac{1}{p}}} = k^{\frac{1}{p}} \quad \text{or} \quad n(f,p) = \left(\frac{1}{\infty}\right) = 0.$$

Thus, for $p < 0$ all measurable functions belong to the space $L_{p(A)}$.

In this paper we proved that function $f(t)$ may not belong to any of the spaces $L_{p(A)}$ for $p > 0$, but the function $n(f,p)$, which is defined for all negative values of the variable p, is analytic at all inner points of the interval in which the value of the function $n(f,p) \neq 0$ except the point $p = 0$, and it is strictly increasing and continuous on this interval.

These remarkable properties of the function $n(f,p)$ allow us to determine for the functions of the spaces $L_{q(A)}$, $0 < q \leq \infty$, that their basic approximative characteristics on interval $[0,q]$ are continuous, non-decreasing, and, in some cases, analytic. This makes it possible to avoid the use of a simple tabular method of estimation of these characteristics.

1. Analyticity of integrals dependent on a parameter

Let us prove auxiliary statements about analyticity of the integrals depending on a parameter. Note, that these statements can be of independent interest.

Lemma 1 (On the analyticity of integrals depending on a parameter). *Let the function $g(z,t)$ of two complex variables z and t be analytic with respect to the variable z for almost all values t belonging to the rectifiable curve $\breve{AB}$, and the integral of the modulus of the function $g(z,t)$ along the curve $\breve{AB}$ is bounded on an arbitrary simple rectifiable closed curve γ which is contained in G, that is*

$$\sup_{u\in\gamma} \int_{\breve{AB}} |g(u,t)|dt \leq M_\gamma. \tag{5}$$

Then the integral $\int_{\breve{AB}} g(z,t)dt = F(z)$, depending on the parameter z is an analytic function in the domain G and in each point $z \in G$, for each natural n the derivative of the n-th order of the integral is equal to the integral along a curve $\breve{AB}$ of the derivative of the n-th order in the variable z of the integrand $g(z,t)$, that is

$$(\forall n \in \mathbb{N}) \wedge (\forall z \in G) \Rightarrow \left(F^{(n)}(z) = \int_{\breve{AB}} g^{(n)}_{z^n}(z,t)dt\right). \tag{6}$$

Proof. Let γ be an arbitrary simple closed rectifiable curve that is contained in G and it is the boundary of the domain D contained in G, that is $\gamma = \partial D$. Let us prove that for the function $F(z)$ the Cauchy integral formula holds at each point $z \in D$, that is

$$(\forall z \in D) \Rightarrow \left(F(z) = \int_{\breve{AB}} g(z,t)dt = \frac{1}{2\pi i}\int_\gamma \frac{F(u)}{u-z}du\right). \tag{7}$$

Let $\rho = \inf_{u\in\gamma} \{|u-z|\}$ be the distance from the point z to the curve γ. Then at each point $u \in \gamma$ the inequality

$$\frac{1}{|u-z|} \leq \frac{1}{\rho} \tag{8}$$

holds.

Using the inequalities (8) and (5) we have

$$\begin{aligned}\int_\gamma \frac{|F(u)|}{|u-z|}du &= \int_\gamma \frac{\left|\int_{\breve{AB}} g(u,t)dt\right|}{|u-z|}du \\ &\le \frac{1}{\rho}\int_\gamma \left(\int_{\breve{AB}} |g(u,t)|dt\right)du \le \frac{1}{\rho}\int_\gamma M_\gamma du = \frac{1}{\rho}M_\gamma l(\gamma),\end{aligned} \tag{9}$$

where $l(\gamma)$ is the length of curve γ. From the inequality (9) follows that for every curve γ there exists a repeated integral

$$\int_\gamma \left(\int_{\breve{AB}} \left|\frac{g(u,t)}{u-z}\right| dt\right)du. \tag{10}$$

From (10) using the corollary to Fubini theorem on the change of the order of integration in repeated integrals (see, e.g. [2, p. 298–300]), we get

$$\frac{1}{2\pi i}\int_\gamma \frac{F(u)}{u-z}du = \frac{1}{2\pi i}\int_\gamma \left(\frac{1}{u-z}\int_{\breve{AB}} g(u,t)dt\right)du = \int_{\breve{AB}} \left(\frac{1}{2\pi i}\int_\gamma \frac{g(u,t)du}{u-z}\right)dt. \tag{11}$$

Since the function $g(z,t)$ is analytic almost for all $t \in \breve{AB}$, then taking into account the Cauchy integral formula, at each point $z \in D$ we have

$$\frac{1}{2\pi i}\int_\gamma \frac{g(u,t)du}{u-z} = g(z,t). \tag{12}$$

From equalities (11), (12) follows (7).

Let us prove that at each point $z \in D$ we have the equality

$$F'(z) = \frac{1}{2\pi i}\int_\gamma \frac{F(u)du}{(u-z)^2}. \tag{13}$$

It follows from the definition of the domain that for each point $z \in D$ there is a curve γ_1 that is a circle with center at a point z such that $\gamma_1 \subset D$ and $\gamma_1 = \partial D_1$, where D_1 is the disk contained in D. Then the equality (13) is valid at every point $z \in D$. Therefore, according to the definition of the analyticity of a function, the function $F(z)$ is analytic on D, and because of the arbitrariness of the curve γ, and hence of the domain D, the function $F(z)$ is also analytic on G.

Using the equality(7), we get

$$\begin{aligned}F'(z) &= \lim_{\Delta z\to 0}\frac{F(z+\Delta z)-F(z)}{\Delta z} = \lim_{\Delta z\to 0}\frac{1}{2\pi i}\frac{1}{\Delta z}\left(\int_\gamma \frac{F(u)\,du}{(u-(z+\Delta z))}\right. \\ &\left.-\int_\gamma \frac{F(u)\,du}{u-z}\right) = \frac{1}{2\pi i}\lim_{\Delta z\to 0}\frac{1}{\Delta z}\int_\gamma \frac{F(u)\Delta z du}{(u-(z+\Delta z))(u-z)} \\ &= \frac{1}{2\pi i}\lim_{\Delta z\to 0}\int_\gamma \frac{F(u)du}{(u-(z+\Delta z))(u-z)}.\end{aligned} \tag{14}$$

It follows from the equalities (13), (14) that it is sufficient to prove equality

$$\lim_{\Delta z\to 0}\int_\gamma \frac{F(u)du}{(u-(z+\Delta z))(u-z)} = \int_\gamma \frac{F(u)du}{(u-z)^2}. \tag{15}$$

The equality (15), according to the properties of limits and properties of integrals, is equivalent to the equalities

$$
\begin{aligned}
&\left(\lim_{\Delta z\to 0}\left(\int_\gamma \frac{F(u)\,du}{(u-(z+\Delta z))\,(u-z)} - \int_\gamma \frac{F(u)\,du}{(u-z)^2}\right) = 0\right)\\
&\Leftrightarrow \left(\lim_{\Delta z\to 0} \Delta z \int_\gamma \frac{F(u)\,du}{(u-z-\Delta z)\,(u-z)^2} = 0\right)\\
&\Leftrightarrow \left(\lim_{\Delta z\to 0} |\Delta z| \left|\int_\gamma \frac{F(u)du}{(u-z-\Delta z)\,(u-z)^2}\right| = 0\right).
\end{aligned}
\tag{16}
$$

Since $\Delta z \to 0$, we can assume that

$$|\Delta z| < \frac{\rho}{2}. \tag{17}$$

Using the inequalities (5), (8) and (17), we have

$$
\begin{aligned}
\left|\int_\gamma \frac{F(u)du}{(u-z-\Delta z)(u-z)^2}\right| &\le \int_\gamma \frac{\left|\int_{\breve{AB}} g(u,t)\,dt\right| du}{|(u-z)-\Delta z|\,|u-z|^2}\\
&\le \int_\gamma \frac{\int_{\breve{AB}} |g(u,t)|\,dt}{\rho^2|\,|u-z|-|\Delta z|\,|}du \le \int_\gamma \frac{M_\gamma du}{\rho^2|\rho-|\Delta z|\,|}\\
&= \frac{M_\gamma l(\gamma)}{\rho^2\,(|\rho-|\Delta z|\,|)} < \frac{M_\gamma l(\gamma)}{\rho^2\frac{\rho}{2}} = M_1.
\end{aligned}
\tag{18}
$$

From (18), according to the theorem on the product of an infinitesimal function by a bounded one, the equalities (14)–(16) follow and thus the analyticity of the function $F(z)$ on the domain G is proved.

Let us prove that at each point $z \in G$ we have the equality

$$F'(z) = \int_{\breve{AB}} g'_z(z,t)dt. \tag{19}$$

Using the inequalities (5), (8), we obtain that at each point $z \in G$ the inequalities

$$\int_\gamma \left|\frac{F(u)du}{(u-z)^2}\right| \le \int_\gamma \left(\int_{\breve{AB}} \frac{|g(u,t)|dt}{|u-z|^2}\right)du \le \frac{M_\gamma l(\gamma)}{\rho^2} \tag{20}$$

hold.

From (20), using the corollary to Fubini theorem on the change of the order of integration in repeated integrals, we have that at each point $z \in G$

$$
\begin{aligned}
F'(z) &= \frac{1}{2\pi i}\int_\gamma \frac{F(u)du}{(u-z)^2} = \frac{1}{2\pi i}\int_\gamma \frac{1}{(u-z)^2}\left(\int_{\breve{AB}} g(u,t)dt\right)du\\
&= \int_{\breve{AB}} \left(\frac{1}{2\pi i}\int_\gamma \frac{g(u,t)}{(u-z)^2}du\right)dt.
\end{aligned}
\tag{21}
$$

Since the function $g(z,t)$ is analytic in the variable z for almost all t, at each point $z \in G$, using the Cauchy integral formula for its derivative, we obtain

$$\frac{1}{2\pi i}\int_\gamma \frac{F(u)du}{(u-z)^2} = g'_z(u,z). \tag{22}$$

It follows from the equalities (21) and (22) that at each point $z \in G$ the equality (19) holds.

Similar to the proof of the equality (19), using the Cauchy integral formula for the n-th order derivative of the function $g(z,t)$ analytic in the domain G, namely

$$g^{(n)}_{z^n}(z,t) = \frac{n!}{2\pi i}\int_\gamma \frac{g(u,t)du}{(u-z)^{n+1}},$$

using the method of mathematical induction, we prove the equality (6). Lemma 1 is proved. □

Remark. In Lemma 1, the curve $\smile{AB}$ can be replaced by an arbitrary set $A \subset \mathbb{R}$ which is Lebesgue measurable and the function $g(z,t)$ of the complex variable t by the function $g(z,t)$ of the real variable $t \in A$. Moreover, the proof of Lemma 1 does not change.

Since every closed simple rectifiable curve $\gamma \subset G$ is a closed and bounded set, the curve γ can be replaced by an arbitrary closed bounded set $F \subset G$.

Thus, the following statement is true.

Corollary 1 (On the analyticity of the integral depending on a parameter). *Let the function $g(z,t)$ of a complex variable z and a real variable t for almost all values t, which belong to the Lebesgue measurable set A, is analytic in the variable z on the domain G and the integral of the modulus of the function with respect to the set A is bounded on an arbitrary closed bounded set $F \subset G$, that is*

$$\sup_{z\in F}\int_A |g(z,t)|dt \le M_F.$$

Then the integral depends on the parameter z

$$\Phi(z) = \int_A g(z,t)dt$$

is an analytic function on the domain G and for each $n \in \mathbb{N}$ and for each $z \in G$ the following equality holds

$$\Phi^{(n)}(z) = \int_A g^{(n)}_{z^n}(z,t)dt.$$

2. Properties of the functions $I(f,z) = I(f,p+iy) = \int_A |f(t)|^{p+iy}\,dt$ and $I(f,p)$

Let $I(f,z) = I(f,p+iy) = \int_A |f(t)|^{p+iy}\,dt$ be the function of complex variable $z = p+iy$, $I(f,p) = \int_A |f(t)|^p dt = \int_{A\setminus B_0} |f(t)|^p dt$ be its restriction on the set of real numbers, where $\mu B_0 = \mu\{t \in A : f(t) = 0\} = 0$.

Then the equality $I(f,0) = \int_A |f(t)|^0\,dt = \mu(A)$ holds for each function $f(t)$ measurable on the set A and the inequality $I(f,p) > 0$ holds for every real p of the domain of $I(f,p)$.

Since the value of the functional $I(f,p)$ does not change with the change of the set B_0, then without loss of generality we can assume that for almost every $t \in A$ and an arbitrary $p \in \mathbb{R}$ the inequality $|f(t)|^p > 0$ is valid.

Lemma 2 (On the properties of the functions $I(f,z)$ and $I(f,p)$ on the intervals $[0,q], [0,q), [0,+\infty)$). *The statements* I–III *are valid.*

I. *If the function $f(t) \in L_{q(A)}$ and for each $p > q > 0$ the function $f(t) \notin L_{p(A)}$, then the statements* I_1–I_3 *hold.*

I_1. *The function $I(f,z)$ is analytic on the domain $G_{(0,q)} = \{z = p+iy : (p \in (0,q)) \wedge (y \in \mathbb{R})\}$ and at each point $z \in G_{(0,q)}$ for each $n \in \mathbb{N}$ the following equality holds:*

$$I^{(n)}(f,z) = \int_A |f(t)|^z \ln^n |f(t)|\,dt. \tag{23}$$

I_2. *The function $I(f,z)$ is continuous on the closure of this domain $\bar{G}_{(0,q)} = G_{[0,q]} = \{z = p + iy : (p \in [0,q]) \wedge (y \in \mathbb{R})\}$.*

I_3. *The function $I(f,p)$ is strictly convex downward and logarithmic convex on the segment $[0,q]$, that is $(\forall p \in (0,q)) \Rightarrow (I''(f,p) > 0) \wedge (\ln I(f,p))'' > 0)$.*

II. *If the function $f(t) \notin L_{q(A)}$ and for each $p \in [0,q)$ the function $f(t) \in L_{p(A)}$, then the statements I_1, II_1–II_3 hold.*

II_1. *The function $I(f,z)$ is continuous on the set $G_{[0,q)} = \{z = p + iy : (p \in [0,q)) \wedge \wedge (y \in \mathbb{R})\}$.*

II_2. *The function $I(f,p)$ is strictly convex downward and logarithmic-ally convex on the $[0,q)$.*

II_3. $\lim\limits_{p \to q-0} I(f,p) = +\infty$.

III. *Let $f(t) \in L_{\infty(A)}$. Then the statements III_1–III_5 are valid.*

III_1. *The function $I(f,z)$ is analytic on the domain $G_{(0,+\infty)} = \{z = p+iy : (p \in (0,+\infty)) \wedge (y \in \mathbb{R})\}$ and at each point of this domain the equality (23) holds.*

III_2. *The function $I(f,z)$ is continuous on the set $G_{[0,+\infty)}$.*

III_3. *The function $I(f,p)$ is strictly convex downward and logarithmic-ally convex on the $[0,+\infty)$.*

III_4. *If $\|f\|_{L_\infty(A)} > 1$, then $\lim\limits_{p \to +\infty} I(f,p) = +\infty$.*

III_5. *If $\|f\|_{L_\infty(A)} \le 1$, then $\lim\limits_{p \to +\infty} I(f,p) = \mu(B_1)$, where $B_1 = \{t \in A : |f(t)| = 1\}$, and for each $p \in (0,+\infty)$ and for any $n \in \mathbb{N}$ the following inequality holds:*

$$(-1)^n I^{(n)}(f,p) > 0,$$

that is the function $I(f,p)$ is regularly monotone on $(0,+\infty)$ (see, e.g. [3, p. 156]).

Proof. Let's prove the statement I_1. To do this, it is sufficient to establish that the function

$$g(z,t) = |f(t)|^z = |f(t)|^{p+iy} = |f(t)|^p (\cos y \ln |f(t)| + i \sin y \ln |f(t)|) \tag{24}$$

satisfies the conditions of Corollary 1 on the domain $G_{(0,q)}$.

For each $t \in A$ the function $|f(t)|^z$ is analytic on the entire complex plane. We prove that the function

$$|g(z,t)| = ||f(t)|^p| \, |\cos y \ln |f(t)| + i \sin y \ln |f(t)|| = |f(t)|^p \tag{25}$$

is bounded on the set A by an integrable function $\varphi(t)$.

If $0 \le p \le q$, then for each $t \in A$ the following relations hold

$$|f(t)|^p \le \max\{1, |f(t)|^p\} \le \varphi(t) = \begin{cases} 1, & t \in B_1^- = \{t \in A : |f(t)| < 1\} \\ |f(t)|^q, & t \in A \setminus B_1^- \end{cases}. \tag{26}$$

Using the additive property of the integral, the properties of the integrals associated with the inequalities and the relations (25), (26), we have

$$\begin{aligned} &\sup_{z \in G_{[0,q]}} \left\{ \int_A |g(z,t)| dt \right\} = \sup_{p \in G_{[0,q]}} \left\{ \int_A |f(t)|^p dt \le \int_A \varphi(t) dt \right\} \\ &= \int_{B_1^-} 1 dt + \int_{A \setminus B_1^-} |f(t)|^q dt \le \mu\left(B_1^-\right) + \int_A |f(t)|^q dt = M_{G_{[0,q]}} = M_F. \end{aligned} \tag{27}$$

Since the function $|f(t)|^z$ is analytic for all complex z, then for each $n \in \mathbb{N}$ and for each z the equality

$$\left(|f(t)|^z\right)^{(n)} = |f(t)|^z \ln^n |f(t)| \tag{28}$$

is valid. From the relations (27), (28) it follows that the function $g(z,t)$ satisfies the conditions of Corollary 1. Therefore, by Corollary 1, the statement I_1 holds.

Let's prove the statement I_2.

It follows from the relations (26) that the conditions of the theorem on an exchanging the limit and the Lebesgue integral are satisfied (see, e.g. [4, p. 120]).

Therefore

$$
\begin{aligned}
&\left(\lim_{p\to 0+0} I(f,p) = \int_A \lim_{p\to 0+0} |f(t)|^p dt = \int_A dt = \mu(A) = I(f,0+0)\right)\\
&\quad\wedge\left(\lim_{p\to q-0} I(f,p) = \int_A \lim_{p\to q-0} |f(t)|^p dt = I(f,q)\right).
\end{aligned} \tag{29}
$$

From the equalities (24), (25), we have

$$
\begin{aligned}
&\Bigg(\lim_{\mathrm{Re}z=p\to 0+0} I(f,z) = \lim_{p\to 0+0}\int_A |f(t)|^p\left(\cos y\ln|f(t)| + i\sin y\ln|f(t)|\right)dt\\
&\quad= \int_A \lim_{p\to 0+0} |f(t)|^p\left(\cos y\ln|f(t)| + i\sin y\ln|f(t)|\right)dt\\
&\quad= \int_A \left(\cos y\ln|f(t)| + i\sin y\ln|f(t)|\right)dt = I(f,0+iy)\Bigg)\\
&\quad\wedge\Bigg(\lim_{\mathrm{Re}z=p\to q-0} I(f,z) = \int_A \lim_{p\to q-0} |f(t)|^p(\cos y\ln|f(t)| + i\sin y\ln|f(t)|)dt\\
&\quad= \int_A |f(t)|^q(\cos y\ln|f(t)| + i\sin y\ln|f(t)|)dt = I(f,q+iy)\Bigg).
\end{aligned} \tag{30}
$$

From the equalities (30), the analyticity of the function $I(f,z)$ on the domain $G_{(0,q)}$ and the definition of the continuity of a function on the set, implies the statement I_2.

Let's prove the statement I_3.

For each $p\in(0,q)$ using the equality (23), we get

$$
I''(f,p) = \int_A |f(t)|^p \ln^2|f(t)|dt. \tag{31}
$$

Let $B_1 = \{f\in A : |f(t)| = 1\}\subseteq A$. Let us prove that $\mu(B_1) < \mu(A)$. If $\mu(B_1) = \mu(A)$, then almost everywhere on the set A $|f(t)| = 1$, but this case is excluded. Thus $\mu(B_1) < \mu(A)$.

Using the additive property of the integral and the properties of integrals associated with inequalities, we obtain

$$
\begin{aligned}
\int_A |f(t)|^p\ln^2|f(t)|dt &= \int_{B_1} |f(t)|^p\ln^2|f(t)|dt + \int_{A\setminus B_1} |f(t)|^p\ln^2|f(t)|dt\\
&= \int_{A\setminus B_1} |f(t)|^2\ln^2|f(t)|dt > 0.
\end{aligned} \tag{32}
$$

It follows from (31), (32) that for each $p\in(0,q)$ the inequality

$$
I''(f,p) > 0 \tag{33}
$$

holds.

It follows from the inequality (33) that the function $I(f,p)$ is strictly convex downwards on $(0,q)$. Therefore, because $I(f,p)$ is continuous on the segment $[0,q]$, the function $I(f,p)$ is strictly convex downward on $[0,q]$.

Since $I(f,p)>0$, for each $p\in(0,q)$ using the equality (23), we obtain

$$(\ln I(f,p))''=\frac{I''(f,p)\cdot I(f,p)-(I'(f,p))^2}{I^2(f,p)} \tag{34}$$
$$=\frac{\int_A|f(t)|^p\ln^2|f(t)|dt\cdot\int_A|f(t)|^pdt-\left(\int_A|f(t)|^p\ln|f(t)|dt\right)^2}{I^2(f,p)}.$$

Using the Cauchy–Bunyakovskii inequality, we have

$$\left(\int_A|f(t)|^p\ln|f(t)|dt\right)^2=\left(\int_A|f(t)|^{\frac{p}{2}}|f(t)|^{\frac{p}{2}}\ln|f(t)|dt\right)^2 \tag{35}$$
$$\le\int_A|f(t)|^pdt\cdot\int_A|f(t)|^p\ln^2|f(t)|dt.$$

In the Cauchy–Bunyakovskii inequality, we have the equality sign if and only if the equalities

$$\left(\lambda|f(t)|^{\frac{p}{2}}=|f(t)|^{\frac{p}{2}}\ln^2|f(t)|\right)\Leftrightarrow\left(\ln^2|f(t)|=\lambda\right)\Leftrightarrow\left(|f(t)|=e^{\sqrt{\lambda}}\right) \tag{36}$$

are valid almost everywhere on the set A.

It follows from the equalities (36) that the function $|f(t)|$ almost everywhere on A is equal to the constant $k=e^{\sqrt{\lambda}}$, but this trivial case is not considered.

Thus, in the inequality (35) equality is impossible. Therefore, it follows from (34), (35) that for each $p\in(0,q)$ the inequality

$$(\ln I(f,p))''>0 \tag{37}$$

holds.

It follows from the inequality (37) that the function $\ln I(f,p)$ is strictly convex downward on $(0,q)$. Therefore, by the continuity of the function $\ln(I(f,p))$ on the segment $[0,q]$ and by definition, the function $I(f,p)$ is logarithmic convex on $[0,q]$. The statement I_3 is proved.

Let's prove the statement II.

Since for every $r\in(0,q)$ the function $f(t)\in L_{r(A)}$, then according to the statement I, the function $I(f,z)$ is analytic on the domain $G_{(0,r)}=\{z=p+iy:(p\in(0,r))\wedge(y\in\mathbb{R})\}$, and equality (23) holds at each point of this domain.

The function $I(f,z)$ is continuous on the set $G_{[0,r]}=\{z=p+iy:(p\in[0,r])\wedge(y\in\mathbb{R})\}$ and the function $I(f,p)$ is strictly convex downward and logarithmic convex on the segment $[0,r]$. Therefore, because of the arbitrariness of r, the statements $\mathrm{I}_1,\mathrm{II}_1,\mathrm{II}_2$ are valid.

It remains to prove the statement II_3.

Since the function $I(f,p)$ is strictly convex downwards and continuous on $[0,q)$, there exists $\delta>0$ such that the function $I(f,p)$ is strictly monotonic on the $(q-\delta,q)\subset[0,q)$. Therefore, by the theorem on the limit of a monotonic function, there exists

$$\lim_{p\to q-0}I(f,p)=\begin{cases}+\infty,\\ I(f,q)=K<\infty.\end{cases}$$

If $I(f,q)=\lim\limits_{p\to q-0}I(f,p)=K$, then the function $f(t)\in L_{q(A)}$, which contradicts the condition of statement II. Thus $\lim\limits_{p\to q-0}I(f,p)=+\infty$ and the statement II_3 is proved.

Let's prove the statement III.

Let us prove that if $f\in L_{\infty(A)}$, then for each $r>0$ the function $f\in L_{r(A)}$. Using the definition of the norm of the space $L_{\infty(A)}$ and the properties of integrals associated with inequalities, we have

$$I(f,r) = \int_A |f(t)|^r dt = \|f\|^r_{L_\infty(A)} \int_A \left(\frac{|f(t)|}{\|f\|_{L_\infty(A)}} \right)^r dt$$
$$\leq \|f\|^r_{L_\infty(A)} \int_A 1^r dt = \|f\|^r_{L_\infty(A)} \mu(A).$$

Owing to the arbitrariness of $r \in (0, +\infty)$, arguing in the same way as in the proof of the statement II, we establish the validity of the statements III_1-III_3.

Let's prove the statement III_4.

Let

$$A_M = \left\{ t \in A : 1 < M \leq |f(t)| \leq \|f\|_{L_\infty(A)} \right\}. \tag{38}$$

Let us prove that there exists M such that the inequality

$$\mu\left(A_M\right) > 0 \tag{39}$$

holds. Assume that for every M such that $1 < M \leq |f(t)| \leq \|f\|_{L\infty(A)}$ equality

$$\mu\left(A_M\right) = 0 \tag{40}$$

is true.

From the equalities (38), (40), using properties of the Lebesgue measure, we obtain

$$\mu(A \backslash A_M) = \mu(A) - \mu\left(A_M\right) = \mu(A)$$

and the following inequality holds almost everywhere on the set A:

$$|f(t)| < M. \tag{41}$$

From (38), (41), on account of the arbitrariness of M, it follows that almost everywhere on A the inequalities $|f(t)| \leq 1$ and $\|f\|_{L_\infty(A)} \leq 1$ are valid, which contradicts the condition of the statement III_4. Thus, the inequality (39) holds.

Using the additive property of the integral, the properties of the integral associated with the inequalities and (38), (39), we obtain

$$\begin{aligned} \lim_{p\to+\infty} I(f,p) &= \lim_{p\to+\infty} \left(\int_{A_M} |f(t)|^p dt + \int_{A\backslash A_M} |f(t)|^p dt \right) \geq \lim_{p\to+\infty} \int_{A_M} |f(t)|^p dt \\ &\geq \lim_{p\to+\infty} M^p \int_{A_M} dt = \mu\left(A_M\right) \lim_{p\to+\infty} M^p = +\infty. \end{aligned} \tag{42}$$

From (42) follows the statement III_4.

Let us now prove the statement III_5.

Using the additive property of the integral, we have

$$\begin{aligned} \int_A |f(t)|^p dt &= \int_{B_1} |f(t)|^p dt + \int_{A\backslash B_1} |f(t)|^p dt \\ &= \int_{B_1} 1^p dt + \int_{A\backslash B_1} |f(t)|^p dt = \mu\left(B_1\right) + \int_{A\backslash B_1} |f(t)|^p dt. \end{aligned} \tag{43}$$

Since $\|f\|_{L_\infty(A)} \leq 1$, then for each $p > 0$ almost everywhere on the set $A\backslash B_1$ the following inequality holds

$$|f(t)|^p < 1. \tag{44}$$

From (43), (44), using the theorem on an exchanging the limit and the Lebesgue integral, we obtain

$$
\begin{aligned}
\lim_{p\to+\infty} I(f,p) &= \lim_{p\to+\infty}\mu(B_1) + \lim_{p\to+\infty}\int_{A\setminus B_1}|f(t)|^p dt \\
&= \mu(B_1) + \int_{A\setminus B_1}\lim_{p\to+\infty}|f(t)|^p dt = \mu(B_1) + \int_{A\setminus B_1} 0dt = \mu(B_1).
\end{aligned} \tag{45}
$$

For each $p > 0$, using the equalities (23), we have

$$
\begin{aligned}
I^{(n)}(f,p) &= \int_A |f(t)|^p \ln^n|f(t)|dt = \int_{B_1}|f(t)|^p \ln^n|f(t)|dt \\
&+ \int_{A\setminus B_1}|f(t)|^p \ln|f(t)|dt = \int_{A\setminus B_1}|f(t)|^p \ln^n|f(t)|dt.
\end{aligned} \tag{46}
$$

Since $\|f\|_{L_\infty(A)} \le 1$, then almost everywhere on the set $A\setminus B_1$ the inequalities

$$
(|f(t)| < 1) \wedge ((-1)^n \ln^n|f(t)| > 0) \tag{47}
$$

are true.

From the equality (46) and inequality (47), we get that for each $p \in (0,+\infty)$ and any $n \in \mathbb{N}$, the following inequality holds

$$
(-1)^n \int_A |f(t)|^p \ln^n|f(t)|dt = (-1)^n I^{(n)}(f,p) > 0. \tag{48}
$$

From (45) and (48) follows the validity of the statement III_5. Lemma 2 is proven. □

Remark. The analyticity of the function $I(f,p)$ could be established using the criterion of analyticity of an infinitely differentiable function of a real variable (see, e.g. [3, p. 142, 143]). But the proof of the infinite differentiability of the function $I(f,p)$, that is $n \in \mathbb{N}$ the equality $I^{(n)}(f,p) = \int_A |f(t)|^p \ln^n|f(t)|dt, n \in \mathbb{N}$, which is a special case of the equality (23), turned out to be more difficult than the proof of the analyticity of the function $I(f,z)$, from which its infinite differentiability follows.

To give examples satisfying the conditions of Lemma 2, we use the following functions defined on the set $(0,1]$:

$$
g_1(t) = \frac{1}{t^{\frac{1}{q}}\left(\ln\frac{t}{e}\right)^{\frac{2}{q}}}, \quad g_2(t) = \frac{1}{t^{\frac{1}{q}}}, \quad g_3(t) = e^{-\frac{1}{t^\gamma}},
$$

where $q > 0$ and $\gamma > 0$.

Then $I(g_1,q) = \int_0^1 |g_1(t)|^q\,dt = \int_0^1 \frac{dt}{t\left(-\ln\frac{t}{e}\right)^2} = 1$ and for each $p > q$ we have $\int_0^1 |g_1(t)|^p\,dt = +\infty$. Thus, $g_1 \in L_{q((0,1])}$ and for each $p > q$ the function $g_1 \notin L_{p((0,1])}$.

For function g_2 the integral $I(g_2,q) = \int_0^1 |g_2(t)|^q dt = \int_0^1 \frac{dt}{t} = +\infty$ and for each $p < q$ we have $I(g_2,p) = \int_0^1 \frac{dt}{t^{\frac{p}{q}}} = \frac{q}{q-p}$. Thus, $g_2 \notin L_{q((0,1])}$ and for each $p < q$ the function $g_2 \in L_{p((0,1])}$.

The function $g_3 \in L_{\infty((0,1])}$ and $\|f\|_{L_{\infty((0,1])}} = e^{-1}$. For each $p > 0$, after changing the variables, we obtain

$$
I(g_3,-p) = \int_0^1 |g_3(t)|^{-p}\,dt = \int_0^1 e^{\frac{p}{t^\gamma}}dt = \frac{1}{\gamma}\int_1^{+\infty}\frac{e^{pu}}{u^{1+\frac{1}{\gamma}}}du = +\infty.
$$

Example 1. Let us prove that the function $f_1(t) = \begin{cases} g_3(t), & t \in (0,1] \\ g_1(t-1), & t \in (1,2] \end{cases}$, which is defined on the set $A = (0,2]$, satisfies the conditions of statement I of Lemma 2. Using the additive property of the integral and changing the variables, we have

$$
\begin{aligned}
I\left(f_1, p\right) &= \int_A \left|f_1(t)\right|^p dt = \int_0^1 \left|g_3(t)\right|^p dt + \int_1^2 \left|g_1(t-1)\right|^p dt \\
&= \int_0^1 e^{-\frac{p}{t^\gamma}} dt + \int_0^1 \left|g_1(t)\right|^p dt = \int_0^1 e^{-\frac{p}{t^\gamma}} dt + \int_0^1 \frac{dt}{t^{\frac{p}{q}} \left(-\ln \frac{t}{e}\right)^{\frac{2p}{q}}}.
\end{aligned} \tag{49}
$$

From (49), according to the examples given earlier, it follows that function $f_1 \in L_{q(A)}$ and $I(f_1,p) = +\infty$ for each $p \in (-\infty, 0) \cup (q, +\infty)$.

Similarly, we can show the validity of the following examples.

Example 2. The function $f_2(t) = \begin{cases} g_3(t), t \in (0,1] \\ g_2(t-1), t \in (1,2] \end{cases}$, which is defined on the set $A = (0,2]$, satisfies the conditions of statement II of Lemma 2.

Example 3. The function $f_3(t) = 3g_3(t)$, which is defined on the set $A = (0,1]$, satisfies the conditions of statements III_1-III_4 of Lemma 2.

Example 4. The function $f_4(t) = g_3(t)$, which is defined on the set $A = (0,1]$, satisfies the conditions of statements III_1-III_3, III_5 of Lemma 2.

Example 5. The function $f_5(t) = \begin{cases} g_3(t), & t \in (0,1] \\ 1, & t \in (1,2] \end{cases}$, which is defined on the set $A = (0,2]$, satisfies the conditions of statements III_1-$\mathrm{III}_3, \mathrm{III}_5$ of Lemma 2.

We denote by $[0,q\rangle$ the intervals $[0,q]$ or $[0,q)$, where $0 < q < \infty$, and $[0,+\infty)$ if $q = +\infty$. Then the following relations hold:

$$
(p \in [0,q\rangle) \Leftrightarrow (-p \in \langle -q, 0]),
$$

$$
\begin{aligned}
&\left(z = p + iy \in G_{[0,q\rangle} = \{p + iy : (p \in [0,q\rangle) \wedge (y \in \mathbb{R})\}\right) \\
&\quad \Leftrightarrow \left(z = -p + iy \in G_{\langle -q,0]} = \{-p + iy : (-p \in \langle -q, 0]) \wedge (y \in \mathbb{R})\}\right),
\end{aligned} \tag{50}
$$

$$
\left(I(f,-p) = I\left(\frac{1}{f}, p\right)\right) \Leftrightarrow \left(I\left(\frac{1}{f}, -p\right) = I(f,p)\right). \tag{51}
$$

If under the conditions of Lemma 2 the function $f(t)$ is replaced by the function $\frac{1}{f(t)}$, then, according to the relations (50), (51), the following statement is true.

Corollary 2 (on the properties of the function $I\left(f,p\right)$ on the intervals $\langle -q, 0]$). *For the function $I\left(f,p\right)$ the statements* I-III *are valid.*

I. *Let the function $\frac{1}{f} \in L_{q(A)}$ and for each $p > q > 0$ the function $\frac{1}{f} \notin L_{p(A)}$. Then the statements* I_1-I_3 *hold.*

I_1. *The function $I(f,z) = I(f, p+iy) = \int_A |f(t)|^z dt$ is analytic on the domain $G_{(-q,0)} = \{z = p + iy : (p \in (-q,0)) \wedge (y \in \mathbb{R})\}$ and for every $z \in G_{(-q,0)}$ and for any natural n following equality holds*

$$
I^{(n)}\left(f,z\right) = \int_A |f(t)|^z \ln^n |f(t)|\, dt. \tag{52}
$$

I_2. *The function $I\left(f,z\right)$ is continuous on the set $G_{[-q,0]} = \big\{z = p + iy : (p \in [-q,0]) \wedge (y \in \mathbb{R})\big\}$.*

I_3. *The function $I(f,p)$ is strictly convex downward and logarithmic convex on the segment $[-q,0]$.*

II. *Let the function $\frac{1}{f} \notin L_{q(A)}$ and for each $p \in (0,q)$ the function $\frac{1}{f} \in L_{p(A)}$. Then the statements* I_1, II_2-II_4 *is true.*

II_2. *The function $I\left(f,z\right)$ is continuous on the set*

$$
G_{(-q,0]} = \{z = p + iy : (p \in (-q,0]) \wedge (y \in \mathbb{R})\}.
$$

II_3. *The function $I(f,p)$ is strictly convex downward and logarithmic convex on $(-q,0]$.*

II_4. $\lim\limits_{p\to -q+0} I(f,p) = +\infty$.

III. *Let the function $\frac{1}{f} \in L_{\infty(A)}$. Then we have the statements* III_1-III_5.

III_1. *The function $I(f,z)$ is analytic on the domain $G_{(-\infty,0)} = \{z = p+iy : (p \in (-\infty,0)) \wedge (y \in \mathbb{R})\}$ and for arbitrary $z \in G_{(-\infty,0)}$ and each $n \in \mathbb{N}$ the equality (52) holds.*

III_2. *The function $I(f,z)$ is continuous on the set*

$$G_{(-\infty,0]} = \{z = p+iy : (p \in (-\infty,0]) \wedge (y \in \mathbb{R})\}.$$

III_3. *The function $I(f,p)$ is strictly convex downward and logarithmic convex on $(-\infty,0]$.*

III_4. *If $\left\|\frac{1}{f}\right\|_{L_{\infty(A)}} > 1$, then $\lim\limits_{p\to-\infty} I(f,p) = +\infty$.*

III_5. *If $\left\|\frac{1}{f}\right\|_{L_{\infty(A)}} \le 1$, then $\lim\limits_{p\to-\infty} I(f,p) = \mu(B_1)$ and for each $p \in (-\infty,0)$ and any $n \in \mathbb{N}$ the inequality*

$$I^{(n)}(f,p) > 0, \tag{53}$$

holds, that is the function $I(f,p)$ is absolutely monotone on $(-\infty,0)$, (see, e.g. [3, p. 156]*).*

It remains to prove only inequality (53).

Since the function $\frac{1}{f(t)}$ satisfies the conditions of statement III_5 of Lemma 2, using this statement and statement III_1, we obtain for any $p \in (0,+\infty)$

$$\begin{aligned}(-1)^n I^{(n)}\left(\frac{1}{f},p\right) &= (-1)^n \int_A \left|\frac{1}{f(t)}\right|^p \ln^n \left|\frac{1}{f(t)}\right| dt \\ &= (-1)^{2n} \int_A |f(t)|^{-p} \ln^n |f(t)|\, dt = I^{(n)}(f,-p) > 0.\end{aligned} \tag{54}$$

From (54) the inequality (53) follows.

From the conditions of Lemma 2 and Corollary 2 it follows that the functions, which satisfy the conditions of Corollary 2, are the functions of a kind $\varphi_i(t) = \frac{1}{f_i(t)}$, where $f_i(t)$ are functions which satisfy the conditions of Lemma 2 and $i \in \{1,2,3,4,5\}$. In consequence of the equality $I(\varphi_i,-p) = I\left(\frac{1}{f_i},-p\right) = \int_A |f_i(t)|^p\, dt = I(f_i,p)$, the graphs of the functions $I(\varphi_i,-p)$ are symmetric about the y-axis to the graphs of the functions $I(f_i,p)$.

Let for each $p > 0$ the function $f(t) \notin L_{p(A)}$ and the function $\frac{1}{f(t)} \notin L_{p(A)}$, that is $I(f,p) = +\infty$ and $I\left(\frac{1}{f},p\right) = +\infty$. Then the function $I(f,p)$ is defined only at the point $p = 0$ and $I(f,0) = \mu(A)$. In what follows the example of such function is given.

Example 6. $f_6(t) = \begin{cases} f_3(t) = e^{-\frac{1}{t^\gamma}}, & \gamma > 0, t \in (0,1] \\ \frac{1}{f_3(t-1)}, & t \in (1,2] \end{cases}$.

We denote by $\langle -q_1, q\rangle$ an arbitrary intervals $[-q_1,q]$, $(-q_1,q)$, $[-q_1,q)$, $(-q_1,q]$, $(-\infty,q)$,$(-\infty,q]$, $(-q_1,+\infty)$, $[-q_1,+\infty)$, $(-\infty,+\infty)$, where $q \ge 0$ and $q_1 \ge 0$, $G_{\langle -q_1,q\rangle} = \{z = p+iy : (p \in \langle -q_1,q\rangle) \wedge (y \in \mathbb{R})\}$ are the sets of complex numbers and $G_{(-q_1,q)} = \{z = p+iy : (p \in (-q_1,q)) \wedge (y \in \mathbb{R})\}$ are the sets of the interior points of the sets $G_{\langle -q_1,q\rangle}$. If the conditions of Lemma 2 and Corollary 2 are fulfilled simultaneously, then combining these statements, we obtain.

Theorem 1(on properties of the functions $I\left(f,z\right)$ and $I\left(f,p\right)$). *The domain of the function $I\left(f,z\right) = I\left(f,p+iy\right)$ as a function of the complex variable z can only be the sets $G_{\langle -q_1,q\rangle}$. The function $I\left(f,z\right)$ is continuous on a domain of definition, analytic at each point of the domain $G_{(-q_1,q)}$ and for each $n\in\mathbb{N}$ and any $z\in G_{(-q_1,q)}$, we have the equality*

$$I^{(n)}\left(f,z\right) = \int_A |f(t)|^z \ln^n |f(t)|\, dt.$$

The function $I\left(f,p\right)$ as a function of the real variable p is strictly convex downwards and logarithmic convex on the domain $\langle -q_1,q\rangle$.

If the end of the interval q does not belong to the domain of definition of the function $I\left(f,p\right)$, that is, $f\notin L_{q(A)}$ and for each $p\in[0,q)$ the function $f\in L_{p(A)}$, then $\lim\limits_{p\to q-0} I\left(f,p\right) = +\infty$. If the end of the interval $-q_1$ does not belong to the domain of definition of the function $I\left(f,p\right)$, that is, $\frac{1}{f}\notin L_{q_1(A)}$ and for each $p\in[0,q_1)$ the function $\frac{1}{f}\in L_{p(A)}$, then $\lim\limits_{p\to -q_1+0} I\left(f,p\right) = +\infty$. If $\|f\|_{L_\infty(A)} > 1$, then $\lim\limits_{p\to+\infty} I\left(f,p\right) = +\infty$. If $\|f\|_{L_\infty(A)} \le 1$, then $\lim\limits_{p\to+\infty} I\left(f,p\right) = \mu(B_1)$, where $B_1 = \{t\in A : |f(t)| = 1\}$ and for each $p>0$ and any $n\in\mathbb{N}$ the following inequality holds:

$$(-1)^n I^{(n)}\left(f,p\right) = (-1)^n \int_A |f(t)|^p \ln^n |f(t)|\, dt > 0. \tag{55}$$

If $\left\|\frac{1}{f}\right\|_{L_\infty(A)} > 1$, then $\lim\limits_{p\to-\infty} I\left(f,p\right) = +\infty$. If $\left\|\frac{1}{f}\right\|_{L_\infty(A)} \le 1$, then $\lim\limits_{p\to-\infty} I\left(f,p\right) = \mu(B_1)$ and for each $n\in\mathbb{N}$ and any $p<0$ the following inequality holds:

$$I^{(n)}\left(f,p\right) = \int_A |f(t)|^p \ln^n |f(t)|\, dt > 0. \tag{56}$$

If the function $I\left(f,p\right)$ is defined on $(-\infty,+\infty)$, that is $f\in L_{\infty(A)}$ and $\frac{1}{f}\in L_{\infty(A)}$, then the statements I – III *are true:*

I. *If $\|f\|_{L_\infty(A)} > 1$ and $\left\|\frac{1}{f}\right\|_{L_\infty(A)} > 1$, then $\lim\limits_{p\to+\infty} I\left(f,p\right) = +\infty$ and $\lim\limits_{p\to-\infty} I\left(f,p\right) = +\infty$.*

II. *If $\|f\|_{L_\infty(A)} > 1$ and $\left\|\frac{1}{f}\right\|_{L_\infty(A)} \le 1$, then $\lim\limits_{p\to+\infty} I\left(f,p\right) = +\infty$, $\lim\limits_{p\to-\infty} I\left(f,p\right) = \mu(B_1)$ and for each $p\in(-\infty,+\infty)$ and any $n\in\mathbb{N}$ the inequality (56) holds.*

III. *If $\|f\|_{L_\infty(A)} \le 1$ and $\left\|\frac{1}{f}\right\|_{L_\infty(A)} > 1$, then $\lim\limits_{p\to+\infty} I\left(f,p\right) = \mu\left(B_1\right)$, $\lim\limits_{p\to-\infty} I\left(f,p\right) = +\infty$ and for each $p\in(-\infty,+\infty)$ and any $n\in\mathbb{N}$ the inequality (55) holds.*

Remark. Let us prove that the inequalities $\|f\|_{L_\infty(A)} \le 1$ and $\left\|\frac{1}{f}\right\|_{L_\infty(A)} \le 1$ can not be true simultaneously. If $\|f\|_{L_\infty(A)} \le 1$, that is, almost everywhere on A the inequality $|f(t)|\le 1$ holds, then almost everywhere on A we have

$$\left|\frac{1}{f(t)}\right| \ge 1. \tag{57}$$

Since $\mu(B_1) = \mu\left\{t\in A : |f(t)| = 1\right\} < \mu(A)$, then it follows from (57) that almost everywhere on $A\setminus B_1$ the inequalities $\left|\frac{1}{f(t)}\right| > 1$ are valid and $\left\|\frac{1}{f}\right\|_{L_\infty(A)} > 1$. If $\left\|\frac{1}{f}\right\|_{L_\infty(A)} \le 1$, then we similarly prove that $\|f\|_{L_\infty(A)} > 1$.

Example 7. Let us prove that the function $h_1(t) = t+\frac{1}{2}$ satisfies conditions of statement I on the set $A=[0,1]$.

Since $\|h_1\|_{L_\infty(A)} = \frac{3}{2} > 1$ and $\left\|\frac{1}{h_1}\right\|_{L_\infty(A)} = 2 > 1$, then the function $h_1(t)$ satisfies conditions of statement I of Theorem 1: $I(h_1, p) = \int_0^1 \left|t + \frac{1}{2}\right|^p dt = \frac{1}{p+1}\left(\left(\frac{3}{2}\right)^{p+1} - \left(\frac{1}{2}\right)^{p+1}\right)$, $p \neq -1$ and $\mu(A) = 1, \mu(B_1) = 0$,

$$I(h_1, p) = \int_0^1 \left|t + \frac{1}{2}\right|^p dt = \ln 3, p = -1.$$

Similarly, we can show the validity of the following examples.

Example 8. The function $h_2(t) = \frac{1}{2}(t+1)$ satisfies conditions of statement II on the set $A = [0,1]$.

Example 9. The function $h_3(t) = \frac{1}{h_2(t)}$ satisfies conditions of statement III of Theorem 1 and the graph of the function $y = I(h_3, p)$ is symmetric to the graph of the function $y = I(h_2, p)$ with respect to the y-axis.

Remark. Since all the functions $I(f,p)$ and $\ln I(f,p)$ are strictly convex downward on the domains of definition $D(I(f))$, which can be arbitrary intervals, then using Jensen's inequality for strictly convex downward functions, we obtain the inequalities:

$$(\forall\{x_1, ..., x_n\} \subset D(I(f)) \wedge \Big(\forall \alpha_k \in (0,1) : \sum_{k=1}^n \alpha_k = 1\Big) \Rightarrow \Bigg(\Bigg(I\Big(f, \sum_{k=1}^n \alpha_k x_k\Big)$$

$$< \sum_{k=1}^n \alpha_k I(f, x_k)\Bigg) \Leftrightarrow \Big(\int_A |f(t)|^{\sum\limits_{k=1}^n \alpha_k x_k} dt < \sum_{k=1}^n \alpha_k \int_A |f(t)|^{x_k} dt\Big)\Bigg)$$

$$\wedge \Bigg(\Bigg(\ln I\Big(f, \sum_{k=1}^n \alpha_k x_k\Big) < \sum_{k=1}^n \alpha_k \ln I\Big(f, x_k\Big)\Bigg)$$

$$\Leftrightarrow \Big(\ln \int_A |f(t)|^{\sum\limits_{k=1}^n \alpha_k x_k} dt < \sum_{k=1}^n \alpha_k \ln \int_A |f(t)|^{x_k} dt\Big)\Bigg).$$

Remark. In connection with the analyticity of the functions $I(f,z)$, interesting questions arise in the constructive theory of functions of a complex variable. Let's formulate one of them.

Let the function $\varphi(z)$ be analytic in the domain $(0,q) \times R$, continuous on the set $[0,q] \times R$, $\varphi(p)$ be its restriction to the set of real numbers $p \in [0,q],$, where $0 < q \leq \infty$. The function $\varphi(p)$ is strictly convex downward, logarithmic convex on the segment $[0,q]$ and all derivatives of even order of the function $\varphi(p)$ are positive on the interval $(0,q)$.

From statement I of Lemma 2 it follows that for each function $f \in L_{q(A)}$, the function $I(f,z) = \int_A |f(t)|^z dt$ has the properties of the function $\varphi(z)$. Is there a function $\varphi(z)$ that is not represented by the integral $I(f,z)$, that is, $\varphi(z) \neq I(f,z)$?

3. Properties of the function $n(f,p)$

Recall that if $f \in L_{q(A)}$ and $q > 0$, then $\lim\limits_{p \to 0+0} n(f,p) = e^{\frac{1}{\mu(A)} \int_A \ln|f(t)| dt}$.

Lemma 3 (on the properties of the function $n(f,p)$ in the case $\frac{1}{f} \notin L_{p(A)}$ for an arbitrary $p > 0$). *If for an arbitrary $p > 0$ the function $\frac{1}{f} \notin L_{p(A)}$, then for all $p < 0$ the function $n(f,p) = 0$ and the statements* I-IV *are true.*

I. *Let the function $f \in L_{q(A)}$ where $q > 0$ and for arbitrary $p > q$ the function $f \notin L_{p(A)}$. Then the statements* I_1-I_3 *hold.*

I_1. *The function $n(f,p)$ is analytic on $(0,q)$.*

I_2. *The function $n(f,p)$ is strictly increasing and continuous on $[0,q]$.*

I_3. *If* $n(f,0+0)=0$, *then the function* $n(f,p)$ *is continuous on* $(-\infty,q]$.

II. *Let the function* $f\notin L_{q(A)}$ *and for any* $p\in(0,q)$ *the function* $f\in L_{p(A)}$. *Then* $\lim\limits_{p\to q-0} n(f,p)=+\infty$ *and statements* $\mathrm{I}_1,\mathrm{II}_1,\mathrm{II}_2$ *hold.*

II_1. *The function* $n(f,p)$ *is strictly increasing and continuous on* $[0,q)$.

II_2. *If* $n(f,0+0)=0$, *then the function* $n(f,p)$ *is continuous on* $(-\infty,q)$.

III. *Let* $f\in L_{\infty(A)}$. *Then* $\lim\limits_{p\to+\infty} n(f,p)=\|f\|_{L_\infty(A)}$ *and statements* III_1-III_3 *are true.*

III_1. *The function* $n(f,p)$ *is analytic on* $(0,+\infty)$.

III_2. *The function* $n(f,p)$ *is strictly increasing and continuous on* $[0,+\infty)$.

III_3. *If* $n(f,0+0)=0$, *then the function* $n(f,p)$ *is continuous on* $(-\infty,+\infty)$.

IV. *If for each* $p>0$ *the function* $f\notin L_{p(A)}$, *then for* $p>0$ *the function* $n(f,p)=+\infty$.

Proof. If for each $-p>0$ the function $\frac{1}{f}\notin L_{-p(A)}$, that is $n\left(\frac{1}{f},-p\right)=+\infty$, then for all $p<0$ we have $n\,(f,p)=\frac{1}{n\left(\frac{1}{f},-p\right)}=\left(\frac{1}{\infty}\right)=0$.

Let us prove the statement I_1. If $f\in L_{q(A)}$ and $q>0$, then from statement I of Lemma 2 follows, that the function $y=I(f,p)$ is analytic on $(0,q)$. Since the function e^t is analytic on $(-\infty,+\infty)$, the function $\frac{1}{p}$ is analytic on $(-\infty,0)$ and $(0,+\infty)$, the function $\ln y$ is analytic on $(0,+\infty)$ and $y=I(f,p)>0$, then from the theorem on the analyticity of a composite function and the analyticity of the product of analytic functions it follows that the function $n\,(f,p)=e^{\frac{1}{p}\ln\frac{1}{\mu(A)}\int_A|f(t)|^p dt}$ is analytic on $(0,q)$.

Let us prove the statement I_2. If $f\in L_{q(A)}$, then (see, e. g. [1, p. 143–145]), the function $n\,(f,p)$ is continuous and strictly increasing on $[0,q]$.

Let us prove the statement I_3. Since for each $p<0$ the function $n\,(f,p)=0$, $n\,(f,0+0)=0$ and $n\,(f,p)$ is the continuous function on $[0,q]$, then $n\,(f,p)$ is continuous function on $(-\infty,q]$.

Let us prove the statement II. Since for each segment $[0,r]\subset[0,q)$ the function $f\in L_{r(A)}$, that is, the conditions of statement I hold, then according to I, the function $n\,(f,p)$ is analytic on $(0,r)$, continuous and strictly increasing on $[0,r]$. As a consequence of arbitrariness of r the statements $\mathrm{I}_1,\mathrm{II}_1,\mathrm{II}_2$ hold.

It remains to prove that $\lim\limits_{p\to q-0} n(f,p)=+\infty$.

Since the function $n\,(f,p)$ is strictly increasing on $[0,q)$, then, by the theorem on the boundary of a monotonic function, we have

$$\lim_{p\to q-0} n\,(f,p)=\sup_{p\in[0,\ q)}\{n\,(f,p)\} \tag{58}$$

Since the function $n\,(f,p)$ is continuous on $[0,q)$ and $f\notin L_{q(A)}$, then from (58) it follows, that $\lim\limits_{p\to q-0} n\,(f,p)=+\infty$.

Let us prove the statement III. The equality $\lim\limits_{p\to+\infty} n\,(f,p)=\|f\|_{L_\infty(A)}$ was proved in work [1, p. 173]. If $f\in L_{\infty(A)}$, then for each segment $[0,r]\subset[0,+\infty)$ the function $f\in L_{r(A)}$, that is the conditions of statement I are fulfilled. The statement III we prove similarly to the statement II. Lemma 3 is proved. □

Since the conditions of Lemma 3 coincide with the conditions of Lemma 2, examples that satisfy the conditions of Lemma 3 can be examples given for Lemma 2.

Example 1. The function $f_1\,(t)=\begin{cases} g_3\,(t)\,, & t\in(0,1]\,,\\ g_1\,(t-1)\,, & t\in(1,2]\end{cases}$ satisfies the conditions I.

Changing the variables and using the additive property of the integral for $p\in(0,q]$ we obtain

$$n\,(f_1,p)=\left(\frac{1}{2}\int_0^2|f_1\,(t)|^p dt\right)^{\frac{1}{p}}=\left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\int_0^1 e^{-\frac{p}{t^\gamma}}dt+\int_0^1\left(t^{-\frac{1}{q}}\Big(-\ln\frac{t}{e}\Big)^{-\frac{2}{q}}\right)^p dt\right)^{\frac{1}{p}},$$

$$n\left(f_1,0+0\right)=e^{\frac{1}{2}\int_0^2 \ln|f_1(t)|dt}=e^{\frac{1}{2}\left(\int_0^1 \ln e^{-\frac{1}{t^\gamma}}\,dt+\int_0^1 \ln t^{-\frac{1}{q}}\left(-\ln\frac{t}{e}\right)^{-\frac{2}{q}}dt\right)}, \tag{59}$$

$$\int_0^1 \ln e^{-\frac{1}{t^\gamma}}dt=-\int_0^1\frac{dt}{t^\gamma}=\begin{cases}-\infty, & \gamma\geq 1\\ -\frac{1}{1-\gamma}, & 0<\gamma<1\end{cases}. \tag{60}$$

Integrating by parts, we get

$$\begin{aligned}\int_0^1 \ln\left(t^{-\frac{1}{q}}\left(-\ln\frac{t}{e}\right)^{-\frac{2}{q}}\right)dt&=-\frac{1}{q}\int_0^1\ln t\,dt-\frac{2}{q}\int_0^1\ln\left(-\ln\frac{t}{e}\right)dt\\&=-\frac{1}{q}\left(t\ln t\Big|_0^1-\int_0^1 dt+2\left(t\ln\left(-\ln\frac{t}{e}\right)\Big|_0^1+\int_0^1\frac{dt}{-\ln\frac{t}{e}}\right)\right)\\&=-\frac{1}{q}\left(-1+2\int_0^1\frac{dt}{-\ln\frac{t}{e}}\right).\end{aligned} \tag{61}$$

Let us prove the inequality

$$\frac{1}{2}<\int_0^1\frac{dt}{-\ln\frac{t}{e}}=\int_0^1 f(t)dt<1, \tag{62}$$

where $f(t)=\left(-\ln\frac{t}{e}\right)^{-1}, f(1)=1, f(0)=\lim_{t\to 0}f(t)=0.$

Since for $t\in(0,1]$ we have the relations

$$f'(t)=\frac{1}{t}\ln^{-2}\frac{t}{e}>0,\quad f'(1)=1,$$

$$\left(f''(t)=-\frac{1}{t^2}\ln^{-2}\frac{t}{e}\left(1+2\ln^{-1}\frac{t}{e}\right)<0\right)\Leftrightarrow\left(0<t<\frac{1}{e}\right),$$

$$\left(f''(t)>0\right)\Leftrightarrow\left(\frac{1}{e}<t\leq 1\right),$$

then the function $f(t)$ is strictly increasing on the segment $[0,1]$, strictly convex upwards on the interval $(0,\frac{1}{e})$, strictly convex downward on the interval $(\frac{1}{e},1]$ and the tangent to the graph of the function at the point $(1,1)$ coincides with the line $y=t$. Therefore, for any $t\in(0,1)$ the following inequality holds

$$t<f(t)<1. \tag{63}$$

From inequality (63), we obtain $\frac{1}{2}=\int_0^1 t\,dt<\int_0^1 f(t)dt<\int_0^1 dt=1$. Thus, the inequality (62) is true.

From inequalities (61) and (62) follows the inequality

$$-\frac{1}{q}<\int_0^1\ln\left(t^{-\frac{1}{q}}\left(-\ln\frac{t}{e}\right)^{-\frac{2}{q}}\right)dt<0. \tag{64}$$

From equalities (59)–(61) and inequality (64) we obtain: if $\gamma\geq 1$, then $n\left(f,0+0\right)=0$ and if $0<\gamma<1$, then $e^{-\frac{1}{2}\left(\frac{1}{1-\gamma}+\frac{1}{q}\right)}<n\left(f,0+0\right)<e^{-\frac{1}{2(1-\gamma)}}$.

Similarly, we can show the validity of the following examples.

Example 2. The function $f_2\left(t\right)=\begin{cases}g_3\left(t\right), & t\in(0,1],\\ g_2\left(t-1\right), & t\in(1,2]\end{cases}$ satisfies the conditions of statement II.

Example 3. The function $f_3\left(t\right)=3g_3\left(t\right)=3e^{-1/t^\gamma}, t\in(0,1], \gamma>0$ satisfies the conditions of statement III.

Example 4. The function $f_4\left(t\right)=\begin{cases}g_3\left(t\right)=e^{-1/t^\gamma}, & t\in(0,1]\\ g_{\gamma_1}\left(t-1\right), & t\in(1,2]\end{cases}$, where $\gamma>0$, $g_{\gamma_1}\left(t\right)=e^{1/t^{\gamma_1}}$, $\gamma_1>0$, satisfies the conditions of statement IV.

If under the conditions of Lemma 3 the function $f(t)$ is replaced by the function $\frac{1}{f(t)}$ and the interval $[0,q\rangle$ by the interval $\langle -q,0]$, then we obtain the following statement.

Corollary 3 (on property of function $n\,(f,p)$ in the case $f \notin L_{p(A)}$ for an arbitrary $p>0$). If the function $f \notin L_{p(A)}$ for an arbitrary $p>0$ and $n\,(f,0) = e^{\frac{1}{\mu(A)}\int_A \ln|f(t)|\,dt} = k < \infty$, $k>0$, then the statements I-III are valid.

I. Let the function $\frac{1}{f} \in L_{q(A)}$, where $q>0$ and for each $p>q$ the function $\frac{1}{f} \notin L_{p(A)}$. Then the statements I_1-I_3 hold.

I_1. The function $n\,(f,p)$ is analytic on $(-q,0)$.

I_2. The function $n\,(f,p)=0$ for each $p<q$.

I_3. The function $n\,(f,p)$ is strictly increasing and continuous on $[-q,0]$.

II. Let the function $\frac{1}{f} \notin L_{q(A)}$ and for each $p \in (0,q)$ the function $\frac{1}{f} \in L_{p(A)}$. Then the statements I_1, II_1-II_3 are true.

II_1. The function $n\,(f,p)=0$ for each $p<q$.

II_2. The function $n\,(f,p)$ is strictly increasing on $[-q,0]$.

II_3. The function $n\,(f,p)$ is continuous on $(-\infty,0]$.

III. Let the function $\frac{1}{f} \in L_{\infty(A)}$. Then $\lim\limits_{p\to-\infty} n\,(f,p) = \frac{1}{\left\|\frac{1}{f}\right\|_{L_\infty(A)}}$ and the statements III_1, III_2 are valid.

III_1. The function $n\,(f,p)$ is analytic on $(-\infty,0)$.

III_2. The function $n\,(f,p)$ is strictly increasing and continuous on $(-\infty,0]$.

Proof. For the proof we use the relation $(p \in [0,q\rangle) \Leftrightarrow (-p \in \langle -q,0])$ and equality

$$\left(n(f,p) = \frac{1}{n\left(\frac{1}{f},-p\right)}\right) \Leftrightarrow \left(n\,(f,-p) = \frac{1}{n\left(\frac{1}{f},p\right)}\right). \tag{65}$$

From (65) it follows that

$$\begin{gathered}\left(\lim_{p\to-\infty} n\,(f,p) = \frac{1}{\lim\limits_{-p\to+\infty} n\left(\frac{1}{f},-p\right)} = \frac{1}{\left\|\frac{1}{f}\right\|_{L_\infty(A)}}\right)\\ \wedge\left(\lim_{p\to-q+0} n\,(f,p) = \frac{1}{\lim\limits_{-p\to q-0} n\left(\frac{1}{f},-p\right)}\right).\end{gathered} \tag{66}$$

If for each $-p>-q>0$ the function $\frac{1}{f} \notin L_{-p(A)}$, that is $n\left(\frac{1}{f},-p\right)=+\infty$, then from (65) we have: if $p<q$, then

$$n\,(f,p)=0. \tag{67}$$

If the function $\frac{1}{f} \notin L_{q(A)}$, then from (66) it follows that

$$n\,(f,-q)=0. \tag{68}$$

If the function $n\left(\frac{1}{f},-p\right)$ is analytic on the intervals $(0,q)$ or $(0,+\infty)$, then from (66) it follows that $n\,(f,p)$ is analytic function on the intervals $(-q,0)$ or $(-\infty,0)$, respectively.

If the function $n\left(\frac{1}{f},-p\right)$ is strictly increasing and continuous on the intervals $[0,q]$ or $[0,+\infty)$, then from (66) it follows that the function $n\,(f,p)$ is strictly increasing and continuous on the intervals $[-q,0]$ or $(-\infty,0]$, respectively.

Since $\frac{1}{f}$ satisfies the conditions of Lemma 3, than from the relations (57)–(68) and the above considerations, according to Lemma 3, we have the validity of Corollary 3. □

Examples of functions satisfying the conditions of Corollary 3 are functions of the kind $\varphi_i(t) = \frac{1}{f_i(t)}$, where $f_i(t)$ are functions satisfying the conditions of Lemma 3, $i \in \{1,2,3\}$ and $0 < \gamma < 1$.

If $i = 1$, then $\varphi_1(t) = \frac{1}{f_1(t)} = \begin{cases} e^{\frac{1}{t^\gamma}}, & t \in (0,1] \\ \frac{1}{g_1(t-1)}, & t \in (1,2] \end{cases}$, where $\gamma > 0$, $\frac{1}{g_1(t)} = t^{\frac{1}{q}}\left(-\ln\frac{t}{e}\right)^{\frac{2}{q}}$. Using the additive property of the integral and changing the variables, we obtain

$$n(\varphi_1, 0) = e^{\frac{1}{2}\left(\int_0^1 \frac{dt}{t^\gamma} + \int_0^1 \ln t^{\frac{1}{q}}\left(-\ln\frac{t}{e}\right)^{\frac{2}{q}} dt\right)}. \tag{69}$$

Using the equality (61), we have

$$\int_0^1 \ln t^{\frac{1}{q}}\left(-\ln\frac{t}{e}\right)^{\frac{2}{q}} dt = \frac{1}{q}\left(-1 + 2\int_0^1 \frac{dt}{-\ln\frac{t}{e}}\right). \tag{70}$$

Using the equalities (60), (61), (69), (70) and inequality (62), we obtain: if $0 < \gamma < 1$, then $e^{\frac{1}{2}\cdot\frac{1}{1-\gamma}} < n(\varphi_1, 0) < e^{\frac{1}{2}\left(\frac{1}{1-\gamma}+\frac{1}{q}\right)}$; if $\gamma \geq 1$, then $n(\varphi_1, 0) = +\infty$. Using examples 2 and 3, we similarly prove that if $i \in \{2,3\}$ and $0 < \gamma < 1$, then $n(\varphi_i, 0) < +\infty$, and if $\gamma \geq 1$, then $n(\varphi_i, 0) = +\infty$.

If $n(\varphi_i, 0) = +\infty$, then, similarly as in the proof of Corollary 3, we prove the validity of the following statement.

Corollary 4. (on property of function $n(f,p)$ in the case $f \notin L_{p(A)}$). *If the function $f \notin L_{p(A)}$ for each $p > 0$ and $n(f,0) = +\infty$, then $\lim_{p\to 0-0} n(f,p) = +\infty$ and the statements* I-III *of Corollary 3, where the intervals $(-q, 0]$ and $(-\infty, 0]$ are replaced by intervals $(-q, 0)$ and $(-\infty, 0)$.*

Examples of functions satisfying the conditions of Corollary 4 are the functions $\varphi_i(t)$, where $i \in \{1,2,3\}$ and $\gamma \geq 1$.

If the conditions of Lemma 3 and Corollary 3 are satisfied simultaneously for the function $f(t)$, then combining these statements we obtain.

Theorem 2. (on the property of function $n(f,p)$). *The statements* I-III *hold.*

I. *If the function $f \in L_{q(A)}$, where $q > 0$, then the statements* I_1-I_3 *are valid.*

I_1. *Let the function $\frac{1}{f} \in L_{q_1(A)}$ and $q_1 > 0$ and for each $p > q_1$ the function $\frac{1}{f} \notin L_{p(A)}$. Then the statements* $\mathrm{I}_1(1)$-$\mathrm{I}_1(3)$ *hold.*

$\mathrm{I}_1(1)$. *The function $n(f,p)$ is analytic on $(-q_1, 0)$ and $(0, q)$.*

$\mathrm{I}_1(2)$. *The function $n(f,p) = 0$ for each $p < -q_1$.*

$\mathrm{I}_1(3)$. *The function $n(f,p)$ is strictly increasing and continuous on $[-q_1, q]$.*

I_2. *Let the function $\frac{1}{f} \notin L_{q_1(A)}$ and for each $p \in (0, q_1)$ the function $\frac{1}{f} \in L_{p(A)}$. Then the statements* $\mathrm{I}_1(1), \mathrm{I}_2(1)$-$\mathrm{I}_2(3)$ *are valid.*

$\mathrm{I}_2(1)$. *The function $n(f,p) = 0$ for each $p \leq -q_1$.*

$\mathrm{I}_2(2)$. *The function $n(f,p)$ is strictly increasing on $[-q_1, q]$.*

$\mathrm{I}_2(3)$. *The function $n(f,p)$ is continuous on $(-\infty, q]$.*

I_3. *Let the function $\frac{1}{f} \in L_{\infty(A)}$. Then $\lim_{p\to-\infty} n(f,p) = \frac{1}{\left\|\frac{1}{f}\right\|_{L_\infty(A)}}$ and the statements* $\mathrm{I}_3(1), \mathrm{I}_3(2)$ *are true.*

$\mathrm{I}_3(1)$. *The function $n(f,p)$ is analytic on the intervals $(-\infty, 0)$ and $(0, q)$.*

$\mathrm{I}_3(2)$. *The function $n(f,p)$ is strictly increasing and continuous on $(-\infty, q)$.*

II. *Let the function $f \notin L_{q(A)}$ and for each $p \in (0, q)$ the function $f \in L_{p(A)}$. Then $\lim_{p\to q-0} n(f,p) = +\infty$ and the statements* II_1-II_3 *hold. The statements* II_1-II_3 *are similar to the statements* I_1-I_3*, only the intervals $(-q_1, q]$ and $(-\infty, q]$ are replaced respectively by intervals $(-q_1, q)$ and $(-\infty, q)$.*

III. *If the function* $f \in L_{\infty(A)}$*, then* $\lim\limits_{p\to+\infty} n(f,p) = \|f\|_{L_\infty(A)}$ *and statements* III_1-III_3 *hold. The statements* III_1-III_3 *are similar to the statements* I_1-I_3*, only the intervals* $[-q_1, q]$ *and* $(-\infty, q]$ *replaced respectively by intervals* $(-q_1, +\infty)$ *and* $(-\infty, +\infty)$*.*

Here are some examples of functions that satisfy the conditions of statement II of Theorem 2.

Example 1. Let

$$\psi_1(t) = \begin{cases} g_2(t) = \frac{1}{t^q}, & t \in (0,1],\ q > 0 \\ g_{q_1}^e(t-1), & t \in (1,2],\ q_1 > 0 \end{cases}, \tag{71}$$

where $g_{q_1}^e(t) = t^{\frac{1}{q_1}}\left(-\ln\frac{t}{e}\right)^{\frac{2}{q_1}}$.

It follows from (71) that the conditions II of Lemma 3 are fulfilled for the function $g_2(t) = \frac{1}{t^q}$, and for the function $g_{q_1}^e(t)$ the conditions II of Corollary 3 are satisfied, that is, the function $\psi_1(t)$ satisfies the conditions II_1 of Theorem 2.

Similarly, we can show the validity of the following examples.

Example 2. The function $\psi_2(t) = \begin{cases} g_2(t) = \frac{1}{t^q}, & t \in (0,1],\ q > 0 \\ g_{q_1}(t-1), & t \in (1,2],\ q_1 > 0 \end{cases}$, where $g_{q_1}(t) = t^{\frac{1}{q_1}}$, satisfies the condition of statement II_2 of Theorem 2.

Example 3. For the function $g_2(t)$ the conditions of statement II of Lemma 3 and of statement III of Corollary 3 are satisfied, that is, the function satisfies the conditions II_3 of Theorem 2.

Here are some examples of functions that satisfy the conditions of statement I of Theorem 2.

Example 4. The function $S_1(t) = \begin{cases} g_1(t) = t^{-\frac{1}{q}}\left(-\ln\frac{t}{e}\right)^{-\frac{2}{q}}, & t \in (0,1],\ q > 0, \\ g_{q_1}^e(t-1), & t \in (1,2],\ q_1 > 0 \end{cases}$ satisfies the condition of statement I_1 of Theorem 2.

Example 5. The function $S_2(t) = \begin{cases} g_1(t), & t \in (0,1], \\ g_{q_1}(t-1), & t \in (1,2] \end{cases}$ satisfies the condition of statement I_2 of Theorem 2.

Remark. All the formulated and proved above statements are also valid for functions

$$I(f,p) = \underbrace{\int\cdots\int}_{A} |f(t_1,\ldots,t_n)|^p\, dt_1\ldots dt_n,$$

$$I(f,z) = \underbrace{\int\cdots\int}_{A} |f(t_1,\ldots,t_n)|^z\, d_1\ldots dt_n,$$

$$n(f,p) = \left(\frac{1}{\mu(A)} \underbrace{\int\cdots\int}_{A} |f(t_1,\ldots,t_n)|^p\, dt_1\ldots dt_n\right)^{\frac{1}{p}},$$

that is, for functions $f(t_1,\ldots,t_n)$ of n variables measurable on the set A of the space R^n.

We denote by $\tilde{L}_\infty^n, \tilde{L}_{\bar{p}}^n$ the spaces of functions defined on the n-dimensional cube $A = \pi^n = [-\pi,\pi)^n$, which are 2π-periodic in each variable, not zero almost everywhere on the set A, essentially bounded and measurable respectively with norms:

$$\|\tilde{f}\|_{L_\infty(A)} = \sup_{\bar{x}\in A} \mathrm{vrai}|\tilde{f}(\bar{x})|, \quad \bar{x} = (x_1, x_2, \ldots, x_n),$$

$$n(\tilde{f}, p) = \left(\frac{1}{\mu(A)} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} |\tilde{f}(x_1, x_2, \ldots, x_n)|^p dx_1 \ldots dx_n\right)^{\frac{1}{p}}$$

$$= \left(\frac{1}{\mu(A)} \int_A |\tilde{f}(\bar{x})|^p d\bar{x}\right)^{\frac{1}{p}} < \infty,$$

where p is an arbitrary real number that is not equal to zero and $\mu(A) = (2\pi)^n$. Then for the functions

$$I(\tilde{f}, z) = \frac{1}{\mu(A)} \int_A |\tilde{f}(\bar{x})|^z d\bar{x}, I(\tilde{f}, p) = \frac{1}{\mu(A)} \int_A |\tilde{f}(\bar{x})|^p d\bar{x},$$

$$n(\tilde{f}, p) = \left(\frac{1}{\mu(A)} \int_A |\tilde{f}(\bar{x})|^p d\bar{x}\right)^{\frac{1}{p}}$$

all the previously formulated statements are valid.

It follows from Theorem 1 and the corollary of the paper [5] that if $1 \le p \le \infty$, then the spaces $\tilde{L}_p^n$ are isometric to the spaces of convolution of these spaces with non-negative delta-like kernels, that is, to the spaces

$$\overline{\tilde{L}_p^n * \tilde{K}_{\bar{y}^k}^n} = \left\{ \tilde{v}\,(\bar{x}, \bar{y}) = \tilde{v}\,(x, \bar{y}) = \left\{ \begin{array}{l} \left(\tilde{f} * \tilde{K}_{\bar{y}^k}\right)(x), \quad \bar{y} > \bar{0}, \\ \tilde{f}(x), \bar{y} = \bar{0} \end{array} \right. : (\tilde{f} \in \tilde{L}_p^n) \right\}$$

and

$$n(\tilde{v}, p) = \sup_{\bar{y} \ge \bar{0}} \left(\frac{1}{\mu(A)} \int_A |\tilde{v}(\bar{x}, \bar{y})|^p \, d\bar{x}\right)^{\frac{1}{p}} = n(\tilde{f}, p) = \left(\frac{1}{\mu(A)} \int_A \left|\tilde{f}(\bar{x})\right|^p d\bar{x}\right)^{\frac{1}{p}}. \tag{72}$$

Then, on the basis of the equality (72), the statement of Theorem 2 holds for the function $n(\tilde{v}, p)$ in the case $\tilde{f} \in \tilde{L}_q^n$, $1 \le q \le \infty$, and in the case $\tilde{f} \notin \tilde{L}_q^n$, where $1 < q \le \infty$ and for each $p \in (0, q)$, $\tilde{f} \in \tilde{L}_p^n$.

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