Methods of Functional Analysis and Topology
Vol. 25 (2019), no. 4, pp. 381–387

APPROXIMATION BY FOURIER SUMS IN CLASSES OF DIFFERENTIABLE FUNCTIONS WITH HIGH EXPONENTS OF SMOOTHNESS

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ABSTRACT. We find asymptotic equalities for the exact upper bounds of approximations by Fourier sums of Weyl-Nagy classes $W^r_{\beta,p}, 1 \le p \le \infty$, for rapidly growing exponents of smoothness r $(r/n \to \infty)$ in the uniform metric. We obtain similar estimates for approximations of the classes $W^r_{\beta,1}$ in metrics of the spaces $L_p, 1 \le p \le \infty$.

Let C be the space of continuous 2π–periodic functions f, in which the norm is defined by the equality

$$\|f\|_C = \max_t |f(t)|,$$

L_p, $1 \le p < \infty$, be the space of 2π–periodic functions f summable to the power p on $[-\pi, \pi)$, in which the norm is given by the formula

$$\|f\|_{L_p} = \|f\|_p = \left(\int_{-\pi}^{\pi} |f(t)|^p dt \right)^{1/p},$$

and L_∞ be the space of measurable and essentially bounded 2π–periodic functions f with the norm

$$\|f\|_{L_\infty} = \|f\|_\infty = \operatorname*{ess\,sup}_t |f(t)|.$$

Further, let $W^r_{\beta,p}, 1 \le p \le \infty$, be the sets of all 2π-periodic functions f, representable as convolutions of the form

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi} \int_{-\pi}^{\pi} \varphi(x-t) B_{r,\beta}(t) dt, \quad a_0 \in \mathbb{R}, \tag{1}$$

where $B_{r,\beta}(\cdot)$ are Weyl-Nagy kernels of the form

$$B_{r,\beta}(t) = \sum_{k=1}^{\infty} k^{-r} \cos\left(kt - \frac{\beta\pi}{2}\right), \quad r > 0, \quad \beta \in \mathbb{R}, \tag{2}$$

and the functions φ satisfy the condition

$$\varphi \in U^0_p = \left\{ \varphi \in L_p : \|\varphi\|_p \le 1, \int_{-\pi}^{\pi} \varphi(t) dt = 0 \right\}. \tag{3}$$

The classes $W^r_{\beta,p}$ are called as Weyl-Nagy classes (see, e.g., [18, 13, 16, 17]).

If $r \in \mathbb{N}$ and $\beta = r$, then the functions of the form (2) are the well-known Bernoulli kernels and the classes $W^r_{\beta,p}$ coincide with the well-known classes W^r_p, which consist of 2π-periodic functions with absolutely continuous derivatives up to $(r-1)$-th order

2010 *Mathematics Subject Classification.* 42A10.
Key words and phrases. Fourier sum, Weyl-Nagy class, asymptotic equality.

inclusive and such that $\|f^{(r)}\|_p \le 1$ and $f^{(r)}(x) = \varphi(x)$ for almost everywhere $x \in \mathbb{R}$, where φ is the function from (1).

For arbitrary $\mathfrak{N} \subset X$, where $X = C$ or $L_p,\ 1 \le p \le \infty$, we consider the quantity

$$\varepsilon_n(\mathfrak{N})_X = \sup_{f\in\mathfrak{N}} \|f(\cdot) - S_{n-1}(f;\cdot)\|_X, \tag{4}$$

where $S_{n-1}(f;x)$ is the partial Fourier sum of order $n-1$ of the function f.

In the case of Weyl-Nagy classes $W^r_{\beta,\infty}$ and $X = C$ for the exact upper bounds (4) the following asymptotic estimate holds

$$\varepsilon_n(W^r_{\beta,\infty})_C = \frac{4}{\pi^2}\frac{\ln n}{n^r} + O\left(\frac{1}{n^r}\right), \quad r > 0, \quad \beta \in \mathbb{R}. \tag{5}$$

For $r \in \mathbb{N}$ and $\beta = r$ this estimate was obtained by A. N. Kolmogorov [2], for arbitrary $r > 0$ by V. T. Pinkevich [6] and S. M. Nikol'skii [4]. In the general case the estimate (5) follows from results, which were obtained in the works of A. V. Efimov [1] and S. A. Telyakovskii [19].

It should be also noticed, that a similar asymptotic equality holds for the classes $W^r_{\beta,1}$ in the metric of the space L_1, namely

$$\varepsilon_n(W^r_{\beta,1})_{L_1} = \frac{4}{\pi^2}\frac{\ln n}{n^r} + O\left(\frac{1}{n^r}\right), \quad r > 0, \quad \beta \in \mathbb{R}, \tag{6}$$

(see [5, 15]).

In these works the parameters r and β of the Weyl-Nagy classes were assumed to be fixed, and the question about the dependence of the remainder term in the estimates (5) or (6) on these parameters was not considered.

The character of the dependence on r and β of the remainder term in estimate (5) was investigated by I. G. Sokolov [12], S. G. Selivanova [7], G. I. Natanson [3], S. A. Telyakovskii [20, 21] and S. B. Stechkin [14].

In the work of S. B. Stechkin [14] the asymptotic behavior, as $n \to \infty$ and $r \to \infty$, of the quantities $\varepsilon_n(W^r_{\beta,\infty})_C$ was completely investigated. Namely, he proved that for arbitrary $r \ge 1$ and $\beta \in \mathbb{R}$ the following equality takes place

$$\varepsilon_n(W^r_{\beta,\infty})_C = \frac{1}{n^r}\left(\frac{8}{\pi^2}\mathbf{K}(e^{-r/n}) + O(1)\frac{1}{r}\right), \tag{7}$$

where

$$\mathbf{K}(q) = \int\limits_0^{\pi/2} \frac{dt}{\sqrt{1 - q^2\sin^2 t}} \tag{8}$$

is a complete elliptic integral of the first kind, and $O(1)$ is a quantity uniformly bounded with respect to n, r and β.

Moreover, S. B. Stechkin [14, theorem 4] proved that for rapidly growing r the remainder in estimate (7) can be improved. Namely, for arbitrary $r \ge n+1$ and $\beta \in \mathbb{R}$ the following equality holds:

$$\varepsilon_n(W^r_{\beta,\infty})_C = \frac{1}{n^r}\left(\frac{4}{\pi} + O(1)\left(1 + \frac{1}{n}\right)^{-r}\right), \tag{9}$$

where $O(1)$ is a quantity uniformly bounded with respect to n, r and β. If $r/n \to \infty$, then the estimate (9) becomes the asymptotic equality.

It also follows from [14] that for the quantities $\varepsilon_n(W^r_{\beta,1})_{L_1}$ the analogous estimates to (7) and (9) take place. Namely, for $r \ge 1$ and $\beta \in \mathbb{R}$ uniformly with respect to the all analyzed parameters the following estimate is true

$$\varepsilon_n(W^r_{\beta,1})_{L_1} = \frac{1}{n^r}\left(\frac{8}{\pi^2}\mathbf{K}(e^{-r/n}) + O(1)\frac{1}{r}\right), \tag{10}$$

where $\mathbf{K}(q)$ is defined by (8), and for $r \geq n+1$ and $\beta \in \mathbb{R}$ uniformly with respect to all analyzed parameters the following estimate holds

$$\varepsilon_n(W^r_{\beta,1})_{L_1} = \frac{1}{n^r}\left(\frac{4}{\pi} + O(1)\left(1+\frac{1}{n}\right)^{-r}\right). \tag{11}$$

Telyakovskii [21] showed that the remainder in formulas (9) and (11) can be replaced by a smaller one, namely, write $O(1)(1+\frac{2}{n})^{-r}$ instead of $O(1)(1+\frac{1}{n})^{-r}$.

In this paper for arbitrary values $1 \leq p \leq \infty$ we establish generalized analogs of estimates (9) and (11) for quantities $\varepsilon_n(W^r_{\beta,p})_C$ and $\varepsilon_n(W^r_{\beta,1})_{L_p}$, respectively. Namely, as a consequence of the main result (Theorem 1) it follows that for $r \geq n+1$, $\beta \in \mathbb{R}$ and $1 \leq p \leq \infty$ the following estimates hold:

$$\varepsilon_n(W^r_{\beta,p})_C = \frac{1}{n^r}\left(\frac{\|\cos t\|_{p'}}{\pi} + O(1)\left(1+\frac{1}{n}\right)^{-r}\right), \tag{12}$$

$$\varepsilon_n(W^r_{\beta,1})_{L_p} = \frac{1}{n^r}\left(\frac{\|\cos t\|_{p}}{\pi} + O(1)\left(1+\frac{1}{n}\right)^{-r}\right), \tag{13}$$

where $1/p + 1/p' = 1$ and $O(1)$ are quantities uniformly bounded in all analyzed parameters. The estimates (12) and (13) are the asymptotic equalities, as $r/n \to \infty$.

The main results of this paper are statements for the sets of functions, which are more general compared to the Weyl-Nagy classes, namely, for the Stepanets classes $L^{\psi}_{\bar\beta,p}$ and $C^{\psi}_{\bar\beta,p}$ [16, 17].

Denote by $L^{\psi}_{\bar\beta,p}$, $1 \leq p \leq \infty$, the set of all 2π-periodic functions f, representable for almost all $x \in \mathbb{R}$ as convolutions of the form

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi}\int_{-\pi}^{\pi} \Psi_{\bar\beta}(x-t)\varphi(t)dt, \quad a_0 \in \mathbb{R}, \tag{14}$$

of functions φ, which satisfy the conditions (3), with kernels $\Psi_{\bar\beta} \in L_1$, which Fourier series have the form

$$S[\Psi_{\bar\beta}](t) = \sum_{k=1}^{\infty} \psi(k)\cos\left(kt - \frac{\beta_k\pi}{2}\right), \tag{15}$$

where $\psi = \{\psi(k)\}_{k=1}^{\infty}$ and $\bar\beta = \{\beta_k\}_{k=1}^{\infty}$ are fixed sequences of real numbers. In the case of $\bar\beta$ is a stationary sequence, i.e. $\beta_k \equiv \beta$, $\beta \in \mathbb{R}$, the kernels $\Psi_{\bar\beta}$ of the form (15) are denoted by Ψ_β, and the classes $L^{\psi}_{\bar\beta,p}$ are denoted by $L^{\psi}_{\beta,p}$.

In this paper we consider the classes $L^{\psi}_{\bar\beta,p}$, which are generated by kernels of the form (15) with coefficients $\psi(k) > 0$ such that

$$\sum_{k=1}^{\infty} \psi(k) < \infty. \tag{16}$$

In this case $S[\Psi_{\bar\beta}] = \Psi_{\bar\beta} \in C$ and, therefore, the embedding $L^{\psi}_{\bar\beta,p} \subset L_\infty$ is true. Since under condition (16) the convolution of kernel $\Psi_{\bar\beta}$ with arbitrary function $\varphi \in U^0_p$ is a continuous function, then the set of all f, representable in the form (14) for all $x \in \mathbb{R}$, is denoted by $C^{\psi}_{\bar\beta,p}$.

For $\psi(k) = k^{-r}, r > 1$, classes $C^{\psi}_{\bar\beta,p}$ are denoted by $W^r_{\bar\beta,p}$. If $\beta_k \equiv \beta,\ \beta \in \mathbb{R}$, then the classes $W^r_{\bar\beta,p}$ are the Weyl-Nagy classes $W^r_{\beta,p}$.

The main result of this paper is the following statement.

Theorem 1. *Let $1 \le p \le \infty,\ n \in \mathbb{N}$ and $\bar\beta = \{\beta_k\}_{k=1}^{\infty}$ be an arbitrary sequence of real numbers. Then for $r \ge n+1$ the following estimates hold:*

$$\varepsilon_n(W^r_{\bar\beta,p})_C = n^{-r}\left(\frac{\|\cos t\|_{p'}}{\pi} + O(1)\left(1+\frac{1}{n}\right)^{-r}\right), \quad \frac{1}{p}+\frac{1}{p'}=1, \tag{17}$$

and

$$\varepsilon_n(W^r_{\bar\beta,1})_{L_p} = n^{-r}\left(\frac{\|\cos t\|_{p}}{\pi} + O(1)\left(1+\frac{1}{n}\right)^{-r}\right), \tag{18}$$

where $O(1)$ are quantities uniformly bounded in all analyzed parameters.

Proof. According to Theorem 4 from the work [8] and Theorem 4 from the work [9] for arbitrary $1 \le p \le \infty,\ \psi(k) > 0,\ \bar\beta = \{\beta_k\}_{k=1}^{\infty}$ and $n \in \mathbb{N}$ taking into account the convergence of the series $\sum\limits_{k=1}^{\infty}\psi(k)$ the following estimates hold:

$$\varepsilon_n(C^{\psi}_{\bar\beta,p})_C = \frac{\|\cos t\|_{p'}}{\pi}\psi(n) + O(1)\sum_{k=n+1}^{\infty}\psi(k), \quad \frac{1}{p}+\frac{1}{p'}=1, \tag{19}$$

and

$$\varepsilon_n(L^{\psi}_{\bar\beta,1})_{L_p} = \varepsilon_n(C^{\psi}_{\bar\beta,1})_{L_p} = \frac{\|\cos t\|_{p}}{\pi}\psi(n) + O(1)\sum_{k=n+1}^{\infty}\psi(k), \tag{20}$$

where $O(1)$ are quantities uniformly bounded in all analyzed parameters. We notice that instead of the condition (16) mentioned theorems have a stronger condition

$$\lim_{k\to\infty}\frac{\psi(k+1)}{\psi(k)} = 0. \tag{21}$$

But, despite of it, the proof of (19) and (20) does not require (21). The condition (21) provides the implementation of the relation

$$\sum_{k=n+1}^{\infty}\psi(k) = o(1)\psi(n), \tag{22}$$

and, therefore, guaranties that the estimates (19) and (20) are asymptotic equalities as $n \to \infty$.

Let put $\psi(k) = k^{-r}, r > 1$. Then, as noted above, $C^{\psi}_{\bar\beta,p} = W^r_{\bar\beta,p},\ 1 \le p \le \infty$, and by virtue of (19)

$$\varepsilon_n(W^r_{\bar\beta,p})_C = \frac{\|\cos t\|_{p'}}{\pi}\frac{1}{n^r} + O(1)\sum_{k=n+1}^{\infty}\frac{1}{k^r}, \quad \frac{1}{p}+\frac{1}{p'}=1. \tag{23}$$

Since for arbitrary $n \in \mathbb{N},\ r \ge n+1$

$$\begin{aligned}\sum_{k=n+1}^{\infty}\frac{1}{k^r} &< \frac{1}{(n+1)^r} + \int_{n+1}^{\infty}\frac{dt}{t^r} = \frac{1}{(n+1)^r} + \frac{1}{(r-1)(n+1)^{r-1}} = \frac{1}{(n+1)^r}\frac{r+n}{r-1} \\ &\le \frac{1}{n^r}\left(1+\frac{1}{n}\right)^{-r}\frac{2r-1}{r-1} \le \frac{1}{n^r}\left(1+\frac{1}{n}\right)^{-r}\left(2+\frac{1}{n}\right) \le 3n^{-r}\left(1+\frac{1}{n}\right)^{-r},\end{aligned} \tag{24}$$

then the estimate (17) follows from (23).

Similarly, in order to obtain the estimate (18) we need to put $\psi(k) = k^{-r}, r > 1$, taking into account the equality $C^{\psi}_{\bar{\beta},1} = W^r_{\bar{\beta},1}$ and estimate (20). We get that

$$\varepsilon_n(W^r_{\bar{\beta},1})_{L_p} = \frac{\|\cos t\|_p}{\pi}\frac{1}{n^r} + O(1)\sum_{k=n+1}^{\infty}\frac{1}{k^r}. \tag{25}$$

From formulas (24) and (25) we obtain (18). Theorem 1 is proved.

Remark 1. *If the conditions of Theorem 1 are satisfied and, in addition,*

$$\frac{r}{n} \to \infty, \tag{26}$$

then formulas (17) and (18) are asymptotic equalities.

Indeed, since the sequence $(1+1/n)^{n+1}$ is monotonically decreasing to number e, we have that

$$\left(1+\frac{1}{n}\right)^{-r} = \left(\left(1+\frac{1}{n}\right)^{n+1}\right)^{-\frac{r}{n+1}} \le e^{-\frac{r}{n+1}}. \tag{27}$$

As it follows from (27) under condition (26) $\left(1+\frac{1}{n}\right)^{-r} \to 0$, and, therefore, the estimates (17) and (18) are asymptotic equalities.

In the case of $\bar{\beta} = \{\beta_k\}_{k=1}^{\infty}$ are the stationary consequences, i.e. $\beta_k \equiv \beta,\ \beta \in \mathbb{R}$, the formulas (12) and (13) follow from (17) and (18), respectively.

The following statement follows from Theorem 1 in the case $p = 2$.

Corollary 1. *Let* $\bar{\beta} = \{\beta_k\}_{k=1}^{\infty}$ *be an arbitrary sequence of real numbers,* $n \in \mathbb{N}$ *and* $r \ge n+1$. *Then*

$$\varepsilon_n(W^r_{\bar{\beta},2})_C = \varepsilon_n(W^r_{\bar{\beta},1})_{L_2} = \frac{1}{n^r}\left(\frac{1}{\sqrt{\pi}} + O(1)\left(1+\frac{1}{n}\right)^{-r}\right), \tag{28}$$

where $O(1)$ *is quantity uniformly bounded in all analyzed parameters.*

In order to prove the first equality in (28) it is enough to use the results of the works [10] and [11]. As it follows from these results for all $r > \frac{1}{2}$ and $n \in \mathbb{N}$

$$\varepsilon_n(W^r_{\bar{\beta},2})_C = \varepsilon_n(W^r_{\bar{\beta},1})_{L_2} = \frac{1}{\sqrt{\pi}}\left(\sum_{k=n}^{\infty}\frac{1}{k^{2r}}\right)^{1/2}. \tag{29}$$

To prove the second equality in (28) it is sufficient to use the estimates (17) and (18) setting $p = p' = 2$. However, the estimate (28) can also be obtained directly from equation (29), taking into account that

$$\frac{1}{\sqrt{\pi}}\left(\sum_{k=n}^{\infty}\frac{1}{k^{2r}}\right)^{1/2} = \frac{1}{\sqrt{\pi}}\left(\frac{1}{n^r} + O(1)\sum_{k=n+1}^{\infty}\frac{1}{k^r}\right), \tag{30}$$

and using relation (24) for $r \ge n+1$.

We notice, since the Horowitz zeta function $\zeta(s,l) = \sum\limits_{m=0}^{\infty}(l+m)^{-s}, Re(s) > 1$, $Re(l) > 0$, has an integral representation

$$\zeta(s,l) = \frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{-lt}}{1-e^{-t}}dt,$$

then the formula (29) can be rewritten in the equivalent form

$$\varepsilon_n(W^r_{\bar{\beta},2})_C = \varepsilon_n(W^r_{\bar{\beta},1})_{L_2} = \frac{1}{\sqrt{\pi\Gamma(2r)}} \left(\int\limits_0^\infty \frac{t^{2r-1}e^{-nt}}{1-e^{-t}} dt \right)^{\frac{1}{2}}.$$

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References

1. A. V. Efimov, *Approximation of continuous periodic functions by Fourier sums*, Izv. Akad. Nauk SSSR, Ser. Mat. **24** (1960), 243–296. (Russian)
2. A. N. Kolmogorov, *On the order of the remainders of the Fourier series of differentiable functions*, in: A. N. Kolmogorov, Selected Works. Mathematics and Mechanics. Nauka, Moscow, (1985), pp. 179–185. (Russian)
3. G. I. Natanson, *Approximation by Fourier sums of functions possessing different structural properties on different parts of the domain of definition*, Vestn. Leningr. Univ. **19** (1961), 20–35. (Russian)
4. S. M. Nikol'skii, *An asymptotic estimation of the remainder under approximation by Fourier sums*, Dokl. Akad. Nauk SSSR **32** (1941), 386–389. (Russian)
5. S. M. Nikol'skii, *Approximation of functions in the mean by trigonometric polynomials*, Izv. Akad. Nauk SSSR, Ser. Mat. **10** (1946), 207–256. (Russian)
6. V. T. Pinkevich, *On the order of the remainders of the Fourier series of functions differentiable in the sense of Weyl*, Izv. Akad. Nauk SSSR, Ser. Mat. **4** (1940), 521–528. (Russian)
7. S. G. Selivanova, *Approximation by Fourier sums of. the functions possessing a derivative satisfying the Lipschitz condition*, Dokl. Akad. Nauk SSSR **105** (1955), 909–912. (Russian)
8. A. S. Serdyuk, *Approximation of classes of analytic functions by Fourier sums in the uniform metric*, Ukrainian Math. J. **57** (2005), no. 8, 1275–1296.
9. A. S. Serdyuk, *Approximation of classes of analytic functions by Fourier sums in the metric of the space L_p*, Ukrainian Math. J., **57** (2005), no. 10, 1635–1651.
10. A. S. Serdyuk and I. V. Sokolenko, *Uniform approximation of classes of $(\psi,\bar{\beta})-$differentiable functions by linear methods*, Approximation Theory of Functions and Related Problems, Zb. prac Inst. mat. NAN Ukr., Kyiv **8** (2011), no. 1, 181–189. (Ukrainian)
11. A. S. Serdyuk and I. V. Sokolenko, *Approximation by linear methods of classes of $(\psi,\bar{\beta})-$differentiable functions*, Approximation Theory of Functions and Related Problems, Zb. prac Inst. mat. NAN Ukr., Kyiv **10** (2013), no. 1, 245–254. (Ukrainian)
12. I. G. Sokolov, *The remainder term of the Fourier series of differentiable functions*, Dokl. Akad. Nauk SSSR **103** (1955), 23–26. (Russian)
13. S. B. Stechkin, *On the best approximation of certain classes of periodic functions by trigonometric polynomials*, Izv. Akad. Nauk SSSR, Ser. Mat. **20** (1956), 643–648. (Russian)
14. S. B. Stechkin, *An estimation of the remainders of the Fourier series of differentiable functions*, Tr. Mat. Inst. Akad. Nauk SSSR **145** (1980), 126–151. (Russian)
15. S. B. Stechkin and S. A. Telyakovskii, *On approximation of differentiable functions by trigonometric polynomials in the L metric*, Tr. Mat. Inst. Akad. Nauk SSSR **88** (1967), 20–29. (Russian)
16. A. I. Stepanets, *Classification and Approximation of Periodic Functions*, Kluwer Academic Publishers, Dordrecht, 1995.
17. A. I. Stepanets, *Methods of Approximation Theory*, Utrecht, VSP, 2005.
18. B. Sz.-Nagy, *Über gewisse Extremalfragen bei transformierten trigonometrischen Entwicklungen. 1. Periodischer Fall*, Ber. Math.-Phys. Kl. Akad. Wiss., Leipzig **90** (1938), 103–134. (German)
19. S. A. Telyakovskii, *On the norms of trigonometric polynomials and approximation of differentiable functions by the linear means of their Fourier series*, Tr. Mat. Inst. Akad. Nauk SSSR **62** (1961), 61–97. (Russian)
20. S. A. Telyakovskii, *Approximation of differentiable functions by the partial sums of their Fourier series*, Math. Notes **4** (1968), 668–673.
21. S. A. Telyakovskii, *Approximation of functions of high smoothness by Fourier sums*, Ukr. Math. J. **41** (1989), 444–451.

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Received 21/05/2019