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NON-AUTONOMOUS SYSTEMS ON LIE GROUPS AND THEIR TOPOLOGICAL ENTROPY

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Abstract. In the present paper we introduce and study the topological entropy of non-autonomous dynamical systems and define the non-autonomous dynamical system on Lie groups and manifolds. Our main purpose is to estimate the topological entropy of the non-autonomous dynamical system on Lie groups. We show that the topological entropy of the non-autonomous dynamical system on Lie groups and induced Lie algebra are equal under topological conjugacy, and a method to estimate the topological entropy of non-autonomous systems on Lie groups is given. To illustrate our results, some examples are presented. Finally some discussions and comments about positive entropy on nil-manifold Lie groups for non-autonomous systems are presented.

1. Introduction

In mathematics, the topological entropy of a topological dynamical system is a non-negative extended real number that is a measure of the system. Topological entropy was first introduced in 1965 by Adler, Konheim and Mc. Andrew [1]. For continuous maps from compact topological space to itself, it is very closely analogous to measure theoretic entropy. On metric spaces, an alternate definition was given by Bowen [3], for uniformly continuous maps and it was proved that, on compact metric spaces, these two definitions are equivalent [4, 6]. The second definition clarified the meaning of the topological entropy. For a system given by an iterated function, the topological entropy represents the exponential growth rate of the number of distinguishable orbit of the iterates.

Choang Peng, discussed the topological entropy of continuous function on Lie groups and proved their shift properties [8]. He proves that, entropy is invariant with isometric endomorphisms of Lie groups. Also he consider algebraic entropy of elementary Abelian groups and Lie groups, and proved that the topological entropy is preserved when projected from Lie group $\mathbb{R}$ to its quotient space compact Lie group S^1 from continuous function lifted from the quotient space [8].

While a discrete autonomous or classical dynamical system is given by the iterations of single map $f : X \to X$ the dynamics of a non-autonomous system is generated by Kolyade and Snoha, in 1996, who extended the concept of topological entropy by composition of different maps [13], they gave two definitions also based on open cover, and separated and spanning sets, separately and proved that these two definitions are equivalent for non-autonomous systems, too. They studied some relations of non-autonomous system and proved that topological entropy is an invariant value of topological equiconjugacy. Although calculating the topological entropy for non-autonomous dynamical systems is not a easy task, one can give the estimation of the topological entropy for some special non-autonomous systems. For example in [14] Kolyada, et al showed that if $f_{1,\infty}$ is a

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finite piecewise monotone, or a bounded totally long-lapped, or a Markov interval non-autonomous dynamical system, then

$$h(f_{1,\infty}) = \lim_{n\longrightarrow\infty} \sup \frac{1}{n}\log(c_{1,n}).$$

Where $h(f_{1,\infty})$ denote entropy and $c_{1,n}$ is the number of the laps of f_1^n. The authors [20] proved that for a sequence of equicontinuous monotone maps on circles,

$$h(f_{1,\infty}) = \lim_{n\longrightarrow\infty} \sup \frac{1}{n}\log\prod_{i=1}^{n}|\deg f_i|),$$

where $\deg f_i$ is the degree of f_i. They also showed that if $f_{1,\infty}$ is a family of homeomorphisms on a finite graph X, then $h(f_{1,\infty}) = 0$ [11].

In this article, our main purpose is to describe the topological entropy of non-autonomous dynamical systems on Lie groups. At first we remind the definitions and main properties of topological entropy and dynamical non-autonomous systems. Then we introduce the entropy of non-autonomous dynamical systems and some of it's properties. Section 3 is devoted to introduce the topological entropy of non-autonomous dynamical system on non-compact spaces. In Section 4, after reminding the Lie groups, we define the non-autonomous dynamical systems on a Lie group and by the exponential map of a Lie group and of the topological conjugacy we introduce induced non-autonomous system on a Lie algebra. Also, topological entropy of non-autonomous systems on simply connected nilpotent Lie groups is considered. In Section 5 we mention linear maps and prove our main theorem which is the relation between the topological entropy of manifold and it's tangent space according to the explained topological conjugate then we calculate it according to the eigenvalues in Theorems 5.2 and 5.1. To illustrate our results some examples, especially on Heisenberg and $SO(2)$ Lie groups, are presented. Finally we end this paper with conclusions and recommendations about positive topological entropy of a flow on a nil-manifold, as a future research work.

2. Preliminaries

In this section we give some basic definitions and notations which are known and we need in the following.

Given a compact metric space (X, d) and a map $f : X \longrightarrow X$, we define the function $d_n : X \times X \longrightarrow \mathbb{R}$ by $d_n(x, y) = \max(f^k(x), (f^k(y))$. For each n, d_n is a metric on X, since d to be a metric on X.

Fix $\epsilon > 0$. Let $n \in \mathbb{N}$. A set $A \subseteq X$ is a $(n, \epsilon)-$spanning set if for every $x \in X$ there exists $y \in A$ such that $d_n(x, y) < \epsilon$. A set $A \subseteq X$ is an $(n, \epsilon)-$separated set if for all distinct points $x, y \in A$ we have $d_n(x, y) \geq \epsilon$.

Let $\mathrm{span}(n, \epsilon, f)$ be the minimum cardinality of a $(n, \epsilon)-$ spanning set, and $\mathrm{sep}(n, \epsilon, f)$ be the maximum cardinality of a $(n, \epsilon) -$ *separated set* [3]. So we have the following notation of metric entropy for f.

Lemma 2.1. [7] *The metric entropy of a map $f : X \to X$ is:*

$$h(f) = \lim_{\epsilon\longrightarrow0^+}\lim_{n\longrightarrow\infty}\frac{1}{n}\log(\mathrm{sep}(n, \epsilon, f)) = \lim_{\epsilon\longrightarrow0^+}\lim_{n\longrightarrow\infty}\frac{1}{n}\log(\mathrm{span}(n, \epsilon, f)).$$

Let X and Y be metric spaces. If the continuous maps $f : X \longrightarrow X$ and $g : Y \longrightarrow Y$ satisfy the relation $foh = hog$, for some homeomorphism $h : Y \longrightarrow X$, then we say that f is topologically conjugated to g. When the relation $foh = hog$ holds for some continuous surjection $h : Y \longrightarrow X$, we say that f is topologically semi-conjugate to g.

We can use of some properties of topological entropy to more easily determine when two systems are not the same.

Some basic properties of topological entropy are [16]:

1− $h(f^n) = n.h(f)$ for any $n \geq 0$.

2− If $Y \subseteq X$ is an invariant closed set then $h(f|_Y) \leq h(f)$.
3− If g is conjugate to f, then $h(g) = h(f)$.
4− $h(f_1 \times f_2) = h(f_1) + h(f_2)$.

After the classical entropy definition, we consider definition and characteristics of topological entropy of non-autonomous dynamical systems which is an extension of this notion.

Let X be a compact topological space and $f_{1,\infty} = \{f_i\}_{i=1}^{\infty}$ a sequence of continuous maps from X to X. The identity map on X will be denoted by id_X. Let $f_i^0 = f_i^{-0} = id_X$, and set $f_i^n = f_{i+(n-1)} o \ldots o f_{i+n} o f_i$ and $f_i^{-n} = (f_i^n)^{-1} = f_i^{-1} o f_{i+1}^{-1} \ldots o f_{i+(n-1)}^{-1}$, for all $i, n \in \mathbb{N}$. Also denote by $f_{1,\infty}^n$ the sequence of maps $\{f_{in+1}^n\}_{i=0}^{\infty}$ and by $f_{1,\infty}^{-1}$ the sequence $\{f_i^{-1}\}_{i=1}^{\infty}$ [13].

Let (X, d) be a compact metric space and $f_{1,\infty}$ be a non-autonomous dynamical system on X. The function d_n given by $d_n(x, y) = \max_{0 \leq j < n} d(f_1^j(x), f_1^j(y))$, for each $n \geq 1$, is a metric equivalent to d [13].

A subset E of the space X is said to be $(n, \epsilon)-$separated if for any two points $x, y \in E$, $d_n(x, y) > \epsilon$ or $x = y$. Let $\text{sep}(n, \epsilon, f_{1,\infty})$ be the largest cardinality of $(n, \epsilon)-$separated set of X.

A subset F of the space X is called $(n, \epsilon)-$spanning if for every $x \in X$, there exists $y \in F$, such that $d_n(x, y) \leq \epsilon$. Take $\text{span}(n, \epsilon, f_{1,\infty})$ is the smallest cardinality of $(n, \epsilon)-$ *spanning* set of X.

Definition 2.1. *Let $f_{1,\infty}$ be a sequence of continuous maps from X to X*

$$h(f_{1,\infty}) = \lim_{\epsilon \longrightarrow 0^+} \lim_{n \longrightarrow \infty} \frac{1}{n} \log(\text{sep}(n, \epsilon, f_{1,\infty})) \tag{1}$$

$$= \lim_{\epsilon \longrightarrow 0^+} \lim_{n \longrightarrow \infty} \frac{1}{n} \log(\text{span}(n, \epsilon, f_{1,\infty})). \tag{2}$$

Denote by $h(f_{1,\infty})$ and called topological entropy of $f_{1,\infty}$ [13].

Now, we give some basic properties of the topological entropy for non-autonomous dynamical systems [13]:

1− For any $K \subseteq X$ with $K = \bigcup_{i=1}^k K_i$, $h(f_{1,\infty}, K) = \max h(f_{1,\infty}, K_i)$.
2− $h(f_{1,\infty}^n) \leq n.h(f_{1,\infty})$, for any $n \geq 1$.
3− If $f_{1,\infty}$ be periodic with period n then $h(f_{1,\infty}^n) = n.h(f_{1,\infty})$.
4− If (X, d) be a compact metric space then $h(f_{1,\infty}^n) = n.h(f_{1,\infty})$, for all $n \geq 1$.

3. Topological entropy of non-autonomous systems on noncompact spaces

Topological conjugacy is an important concept for determining when two systems are dynamically equivalent. Suppose that $\pi_{1,\infty}$ is a sequence of equicontinuous maps from X to Y, such that the following diagram is commutative [13].

$$\begin{array}{ccccc} X & \xrightarrow{f_i} & X & \xrightarrow{f_{i+1}} & X\ldots\,. \\ \downarrow{\scriptstyle \pi_i} & & \downarrow{\scriptstyle \pi_{i+1}} & & \downarrow{\scriptstyle \pi_{i+2}} \\ Y & \xrightarrow[g_i]{} & Y & \xrightarrow[g_{i+1}]{} & Y\ldots \end{array}$$

There are two special cases for $\pi_{1,\infty}$:
1− When $\pi_{1,\infty}$ is a sequence of equicontinuous surjective maps from X onto Y. In this case we say that $\pi_{1,\infty}$ is an equisemiconjugacy between $f_{1,\infty}$ and $g_{1,\infty}$.
2− Also when $\pi_{1,\infty}$ and $\pi_{1,\infty}^{-1}$ are two equicontinuous sequence of homeomorphisms. In this case we say that $\pi_{1,\infty}$ is an equiconjugacy between $f_{1,\infty}$ and $g_{1,\infty}$.

Theorem 3.1. [13] *Let (X, ρ) and (Y, τ) be compact metric spaces, $f_{1,\infty}$ be a sequence of continuous maps from X into itself, $g_{1,\infty}$ be a sequence of continuous maps from Y into itself. If the system $(X; f_{1,\infty})$ is the equisemiconjugate with $(Y; g_{1,\infty})$ then*

$$h(Y; g_{1,\infty}) \leq h(X; f_{1,\infty}).$$

Corollary 3.1. [13] *Let (X, ρ) and (Y, τ) be a compact metric space, $f_{1,\infty}$ be a sequence of continuous maps from X into itself and $g_{1,\infty}$ be a sequence of continuous maps from Y into itself. If the system $(X; f_{1,\infty})$ is equiconjugate with $(Y; g_{1,\infty})$ then*

$$h(Y; g_{1,\infty}) = h(X; f_{1,\infty}). \tag{3}$$

In the following, topological entropy of non-autonomous dynamical systems on noncompact metric spaces is investigated. In [19] Walter defined topological entropy for dynamical systems on noncompact metric spaces. Now, we use of this idea and give similar argument for non-autonomous dynamical systems.

Let (X, d) be a metric space and $f_{1,\infty}$ be a non-autonomous system on X.

Definition 3.1. *Let $n \in N$, $\epsilon > 0$ and K be a compact subset of X.*

A subset E of K is said to be (n, ϵ) separated subset of K with respect to $f_{1,\infty}$ if for every distinct points $x, y \in E$, $d_n(x, y) > \epsilon$.

A subset F of K is said to be (n, ϵ) spanning subset of K with respect to $f_{1,\infty}$ if for every $x \in K$, there exists $y \in F$ such that $d_n(x, y) \leq \epsilon$.

Let $\mathrm{span}(n, \epsilon, f_{1,\infty}, K)$ denote the smallest cardinality of any (n, ϵ) spanning sets and $\mathrm{sep}(n, \epsilon, f_{1,\infty}, K)$ denote the largest cardinality of any (n, ϵ) separated sets for K with respect to $f_{1,\infty}$.

By a similar arguments of Chapter 7 of [19] we can obtain the following result.

Lemma 3.1. *Let (X, d) be a metric space and $f_{1,\infty}$ be a non-autonomous system on X and K be a compact subset of X. For every $n \in N$ and $\epsilon > 0$,* $\mathrm{span}(n, \epsilon, f_{1,\infty}, K) < \infty$ *and*

$$\mathrm{span}(n, \epsilon, f_{1,\infty}, K) \leq \mathrm{sep}(n, \epsilon, f_{1,\infty}, K) \leq \mathrm{span}(n, \frac{\epsilon}{2}, f_{1,\infty}, K).$$

Also, if $\epsilon < \delta$ then $\mathrm{sep}(n, \epsilon, f_{1,\infty}, K) \leq \mathrm{sep}(n, \delta, f_{1,\infty}, K)$.

Then we can introduce the following definition of topological entropy on noncompact metric spaces.

Definition 3.2. *Let $f_{1,\infty}$ be a sequence of continuous maps from X to X and K be a compact subset of X.*

$$\begin{aligned} h(K; f_{1,\infty}) &= \lim_{\epsilon \longrightarrow 0^+} \lim_{n \longrightarrow \infty} \frac{1}{n} \log(\mathrm{sep}(n, \epsilon, f_{1,\infty}, K)), \\ &= \lim_{\epsilon \longrightarrow 0^+} \lim_{n \longrightarrow \infty} \frac{1}{n} \log(\mathrm{span}(n, \epsilon, f_{1,\infty}, K)). \end{aligned}$$

The topological entropy of $f_{1,\infty}$ (on X) is $h(X; f_{1,\infty}) = \sup_K h(K; f_{1,\infty})$. Where the supremum is taken over the collection of all compact subsets of X.

In the following theorem we show that under a simple assumption Theorem 3.1 is satisfied for noncompact metric spaces.

Theorem 3.2. *Suppose that (X, ρ) and (Y, τ) are metric spaces, $f_{1,\infty}$ is a sequence of continuous maps from X into itself, $g_{1,\infty}$ is a sequence of continuous maps from Y into* *itself. If the system* $(X; f_{1,\infty})$ *is the semiconjugate with* $(Y; g_{1,\infty})$ *as the following*

$$\begin{array}{ccccc} X & \xrightarrow{f_i} & X & \xrightarrow{f_{i+1}} & X... \\ \pi\downarrow & & \pi\downarrow & & \pi\downarrow \\ Y & \xrightarrow[g_i]{} & Y & \xrightarrow[g_{i+1}]{} & Y... \end{array}$$

where $\pi : X \to X$ *is a continuous map. Then*

$$h(Y; g_{1,\infty}) \leq h(X; f_{1,\infty}).$$

Proof. Let K be a compact subset of Y and consider the numbers $\epsilon > 0$ and $n \in N$. Take an (n, ϵ) separated subset E of K with the largest cardinality $sep(n, \epsilon, f_{1,\infty}, K)$. By axiom of choice, we can construct the set $E' \subset X$ containing only one point from each $\pi^{-1}(e)$, $e \in E$ and no other points. This is clear that E' is a finite and consequently compact subset of X. This implies that there exists $\delta < \epsilon$ such that for every $a, b \in E'$, if $\tau(\pi(a), \pi(b)) > \epsilon$ then $\rho(a, b) > \delta$. Hence E' is an (n, ϵ) separated subset of itself as a compact subset of X. Then $h(K; g_{1,\infty}) \leq h(X; f_{1,\infty})$. Since K is an arbitrary compact subset of Y, we easily conclude that $h(Y; g_{1,\infty}) \leq h(X; f_{1,\infty})$. □

4. Topological entropy of Lie group

In this section, we introduce non-autonomous dynamical system on Lie groups and estimate their topological entropy. Lie group theorems on entropy of classical dynamical system have been discussed [3], our aims is to consider topological entropy of non-autonomous systems on Lie groups.

A *Lie group* is a smooth manifold G that is also a group in the algebraic sense, with the property that the multiplication map $m : G \times G \longrightarrow G$ and inversion map $i : G \longrightarrow G$ given by $m(g, h) = gh$ and $i(g) = g^{-1}$ are differentiable [15].

Suppose that G is a Lie group and $g \in G$. Consider the maps

$$\begin{aligned} l_g : G &\longrightarrow G \\ h &\longmapsto gh, \\ r_g : G &\longrightarrow G \\ h &\longmapsto hg^{-1}. \end{aligned}$$

These are both diffeomorphisms by the definition of Lie group[18]. The Lie algebra of all smooth left-invariant vector field on Lie group G(i.e. vector fields X such that $l_{g*}X = X$ for all $g \in G$) is called the Lie algebra of G, and denote by $\mathfrak{g}$. The dimension of $\mathfrak{g}$ is finite as well as the dimension of G.

Let G be a Lie group, the evaluation map $\varepsilon : \mathfrak{g} \longrightarrow T_eG$, given by $\varepsilon(x) = x_e$, is a vector space isomorphism.

Given a Lie group G with Lie algebra $\mathfrak{g}$, define map $exp : \mathfrak{g} \longrightarrow G$, called the exponential map of G, by letting $expX = F(1)$, where F is the one parameter subgroup generated by X, or equivalently the integral curve of X starting at the identity. For example the exponential of $gl(n, \mathbb{R})$ is given by $expA = e^A$, if V is a finite dimensional real vector, then the exponential map of $GL(V)$ can be written as follows:

$$expA = \sum_{k=0}^{\infty} \frac{1}{k!} A^k.$$

Now, we remember some properties of exponential map that are needed in the rest of the paper [15]:

1− The exponential map is a smooth map from $\mathfrak{g}$ to G.
2− For any $X \in \mathfrak{g}, exp(s + t)X = exp(sX)exp(tX)$.
3−The exponential map for a Lie group does not necessarily injective or surjective. But,

it is locally injective and surjective, indeed the exponential map is a diffeomorphism from some neighborhood of 0 in $\mathfrak{g}$ to a neighborhood of e in G.

Recall that given a smooth map between manifolds $\psi : X \longrightarrow N$ we get a linear map called pushing forward along ψ: $\psi_* = d\psi_x : T_xX \longrightarrow T_{\psi(x)N}$ [10]. We use the following lemma to prove Theorem 4.1.

Lemma 4.1. *If A is an endomorphism of a Lie group G and d is a right invariant metric such that*

$$\begin{array}{ccc} T_eG & \xrightarrow{dA} & T_eG \\ {\scriptstyle exp}\downarrow & & \downarrow{\scriptstyle exp} \\ G & \xrightarrow[A]{} & G \end{array}$$

then $h_d(R_g o A) = h_d(A) = h_d(dA \mid T_eG)$ [3].

Now, by Theorem 3.2 we have the following generalization of Corollary 3.1 which is one of the main results of this paper. This theorem obtains that topological entropy is an invariant object under topological conjugacy of non-autonomous system on Lie groups and associated non-autonomous system on Lie Algebra.

Theorem 4.1. *Suppose that G is a Lie group and T_eG tangent space its. Let $A_{1,\infty} = \{A_i\}_{i\geqslant 1}$ be a non-autonomous system on G, where A_i is an endomorphism of a Lie group G and dA_i is the push-forward of the map A_i, for $i \geq 1$, let $f_{1,\infty} = dA_1, dA_2 \ldots$ and $g_{1,\infty} = A_1, A_2, \ldots$, then*

$$\begin{array}{ccccc} T_eG & \xrightarrow{dA_1} & T_eG & \xrightarrow{dA_2} & T_eG... \\ {\scriptstyle exp}\downarrow & & {\scriptstyle exp}\downarrow & & {\scriptstyle exp}\downarrow \\ G & \xrightarrow[A_1]{} & G & \xrightarrow[A_2]{} & G... \end{array} \tag{4}$$

consequently, we have

$$h(G; g_{1,\infty}) \leq h(T_eG; f_{1,\infty}).$$

Moreover, the equality holds, if the exponential map is bijective.

Proof. Let G be a Lie group and T_eG be it's tangent space which is isomorphism with associated Lie algebra. This is clear that $exp : T_eG \longrightarrow G$ from some neighborhood of 0 in T_eG to a neighborhood of e in G is injective and homeomorphism [15, 2]. Then for every $i \geq 1$ the following commutative diagram

$$\begin{array}{ccc} T_eG & \xrightarrow{dA_i} & T_eG \\ {\scriptstyle exp}\downarrow & & \downarrow{\scriptstyle exp} \\ G & \xrightarrow[A_i]{} & G \end{array} \tag{5}$$

satisfied in Proposition 10 of [3] and Lemma 4.1. Then Diagram 4 is commutative and exp is an equi-semiconjugacy between $g_{1,\infty}$ and $f_{1,\infty}$.

From this fact together with Theorem 3.2, we have that $h(G; g_{1,\infty}) \leq h(T_eG; f_{1,\infty})$. □

In [17], the author consider the topological entropy for automorphisms of simply connected nilpotent Lie groups and proved the following theorem.

Theorem 4.2. [17] *Let $f : G \longrightarrow G$ be an automorphism, where G is a simply connected nilpotent Lie group. Then this implies that $h(f) = 0$.*

By the above theorems, topological entropy of a Lie group is equal to topological entropy of the related Lie algebra, hence the entropy on the nilpotent Lie group is zero. In the next theorem we introduce a category of nontrivial non-autonomous systems on Lie groups with zero entropy.

Theorem 4.3. *Let $f_i : G \longrightarrow G$ be an automorphism for all $i \geq 1$ and the sequence $\{f_i\}$ converging uniformly to f, where G is a simply connected nilpotent Lie group. Then it follows that $h(f_{1,\infty}) = 0$.*

Proof. Theorem 4.2 implies that $h(f) = 0$, as a classic discrete dynamical system and by Theorem E of [13] we have that, $h(f_{1,\infty}) \leq h(f)$, consequently $h(f_{1,\infty}) = 0$. □

5. Topological entropy of linear non-autonomous systems

We start this section by introducing linear maps theorems on non-autonomous dynamical systems, which are satisfied for autonomous dynamical systems. Then we apply the results to calculate the entropy of linear maps. In the rest of the paper we assume that all matrices are diagonalizable.

The explanation of some definitions and lemmas are necessary in the process of proving of Theorem 5.1.

Remark 5.1. *Let M be a manifold of dimension m and T_xM be its tangent space. Suppose that $T_i : M \longrightarrow M$ is a differentiable map and $\{A_1, \dots, A_t\}$ is the set of all functions where $T_i \in \{A_1, \dots, A_t\}$. Consider*

$$a_i = \sup_{x\in M} \|dA_i|A_{i_x}M\|, \quad b_i = \sup_{x\in M} \|dT_i|T_{i_x}M\| \quad \text{for all} \quad 1 \leq i \leq t.$$

Since $T_1^n = T_n o T_{n-1} o \dots o T_1$, then

$$|d(T_1^n)| = |dT_n o T_{n-1} o \dots o T_1| \leq |dT_n||dT_{n-1}| \dots |dT_1| < b_n b_{n-1} \dots b_1.$$

Also, $\lim\limits_{n\longrightarrow\infty} (b_n b_{n-1} \dots b_1)^{\frac{1}{n}} = (a_1^{n_1} \dots a_t^{n_t})^{\frac{1}{n}} = a_1^{\frac{n_1}{n}} \dots a_t^{\frac{n_t}{n}} = a_1^{N_1} \dots a_t^{N_t} = a.$
Where $n_k = \{|i|$ such that $1 \leq i \leq n$, $T_i = A_k\}$. And put N_t be density presence of A_t in $\{T_i\}_{i=1}^{\infty}$, in other words $\lim\limits_{n\longrightarrow\infty} \dfrac{n_k}{n} = N_k$. *It is obvious that $\sum_{k=1}^{t} N_k = 1$.*

The concepts introduced in this remark are used in the rest of this paper.

Lemma 5.1. *Suppose that $\{A_1, \dots, A_t\}$ is the set of all functions in $T_{1,\infty}$. With the assumptions given in the previous remark*

$$d(T_1^n(x), T_1^n(y)) \leq a_1^{n_1} \dots a_t^{n_t} d(x, y) = \alpha_n d(x, y).$$

Where $n_k = \{i : 1 \leq i \leq n,\ T_i = A_k\}$.

Proof. Let $b_i \in \{a_1^{n_1}, \dots, a_m^{n_m}\}$, where $a_i = \sup\limits_{x\in M} \|dA_i|A_{i_x}M\|, T_i \in \{A_1, \dots, A_t\}$, and $T_{1,\infty} = T_1,\ T_2,\ \dots$ By choose of b_i, we prove with induction

$$d(T_1(x), T_1(y)) \leq b_1 d(x, y),$$

$$d(T_2 o T_1(x), T_2 o T_1(y)) \leq b_2 d(T_1(x), T_1(y)) \leq b_2 b_1 d(x, y),$$

so, by induction on n, we have

$$d(T_1^n(x), T_1^n(y)) \leq b_n \dots b_1 d(x, y) = a_1^{n_1} \dots a_t^{n_t} d(x, y) = \alpha_n d(x, y).$$

Hence we obtain $d(T_1^n(x), T_1^n(y)) \leq \alpha_n d(x, y)$. □

Next Theorem is one of main results of this paper. This theorem gives an estimate of topological entropy of non-autonomous systems. The basic idea of the proof comes from Theorem 4.1 of [3].

Theorem 5.1. *Let M be a $m-$dimensional Riemannian manifold and $T_{1,\infty} = \{T_i\}_{i=1}^{\infty}$ be a non-autonomous system such that, $T_i : M \longrightarrow M$ is a differentiable map, for all $i \geq 1$. Assume that all maps T_i are from the set $\{A_1, \dots, A_t\}$. then*

$$h(T_{1,\infty}) \leq \max\{o, m\,[N_1 \log a_1 + \cdots + N_t \log a_t]\}.$$

Proof. If $a_i = \infty$ for some $1 \leq i \leq t$ then it is nothing to prove. Suppose that $a_i < \infty$ for all $1 \leq i \leq t$ and $K \subseteq M$ is a compact set. Let $f_1, \dots, f_r : B_3(0) \subseteq \mathbb{R}^m \longrightarrow M$ are differentiable maps such that the $f_i(B_1(0))$ cover K and choose $A > 0$ such that $d(f_i(x), f_i(y)) \leq Ad(x, y)$ for all $x, y \in B_2(0)$ and $1 \leq i \leq r$.

As in Remark 5.1, put $a = a_1^{N_1} \dots a_t^{N_t}$, if $a \leq 1$, then Lemma 5.1 implies that, for all $n \geq 1$ and for each $x, y \in X$,

$$d(T_1^n(x), T_1^n(y)) \leq ad(x, y) \leq d(x, y).$$

Then $T_{1,\infty}$ never expands distances, in this case every $(1, \epsilon)$-spanning set is a (n, ϵ)-spanning set and we get $h(T_{1,\infty}) = 0$.

Thus we may assume that $a \geq 1$. Let $E(\delta) = \{(r_1\delta, \dots, r_m\delta) | r_i \in \mathbb{Z}, |r_i| < 2\}$, for each $0 < \delta \leq 1$. Then $cardE(\delta) \leq (\frac{5}{\delta})^m$ and there is a constant $\beta > 0$ (depending on metrice d used for $\mathbb{R}^m$) such that for each $y \in B_1(0)$, there is a $x \in E(\delta)$ with $d(x, y) \leq \beta\delta$.

Since $d(f_i(x), f_i(y)) \leq Ad(x, y)$ and $d(x, y) \leq \beta\delta$, by Lemma 5.1 we have that

$$d(T_1^n(x), T_1^n(y)) \leq \alpha_n d(x, y).$$

Thus $d(T_1^n(x), T_1^n(y)) \leq \alpha_n AB\delta = \epsilon$, this is clear that $T(\delta) = \bigcup_{1\leq i\leq r} f_i E(\delta)$ is a $(n, \alpha_n AB\delta)$-spanning set for K and $cardT(\delta) \leq r(\frac{5}{\delta})^m$.

Hence, considering $\delta = \dfrac{\epsilon}{\alpha_n AB}$. These statements imply that

$$cardT(\delta) \leq r\left[\frac{5}{\dfrac{\epsilon}{\alpha_n AB}}\right]^m = r(\frac{5\alpha_n AB}{\epsilon})^m = r\left[\frac{5AB}{\epsilon}\right]\alpha_n^m$$

and T is a spanning set, therefore $r_n(\epsilon, K) \leq \left[\dfrac{5AB}{\epsilon}\right]\alpha_n^m$. Consequently, by Remark 5.1 we have

$$\begin{aligned}
h(T_{1,\infty}) &= \frac{1}{n}\log r_n(\epsilon, K) \leq \frac{1}{n}\log\left[(\frac{5AB}{\epsilon})^m\right]\alpha_n^m \\
&= \log(\left[(\frac{5AB}{\epsilon})^m\right])^{\frac{1}{n}}(a_1^{n_1}\dots a_t^{n_t})^{\frac{m}{n}} = m(\log a_1^{\frac{n_1}{n}} + \cdots + \log a_t^{\frac{n_t}{n}}) \\
&= m(\log a_1^{N_1} + \cdots + \log a_t^{N_t}) = m(N_1 \log a_1 + \cdots + N_t \log a_t).
\end{aligned}$$

This implies that, we have the following inequality, which is a strong tool to estimate a upper bounded for topological entropy of a non-autonomous systems:

$$h(T_{1,\infty}) \leq \max\{0, m\,[N_1 \log a_1 + \cdots + N_t \log a_t]\}.$$

□

Let A be a matric and $e_1, \dots, e_n$ are eigenvectors corresponding to the eigenvalues $\lambda_1, \dots, \lambda_n$, respectively and $\lambda_1 = \lambda$ be the maximum eigenvalue. Then for all vector, $V = v_1e_1 + v_2e_2 + v_3e_3 + \cdots + v_ne_n$ we have

$$\|AV\| = (\lambda_1 v_1)^2 + (\lambda_2 v_2)^2 + \cdots + (\lambda_n v_n)^2 \leq \lambda_1(v_1^2 + v_2^2 + \cdots + v_n^2) \leq \lambda_1.$$

So, we reach to the following result, that estimate the topological entropy of a linear non-autonomous system by maximum eigenvalue of matrices that construct our non-autonomous system.

Corollary 5.1. *Consider the non-autonomous system* $T_{1,\infty} = \{T_i\}_{i=1}^{\infty}$*, such that each* $T_i : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ *is a linear map belong to the set* $\{A_1, \ldots, A_t\}$*. Put* μ_j *is an eigenvalue of* A_j *with maximum absolute value, for all* $1 \le j \le t$*, then*

$$h(T_{1,\infty}) \le \max\{0, m \sum_{i=1}^{t} N_i \log|\mu_i|\}. \tag{6}$$

Proof. Let μ_j be an eigenvalue of A_j with maximum absolute value. Since A_j is linear then $dA_j = A_j$. As for Theorem 5.1, we have

$$h(T_{1,\infty}) \le \max\{0, m\,[N_1 \log a_1 + \cdots + N_t \log a_t]\}, \quad \text{where} \quad a_i = \sup\|dA_j\|.$$

Then it is not difficult to show that

$$\begin{aligned} h(T_{1,\infty}) &\le \max\{0, m\,[N_1 \log\|a_1\| + \cdots + N_t \log\|a_t\|]\} \\ &\le \max\{0, m\,[N_1 \log|\mu_1| + \cdots + N_t \log|\mu_t|]\}. \end{aligned}$$

Thus we have the following important inequality:

$$h(T_{1,\infty}) \le \max\{0, m \sum_{i=1}^{t} N_i \log|\mu_i|\}. \qquad \square$$

In [12] the authors introduce product of non-autonomous systems and prove the following proposition.

Proposition 5.1. [12] *If* $(X, f_{1,\infty})$ *and* $(Y, g_{1,\infty})$ *be topological space then for product system* $(X \times Y, f_{1,\infty} \times g_{1,\infty})$ *defined*

$$h(f_{1,\infty} \times g_1, \infty) \le h(f_{1,\infty}) + h(g_{1,\infty}).$$

For example suppose that $f_{1,\infty} = (2x, 3x, 2x, 3x, \ldots)$ and $g_{1,\infty} = (4x, 5x, 4x, 5x, \ldots)$, then

$$f_{1,\infty} \times g_{1,\infty} = ((2x, 4x), (3x, 5x), (2x, 4x), (3x, 5x), \ldots).$$

Now we use this proposition to prove the following theorem which is the main result of this paper. In this theorem we consider linear maps with the same eigenspaces.

Theorem 5.2. *Suppose that* $\{A_1, A_2, \ldots, A_t\}$ *is a finite set of matrices and* μ_i^j *is eigenvalue of* A_i *with multiplicity* m_j *and the same associated eigenspace* E_j*, for all* $1 \le i \le t$*. Assume that* $f_{1,\infty} = f_1, f_2, \ldots$ *is a non-autonomous system consists of* $\{A_1, A_2, \ldots, A_t\}$ *and* $N_1, \ldots, N_t$ *are multiplicity* $A_1, \ldots, A_t$ *in* $f_{1,\infty}$*, respectively. Then*

$$h(f_{1,\infty}) \le \sum_{j=1}^{s} m_j (\sum_{i=1}^{t} N_i \log \mu_i^j), \tag{7}$$

where the summation is on all eigenvalues with value greater than one.

Proof. Let E_j is the common associated eigenspace respect to eigenvalue μ_i^j, for all $1 \le i \le t$. This is clear that $A_i(E_j) \subseteq E_j$, for all A_i and E_j, then we put $T_{i_j} = f_i|_{E_j}$ and $T_{j,\infty} = \{T_{1_j}, T_{2_j}, T_{3_j}, \ldots\}$. Hence,

$$f_i = T_{i_1} \times T_{i_2} \times \cdots \times T_{i_s} : E_1 \times E_2 \times \cdots \times E_s \longrightarrow E_1 \times E_2 \times \cdots \times E_s$$

and $f_{1,\infty} = T_{1,\infty} \times T_{2,\infty} \times \cdots \times T_{s,\infty}$.

Proposition 5.1 implies that $h(T_{j,\infty}) \le m_j \sum_{i=1}^{t} N_i \log|\mu_i^j|$, for all $1 \le j \le s$.

Consequently, Proposition 5.1 and Corollary 5.1 imply that

$$h(f_{1,\infty}) = h(T_{1,\infty} \times T_{2,\infty} \times \cdots \times T_{s,\infty}) \leq \tag{8}$$

$$h(T_{1,\infty}) + \cdots + h(T_{s,\infty}) \leq \tag{9}$$

$$m_1(N_1 \log(\mu_1^1) + N_2 \log(\mu_2^1) + \cdots + N_t \log(\mu_t^1)) + \tag{10}$$

$$m_2(N_1 \log(\mu_1^2) + N_2 \log(\mu_2^2) + \cdots + N_t \log(\mu_t^2)) + \cdots + \tag{11}$$

$$m_s(N_1 \log(\mu_1^s) + N_2 \log(\mu_2^s) + \cdots + N_t \log(\mu_t^s)). \tag{12}$$

Therefore, we achieved an important and useful relationship, which is the main result of this paper

$$h(f_{1,\infty}) \leq \sum_{j=1}^{s} m_j (\sum_{i=1}^{t} N_i \log \mu_i^j).$$

Then, the proof is completed. □

Corollary 5.2. *Suppose that $\{A_1, A_2, \dots, A_t\}$ is a finite set of endomorphisms of a Lie group G such that their tangent maps on T_eG have the previous theorem conditions. Consider the non-autonomous system $f_{1,\infty} = f_1, f_2, \dots$ on G consists of $\{A_1, A_2, \dots, A_t\}$ Then*

$$h(f_{1,\infty}) \leq \sum_{j=1}^{s} m_j (\sum_{i=1}^{t} N_i \log \mu_i^j),$$

where μ_i^j is eigenvalue of $dA_i|_{T_eG}$ with multiplicity m_j and the same associated eigenspace E_j, for all $1 \leq i \leq t$.

6. Some examples

In this section we are going to introduce some examples of non-autonomous systems on Lie groups and estimate their topological entropy.

Example 6.1. *The set of all three dimensional real matrices A of the form*

$$A = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix},$$

where a, b and c are real numbers, is the Heisenberg group, H. This is well known that it is a simply connected and nilpotent Lie group. Consider the matrices

$$B_n = \begin{pmatrix} 1 & 2+\frac{1}{n} & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

for all $n \in N$. The sequence $\{f_n\}_{n\geq 1}$ of automorphisms $f_n : H \to H$ defined by $f_n(A) = B_n A B_n^{-1}$ is uniformly convergence to $f : H \to H$ defined by $f(A) = BAB^{-1}$. Then, by Theorem 4.3 the non-autonomous dynamical system $f_{1,\infty}$ on H has zero topological entropy.

To use Theorem 4.1 it is necessary that the exponential map be a surjective map. The exponential map is not always surjective, but if the Lie group G is connected and compact or connected and nilpotent, then its exponential map is surely surjective.

Example 6.2. *Let $SO(2)$ denoted the set of all 2×2 orthogonal matrices with determinant one. These group elements are often represented as two dimensional matrices of the form*

$$A(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The Lie algebra, $\mathbf{so}(2)$*, is the set of* 2×2 *skew-symmetric matrices. The single generator of* $\mathbf{so}(2)$ *corresponds to the derivative of* $2-D$ *rotation, evaluated at the identity:*

$$E=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Hence, elements of $\mathbf{so}(2)$ *are represented as*

$$\begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}. \tag{13}$$

The exponential map $\exp : SO(2) \to \mathbf{so}(2)$ *is defined by*

$$\exp\begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} = I+\begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}+\frac{1}{2!}\begin{pmatrix} -\theta^2 & 0 \\ 0 & -\theta^2 \end{pmatrix}\frac{1}{3!}\begin{pmatrix} 0 & \theta^3 \\ -\theta^3 & 0 \end{pmatrix}+\cdots$$
$$=\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

This Lie group is connected and compact, then its exponential map is surjective. Consider the endomorphisms $T, S : SO(2) \to SO(2)$ *defined by* $T(g) = g^k$ *and* $S(g) = g^m$*, where* k, m *are two positive integer numbers. Indeed for every*

$$g=\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

we have that

$$T(g)=\begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix}, \quad S(g)=\begin{pmatrix} \cos m\theta & -\sin m\theta \\ \sin m\theta & \cos m\theta \end{pmatrix}.$$

Applying this fact, then we have that the tangent maps $dT : R \to R$ *and* $dS : R \to R$ *are defined by* $dT(\theta) = k\theta$ *and* $dS(\theta) = m\theta$*. Now, consider a non-autonomous system* $f_{1,\infty} = \{f_n\}_{n\geq 1}$ *such that* $f_n \in \{T, S\}$*, for all* $n \geq 1$*. It is follows from Theorem 4.1 and consequently Theorem 5.2 that* $h(f_{1,\infty}) \leq N_T \log k + N_S \log m$*, where* N_T, N_S *denoted the probability density of the functions* T *and* S *in* $f_{1,\infty}$*, respectively.*

Example 6.3. *Let*

$$T : \mathbb{R}^4 \longrightarrow \mathbb{R}^4,$$
$$(x, y, z, w) \longmapsto (2x, 2y, 2z, -2w)$$

and

$$S : \mathbb{R}^4 \longrightarrow \mathbb{R}^4,$$
$$(x, y, z, w) \longmapsto (3x, 3y, -\frac{1}{2}z+\frac{7}{2}w, \frac{7}{2}z-\frac{1}{2}w).$$

By Theorem 5.1 for estimate the entropy of the linear non-autonomous systems we need eigenvalues of their maps

$$T=\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}, \quad S=\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{7}{2} \\ 0 & 0 & \frac{7}{2} & -\frac{1}{2} \end{pmatrix}$$

$\lambda_T = 2, -2$ *where we have* 2 *three times and* -2 *once,*
$\lambda_S = 3, -4$ *where we have* 3 *three times and* -4 *once.*

Suppose that $f_{1,\infty} = T, S, T, S, T, S, \ldots$ *This is clear that for* $f_{1,\infty}$ *the probability density of* T *and* S *is the same number* $\frac{1}{2}$*. By Relation 8 in Theorem 5.2 we have that*

$$h(f_{1,\infty}) \leq 1(\frac{1}{2}\log|-2| + \frac{1}{2}\log|-4|) + 3(\frac{1}{2}\log|2| + \frac{1}{2}\log|3|)\cdots$$
$$= 4\log 2 + \frac{3}{2}\log 3 = \log 2^4 + \log 3^{\frac{3}{2}} = \log(2^4 \times 3^{\frac{3}{2}}) \backsimeq 1.9.$$

Now, consider the system $f_{1,\infty} = T, T, S, T, T, S, \ldots$*, where* T *and* S *are defined before. For this system the density of frequency* T *is* $\frac{2}{3}$ *and density of frequency* S *is* $\frac{1}{3}$ *hence*

$$h(f_{1,\infty}) = 1(\frac{2}{3}\log|-2| + \frac{1}{3}\log|-4|) + 3(\frac{2}{3}\log|2| + \frac{1}{3}\log|3|)$$
$$= \frac{10}{3}\log 2 + \log 3 = \log 2^{\frac{10}{3}} + \log 3 = \log(2^{\frac{10}{3}} \times 3) \backsimeq 1.5.$$

Now, we generalize the method for $f_{1,\infty} = T, T, \ldots, T, S, T, T, \ldots, S, \ldots$ *where the density of frequency* T *is* $\frac{n-1}{n}$ *and density of frequency* S *is* $\frac{1}{n}$*, then we have*

$$h(f_{1,\infty}) = 1(\frac{n-1}{n}\log|-2| + \frac{1}{n}\log|-4|) + 3(\frac{n-1}{n}\log|2| + \frac{1}{n}\log|3|)$$
$$= \frac{4n-2}{n}\log 2 + \frac{3}{n}\log 3 = \log 2^{\frac{4n-2}{n}} + \log 3^{\frac{3}{n}} = \log(2^{4n-2} \times 3^3)^{\frac{1}{n}}.$$

Given the examples in which the multiplicity of a function extends to infinity, we obtain a main result.

Corollary 6.1. *The quantity is taken finite in this paper; however in infinite case when in the non-autonomous system* $f_{1,\infty} = T, T, \ldots, T, S, T, T, \ldots, S, \ldots,$ *the number* n *convergence to infinitely the frequency of* S *is ignored in calculating the entropy. Because probability density of* S *is* $\frac{1}{n}$ *and* $\lim_{n\longrightarrow\infty}\frac{1}{n} = 0$*. But the density of frequency* T *is* $\frac{n-1}{n}$ *that is* $\lim_{n\longrightarrow\infty}\frac{n-1}{n} = 1$*.*

7. Discussions and future directions

In this work we introduced non-autonomous dynamical systems on Lie groups as well as Lie algebras. The topological entropy for this type of systems is considered. This is well known that topological entropy is a number which is usually taken as a measure of complexity of a topological dynamical system. In particular, positive topological entropy is an important property to characteristic feature of chaos. It is well-known that positive topological entropy implies Li-Yorke chaos for any surjective continuous map on a compact metric space. In accordance with what was said, zero topological entropy implies that the system has a simple dynamical behavior.

So this is very interesting to investigate positive topological entropy and chaos for non-autonomous dynamical systems on Lie groups. In this direction, considering non-autonomous dynamical systems on nil-manifold Lie groups and studying their topological entropy is an attractive topic for future studies. More precisely, Given a simply connected nilpotent Lie groups G and a compact discrete subgroup $\Gamma < G$, the closed manifold $(\Gamma\backslash G)^k$ is called a nilmanifold. When G is $K-$ step, the nilmanifold M is called $K-$ step. A metric g or magnetic field σ on M is called left–invariant if its pullback to G is left–invariant.

Butler in [5] has shown that the topological entropy of a left-invariant geodesic flow on a two-step nilmanifold vanishes.

Also, Epstein In [9] has shown that there is a magnetic field on a two-step nilmanifold that has positive topological entropy for arbitrarily high energy levels. More precisely,

Theorem 7.1. [9] *There exists a* 2*–step nil-manifold* $\Gamma\backslash G$ *with left–invariant metric* g *and left–invariant* σ *such that the flow of the magnetic system* $(\Gamma\backslash G, g, \sigma)$ *has positive topological entropy on arbitrarily high energy levels.*

We conjecture that the technique developed here for topological entropy of non-autonomous systems could be applied for the topics mentioned above. But, Since this requires a lot of definitions and preliminaries; We would prefer to do it in a separate work.

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References

1. R. Adler, A. Konheim, M. McAndrew, *Topological entropy*, Trans. Amer. Math. Soc. **114** (1965), 309–319.
2. A. L. Onishchik, E. B. Vinberg, *Lie Groups and Algebraic Groups*, Nuka, Moscow, 1988.
3. R. Bowen, *Entropy for group endomorphism and homogeneous space*, Trans. Amer. Math. Soc. **153** (1971), 401–414.
4. R. Bowen, *Periodic points and measures for Axiom A diffeomorphisms*, Trans. Amer. Math. Soc. **154** (1971), 377–397.
5. L. Butler, *Integrable geodesic flows with wild first integrals: the case of two-step nilmanifolds*, Ergodic Theory Dynam. Systems **23** (2003), 771–797.
6. J. S. Canovas, *Resent results on non-autonomous discrete systems*, Bol. Soc. Esp. Mat. Apl. **51** (2010), 33–41.
7. C. Corda, M. Fatehi Nia, M. R. Molaei, Y. Sayyari, *Entropy of iterated function systems and their relations with black holes and bohr-like black holes entropies*, Entropy **20** (2018), 56–72.
8. C. Peng, *Algebraic and topological entropy on Lie groups*, Mathematical and Computer Modelling **39** (2004), 13–19.
9. J. Epstein, *Topological entropy of left-invariant magnetic flows on 2-step nilmanifolds*, London Mathematical Society **30** (2016), 1–12.
10. J. Baez, *Classical Mechanics*, Lecture, **14**, 2008.
11. J. Zhang, L. Chen, *Lower bounds of the topological entropy for non-autonomous dynamical systems*, Appl. Math. J. Chinese Univ. **24** (2009), 76–82.
12. C. Kawan, Y. Latushkin, *Some results on the entropy of non-autonomous dynamical systems*, Dynamical Systems **31** (2016), 251–279.
13. S. Kolyada, L. Snoha, *Topological entropy of non-autonomous dynamical system*, Random and Computational Dynamics **4** (1996), 205–233.
14. S. Kolyada, M. Misiurewicz, L. Snoha, *Topological entropy of non-autonomous piecewise monotone dynamical systems on the interval*, Fund. Math. **160** (1999), 161–181.
15. J. M. Lee, *Introduction to Smooth Manifolds*, Graduate Texts in Mathematics, **218**, 2002.
16. Z. H. Nitecki, *Topological entropy and the preimage structure of maps*, Real Anal. Exchange **29** (2003), no. 4, 9–41.
17. M. Patrao, *Entropy and its variational principle for noncompact metric space*, Ergodic Theory and Dynamical Systems **30** (2010), 1529–1542.
18. A. Razavi, *Lie Groups and Lie Algebras*, Amirkabir University of Technology, 2008.
19. P. Walters, *An Introduction to Ergodic Theory*, Springer Verlag, New York, 1982.
20. Y. Zhu, Z. Jinlian, H. Lianfa, *Topological entropy of a sequence of monotone maps on circles*, Journal of the Korean Mathematical Society **43** (2006), 373–382.

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