

ENTROPY FUNCTIONALS AND THEIR EXTREMAL VALUES FOR SOLVING THE STIELTJES MATRIX MOMENT PROBLEM

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Dedicated to Yuri Arlinskii on occasion of his 70th birthday

ABSTRACT. Entropy functionals and their extremal values for solving the Stieltjes matrix moment problem are defined and investigated for the first time. Explicit formulas for the extremal values of the entropy over the set of solutions of the Stieltjes matrix moment problem are obtained. A geometric interpretation in terms of Weyl matrix intervals is presented.

1. INTRODUCTION

Entropy functionals and their extremal values have been studied by many authors (see, for example, [1], [2], [3], [5]). But similar functionals were not considered for solution interpolation problems in the matrix Stieltjes class. In this paper, entropy functionals over solutions of the Stieltjes matrix moment problem are defined and studied for the first time.

Given integers $m, n \geq 1$, we let \mathbb{C}^m denote the linear space of columns of complex numbers $x = \text{col}(x_1 \ x_2 \ \dots \ x_m)$ of size m equipped with the inner product $(x, y) = \sum_{j=1}^m \bar{x}_j y_j$. Let $\mathbb{C}^{m \times n}$ be the set of complex matrices with m rows and n columns. Denote by $\mathbb{C}_H^{m \times m}$ the set of all Hermitian matrices. A Hermitian matrix A is called nonnegative if $(x, Ax) \geq 0 \ \forall x \in \mathbb{C}^m$. By $\mathbb{C}_{\geq}^{m \times m}$ denote the set of nonnegative matrices. A nonnegative matrix A is called positive if $(x, Ax) > 0$ for any nonzero vector $x \in \mathbb{C}^m$. Let $\mathbb{C}_{>}^{m \times m}$ be the set of positive matrices. By $I_m \in \mathbb{C}^{m \times m}$ denote the identity matrix and by $O_{m \times n} \in \mathbb{C}^{m \times n}$ denote the zero matrix. We will often omit the subscripts of the identity matrix and the zero matrix if these subscripts are clear from the context. For Hermitian matrices A, B we write $A > B$ ($A \geq B$) if $A - B \in \mathbb{C}_{>}^{m \times m}$ ($A - B \in \mathbb{C}_{\geq}^{m \times m}$). If the matrix A is invertible then by A^{-*} denote the matrix $(A^{-1})^*$. If $f(z)$ is a matrix function (MF) then by $f^*(z)$ denote the MF $(f(z))^*$. Let $f(z)$ be an invertible MF. By $f^{-1}(z)$ and $f^{-*}(z)$ denote MFs $(f(z))^{-1}$ and $((f(z))^{-1})^*$ respectively. By definition $z = |z| \exp(i \arg z)$, $-\pi < \arg z \leq \pi$ and $\sqrt{z} = \sqrt{|z|} \exp\left(\frac{i \arg z}{2}\right)$.

We will also write $\mathbb{C}_+ = \{z \in \mathbb{C} : \text{Im } z > 0\}$, $\mathbb{C}_- = \{z \in \mathbb{C} : \text{Im } z < 0\}$, $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ and $\mathbb{R}_- = \{x \in \mathbb{R} : x < 0\}$.

Denote by \mathfrak{B} the σ -algebra of Borel subsets of the real line \mathbb{R} . A mapping $\sigma : \mathfrak{B} \rightarrow \mathbb{C}_{\geq}^{m \times m}$ is called a nonnegative matrix measure if

$$\sigma\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \sigma(A_j)$$

for any infinite sequence $(A_j)_{j=1}^{\infty}$ of pairwise disjoint Borel subsets of \mathbb{R} .

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We first recall some facts about the Stieltjes matrix moment problem (see [6]–[10]). Let $(s_j)_{j=0}^{2n+1}$ be an arbitrary sequence of complex $m \times m$ matrices. We consider the following block matrices:

$$H_1 = (s_{j+k})_{j,k=0}^n, \quad H_2 = (s_{j+k+1})_{j,k=0}^n, \quad T_1 = \begin{pmatrix} O_{mn \times m} & I_{mn} \\ O_{m \times m} & O_{m \times mn} \end{pmatrix},$$

$$R_1(z) = (I - zT_1)^{-1}, \quad u_2 = \text{col}(-s_0 \ -s_1 \ \dots \ -s_n), \quad u_1 = T_1' u_2, \quad v_1 = \begin{pmatrix} I_m \\ O_{mn \times m} \end{pmatrix}.$$

Assume that the block matrices H_1 and H_2 satisfy the following conditions:

$$H_1 > O, \quad H_2 > O.$$

In the Stieltjes matrix moment problem it is required to describe all matrix-valued non-negative measures σ on the half-axis \mathbb{R}_+ such that

$$s_j = \int_{\mathbb{R}_+} t^j \sigma(dt), \quad 0 \leq j \leq 2n, \quad s_{2n+1} \geq \int_{\mathbb{R}_+} t^{2n+1} \sigma(dt).$$

Let \mathcal{M}_+ denote the set of all solutions σ to the Stieltjes matrix moment problem. Under the above assumptions it is known that $\mathcal{M}_+ \neq \emptyset$. With each solution of the Stieltjes matrix moment problem we associate a MF as follows:

$$(1) \quad s(z) = \int_{\mathbb{R}_+} \frac{\sigma(dt)}{t-z}, \quad \sigma \in \mathcal{M}_+.$$

By \mathcal{F}_+ denote the set of associated MFs. It is obvious that associated MFs are holomorphic MFs in $\mathbb{C} \setminus \mathbb{R}_+$. The Stieltjes inversion formula establishes a one-to-one correspondence between \mathcal{F}_+ and \mathcal{M}_+ .

Let

$$J = \begin{pmatrix} O_{m \times m} & -iI_{m \times m} \\ iI_{m \times m} & O_{m \times m} \end{pmatrix}, \quad J_\pi = \begin{pmatrix} O_{m \times m} & I_{m \times m} \\ I_{m \times m} & O_{m \times m} \end{pmatrix}.$$

The pair of meromorphic $m \times m$ MF $\text{col}(p(z) \ q(z))$ in $\mathbb{C} \setminus \mathbb{R}_+$ is said to be *Stieltjes* if for this pair there exists a discrete the set of points \mathcal{D}_{pq} in $\mathbb{C} \setminus \mathbb{R}_+$ such that

- (1) $p^*(z)p(z) + q^*(z)q(z) > O, \quad z \in \mathbb{C} \setminus \{\mathbb{R}_+ \cup \mathcal{D}_{pq}\}.$
- (2) $(p^*(z), q^*(z)) \frac{J}{i(\bar{z}-z)} \begin{pmatrix} p(z) \\ q(z) \end{pmatrix} \geq O, \quad z \in \mathbb{C} \setminus \{\mathbb{R}_+ \cup \mathcal{D}_{pq}\}.$
- (3) $(p^*(z), q^*(z)) J_\pi \begin{pmatrix} p(z) \\ q(z) \end{pmatrix} \geq O, \quad z \in \{z \in \mathbb{C} : \Re z < 0\} \setminus \mathcal{D}_{pq}.$

On the set of Stieltjes pairs, we introduce the equivalence ratio: the pairs $\text{col}(p_1(z) \ q_1(z))$ and $\text{col}(p_2(z) \ q_2(z))$ are said to be equivalent if there exists a MF $Q(z)$ such that the MF $Q(z), (Q(z))^{-1}$ are both meromorphic in $\mathbb{C} \setminus \mathbb{R}_+$ and

$$p_1(z) = p_2(z)Q(z), \quad q_1(z) = q_2(z)Q(z).$$

The set of equivalence classes of Stieltjes pairs will be denoted by \mathcal{S}_∞ .

A polynomial MF

$$U_1(z) = \begin{pmatrix} \alpha_1(z) & \beta_1(z) \\ \gamma_1(z) & \delta_1(z) \end{pmatrix} = \begin{pmatrix} I + zv_1^* R_1(z) H_2^{-1} u_2 & -zv_1^* R_1(z) H_1^{-1} v_1 \\ u_2^* R_1(z) H_2^{-1} u_2 & I - zu_1^* R_1(z) H_1^{-1} v_1 \end{pmatrix}$$

is called *the resolvent matrix* the Stieltjes moment problem.

The formula

$$s(z) = (\gamma_1(z)p(z) + \delta_1(z)q(z)) (\alpha_1(z)p(z) + \beta_1(z)q(z))^{-1}$$

establishes a bijective correspondence between \mathcal{F}_+ and \mathcal{S}_∞ .

Substituting the Stieltjes pairs $\text{col}(I \ O)$ and $\text{col}(O \ I)$ in (1), we obtain extremal MFs

$$(2) \quad s_F(z) = \gamma_1(z)\alpha_1^{-1}(z) \in \mathcal{F}_+, \quad s_K(z) = \delta_1(z)\beta_1^{-1}(z) \in \mathcal{F}_+.$$

Suppose that $t_0 \in \mathbb{R}_-$. The matrix interval $[s_F(t_0), s_K(t_0)]$ is called *the matrix Weyl interval* (see [6]–[10]). We can prove that $\{s(x_0) : s \in \mathcal{F}_+\} = [s_F(t_0), s_K(t_0)]$.

From (1) it follows that $(s(z) - s^*(z))/i \geq O$, $z \in \mathbb{C}_+$. Consequently, at almost all points $t \geq 0$ exist nontangential limits $s(t) = \lim_{y \rightarrow +0} s(t + iy)$. Let t_0 be a point belongs to \mathbb{R}_- . For any s belongs to \mathcal{F}_+ the entropy functional $\mathcal{I}(s; t_0)$ is defined by the formula

$$\mathcal{I}(s; t_0) = \int_0^{+\infty} \ln \left(\frac{s(t) - s^*(t)}{2i} \right) \frac{\sqrt{-t_0}}{\pi \sqrt{t(t-t_0)}} dt.$$

The main result in our paper is as follows.

Theorem 1. *The entropy functional has an upper bound*

$$\mathcal{I}(s; t_0) \leq \ln \det \frac{s_K(t_0) - s_F(t_0)}{4} \quad \forall s \in \mathcal{F}_+,$$

with equality if and only if

$$\tilde{s}(z) = (\gamma_1(z)\tilde{p}(z) + \delta_1(z)\tilde{q}(z))(\alpha_1(z)\tilde{p}(z) + \beta_1(z)\tilde{q}(z))^{-1} \in \mathcal{F}_+,$$

$$\begin{pmatrix} \tilde{p}(z) \\ \tilde{q}(z) \end{pmatrix} = \begin{pmatrix} \beta_1^*(t_0)/\sqrt{t_0} \\ \alpha_1^*(t_0)/\sqrt{z} \end{pmatrix} Q(z) \in \mathcal{S}_\infty.$$

Moreover, the matrix $\tilde{s}(t_0)$ coincides with the center of the matrix Weyl interval $[s_F(t_0), s_K(t_0)]$, i.e.,

$$\tilde{s}(t_0) = \frac{s_K(t_0) + s_F(t_0)}{2}.$$

2. ENTROPY FUNCTIONALS FOR THE HAMBURGER MATRIX MOMENT PROBLEM

We first recall some facts about the Hamburger matrix moment problem (see, for example, [4], [5], [12], [13]). Let $(w_j)_{j=0}^{2n}$ be an arbitrary sequence of complex $m \times m$ matrices. We consider the following block matrices

$$H = (w_{j+k})_{j,k=0}^n, \quad T = \begin{pmatrix} O_{mn \times m} & I_{mn} \\ O_{m \times m} & O_{m \times mn} \end{pmatrix}, \quad R(w) = (I - wT)^{-1},$$

$$u = \text{col}(O - w_0 - w_1 \dots - w_{n-1}), \quad v = \begin{pmatrix} I_m \\ O_{mn \times m} \end{pmatrix}.$$

Assume that the block matrix H is positive. In the Hamburger matrix moment problem it is required to describe all the matrix-valued nonnegative measures τ such that

$$(3) \quad w_j = \int_{\mathbb{R}} t^j \tau(dt), \quad 0 \leq j \leq 2n-1, \quad w_{2n} \geq \int_{\mathbb{R}} t^{2n} \tau(dt).$$

Let \mathcal{M} denote the set of solutions to problem (3). Under the above assumptions, $\mathcal{M} \neq \emptyset$. With each matrix measure $\tau \in \mathcal{M}$ we associate a MF as follows:

$$(4) \quad f(w) = \int_{\mathbb{R}} \frac{\tau(dt)}{t-w}.$$

By \mathcal{F} denote the set of associated MFs f . It is obvious that associated MFs are holomorphic MFs in \mathbb{C}_+ . The Stieltjes inversion formula establishes a one-to-one correspondence between \mathcal{F} and \mathcal{M} .

A pair of meromorphic $m \times m$ MF $\text{col}(p(w) \ q(w))$ in \mathbb{C}_+ is said to be *Nevanlinna* if for this pair there exists the discrete set of points \mathcal{D}_{pq} in \mathbb{C}_+ such that

- (1) $p^*(w)p(w) + q^*(w)q(w) > O$, $w \in \mathbb{C}_+ \setminus \mathcal{D}_{pq}$.
- (2) $(p^*(w)q^*(w)) \frac{J}{i(\bar{z}-z)} \begin{pmatrix} p(w) \\ q(w) \end{pmatrix} \geq O$, $w \in \mathbb{C}_+ \setminus \mathcal{D}_{pq}$.

On the set of Nevanlinna pairs, we introduce the equivalence ratio: the pairs

$$\operatorname{col}(p_1(w) \ q_1(w)) \quad \text{and} \quad \operatorname{col}(p_2(w) \ q_2(w))$$

are said to be equivalent if there exists a MF $Q(w)$ such that the MF $Q(w)$, $(Q(w))^{-1}$ are both meromorphic in \mathbb{C}_+ and

$$p_1(w) = p_2(w)Q(w), \quad q_1(w) = q_2(w)Q(w).$$

The set of equivalence classes of Nevanlinna pairs will be denoted by \mathcal{R}_∞ .

A polynomial MF

$$(5) \quad U(w) = \begin{pmatrix} \alpha(w) & \beta(w) \\ \gamma(w) & \delta(w) \end{pmatrix} = \begin{pmatrix} I + zv^*R(w)H^{-1}u & -zv^*R(w)H^{-1}v \\ zu^*R(w)H^{-1}u & I - zu^*R(w)H^{-1}v \end{pmatrix}$$

is called *the resolvent matrix* the Hamburger moment problem.

The formula

$$(6) \quad f(w) = (\gamma(w)p(w) + \delta(w)q(w)) (\alpha(w)p(w) + \beta(w)q(w))^{-1}$$

establishes a bijective correspondence between \mathcal{F} and \mathcal{R}_∞ .

By \mathbf{S} denote the set of $m \times m$ MFs $S(w)$ which are analytic and contractive (i.e., $S^*(w)S(w) \leq I_m$) in \mathbb{C}_+ .

It is well known that the formula

$$(7) \quad p(w) = (I + S(w))Q(w), \quad q(w) = i(I - S(w))Q(w)$$

establishes a bijective correspondence between \mathbf{S} and \mathcal{R}_∞ . Here MFs $Q(z)$, $(Q(z))^{-1}$ are both meromorphic in \mathbb{C}_+ .

By definition, put

$$(8) \quad E(w) = \alpha(w) + i\beta(w), \quad F(w) = \alpha(w) - i\beta(w),$$

$$(9) \quad G(w) = \gamma(w) + i\delta(w), \quad H(w) = \gamma(w) - i\delta(w).$$

Combining (6), (7), (8), and (9), we get

$$f(w) = (G(w) + H(w)S(w)) (E(w) + F(w)S(w))^{-1}.$$

From (4) follows that $(f(w) - f^*(w))/i \geq O$, $w \in \mathbb{C}_+$. Consequently, at almost all points $x \in \mathbb{R}$ exist nontangential limits $f(x) = \lim_{y \rightarrow +0} f(x + iy)$. Let $w_0 = x_0 + iy_0$ be a point belongs to \mathbb{C}_+ . For any f belongs to \mathcal{F} the entropy functional $\mathcal{I}(f; w_0)$ is defined by the formula

$$(10) \quad \mathcal{I}(f; w_0) = \int_{\mathbb{R}} \ln \left(\det \frac{f(x) - f^*(x)}{2i} \right) \frac{y_0}{\pi((x - x_0)^2 + y_0^2)} dx.$$

The following theorem was proved in [5].

Theorem 2. *The entropy functional has an upper bound*

$$(11) \quad \mathcal{I}(f; w_0) \leq \ln \det \left(2i(\bar{w}_0 - w_0)v^*R(w_0)H^{-1}R^*(w_0)v \right)^{-1} \quad \forall f \in \mathcal{F},$$

with equality if and only if

$$\begin{aligned} \tilde{f}(w) &= (G(w) + H(w)\tilde{S}) (E(w) + F(w)\tilde{S})^{-1} \in \mathcal{F}, \\ \tilde{S} &= -F^*(w_0)E^{-*}(w_0) \in \mathbf{S}. \end{aligned}$$

In other words,

$$\begin{aligned} \tilde{f}(w) &= (\gamma(w)\tilde{p}(w) + \delta(w)\tilde{q}(w)) (\alpha(w)\tilde{p}(w) + \beta(w)\tilde{q}(w))^{-1} \in \mathcal{F}, \\ \begin{pmatrix} \tilde{p}(w) \\ \tilde{q}(w) \end{pmatrix} &= \begin{pmatrix} -\beta^*(w_0) \\ \alpha^*(w_0) \end{pmatrix} Q(w) \in \mathcal{R}_\infty. \end{aligned}$$

3. ENTROPY FUNCTIONALS FOR THE SYMMETRIC HAMBURGER MOMENT PROBLEM

Let $(w_j)_{j=0}^{4n+2}$ be a sequence of complex $m \times m$ matrices such that

$$H = (w_{j+k})_{j,k=0}^{2n+1} > O.$$

Corresponding moment problem (3) is said to be the *symmetric* Hamburger matrix moment problem if $w_{2j+1} = O_{m \times m}$, $j = 0, \dots, 2n$. The symmetric moment problem we will study in these chapter.

If we replace w_{2j} , $j = 0, \dots, 2n + 1$ by s_j , we obtain a sequence

$$(12) \quad s_0, O, s_1, O, s_2, O, \dots, O, s_{2n+1}.$$

Given the sequence (12) of $m \times m$ matrices, we construct the following block matrices:

$$(13) \quad H = \begin{pmatrix} s_0 & O & s_1 & O & \dots & s_n & O \\ O & s_1 & O & s_2 & \dots & O & s_{n+1} \\ s_1 & O & s_2 & O & \dots & s_{n+1} & O \\ O & s_2 & O & s_3 & \dots & O & s_{n+2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ s_n & O & s_{n+1} & O & \dots & s_{2n} & O \\ O & s_{n+1} & O & s_{n+2} & \dots & O & s_{2n+1} \end{pmatrix} \in \mathbb{C}^{(2n+2)m \times (2n+2)m},$$

$$T = \begin{pmatrix} O_{(2n+1)m \times m} & I_{(2n+1)m} \\ O_{m \times m} & O_{m \times (2n+1)m} \end{pmatrix}, \quad v = \begin{pmatrix} I_m \\ O_{(2n+1)m \times m} \end{pmatrix},$$

$$u = \text{col}(O - s_0 \ O - s_1 \ \dots - s_n) \in \mathbb{C}^{m \times (2n+2)m}, \quad R(w) = (I_{(2n+2)m} - wT)^{-1},$$

$$H_1 = (s_{j+k})_{j,k=0}^n, \quad H_2 = (s_{j+k+1})_{j,k=0}^n,$$

$$T_1 = \begin{pmatrix} O_{mn \times m} & I_{mn} \\ O_{m \times m} & O_{m \times mn} \end{pmatrix}, \quad v_1 = \begin{pmatrix} I_m \\ O_{mn \times m} \end{pmatrix},$$

$$R_1(w) = (I_{(n+1)m} - wT_1)^{-1}, \quad u_2 = \text{col}(-s_0 - s_1 \ \dots - s_n), \quad u_1 = T_1^* u_2.$$

It is easily verified that the block matrices defined above satisfy the main identity

$$(14) \quad T_1^* H_2 - H_1 = v_1 u_2^*.$$

By P denote block matrix

$$P = \begin{pmatrix} I & O & O & O & \dots & O & O \\ O & O & I & O & \dots & O & O \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & O & \dots & I & O \\ O & I & O & O & \dots & O & O \\ O & O & O & I & \dots & O & O \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & O & \dots & O & I \end{pmatrix} \in \mathbb{C}^{(2n+2)m \times (2n+2)m}.$$

These matrix has exactly one entry of 1 in each row and each column and 0s elsewhere, i.e., P is a permutation matrix. In particular, the matrix P is orthogonal $P'P = PP' = I$.

It is easy to see that

$$(15) \quad Pv = \begin{pmatrix} v_1 \\ O_{(n+1)m \times m} \end{pmatrix}, \quad Pu = \begin{pmatrix} O_{(n+1)m \times m} \\ u_2 \end{pmatrix},$$

$$(16) \quad PHP' = \begin{pmatrix} H_1 & O_{(n+1)m \times (n+1)m} \\ O_{(n+1)m \times (n+1)m} & H_2 \end{pmatrix},$$

$$(17) \quad v^*R(w)P' = (v_1^*R_1(w^2) \quad wv_1^*R_1(w^2)), \quad u^*R(w)P' = (A(w) \quad u_2^*R_1(w^2)).$$

Here by $A(w)$ denote some polynomial $m \times m(n+1)$ MF.

It follows from (13), (16) that

$$(18) \quad H > O \Leftrightarrow H_1 > O \wedge H_2 > O.$$

This implies that

$$H^{-1} = P' \begin{pmatrix} H_1^{-1} & O_{(n+1)m \times (n+1)m} \\ O_{(n+1)m \times (n+1)m} & H_2^{-1} \end{pmatrix} P.$$

Theorem 3. *Let*

$$U(w) = \begin{pmatrix} \alpha(w) & \beta(w) \\ \gamma(w) & \delta(w) \end{pmatrix} = \begin{pmatrix} I + wv^*R(w)H^{-1}u & -wv^*R(w)H^{-1}v \\ wu^*R(w)H^{-1}u & I - wu^*R(w)H^{-1}v \end{pmatrix}$$

be a resolvent matrix for the Hamburger symmetric matrix moment problem that corresponds to sequence (12). Then

$$(19) \quad \begin{aligned} U(w) &= \begin{pmatrix} I + w^2v_1^*R_1(w^2)H_2^{-1}u_2 & -wv_1^*R_1(w^2)H_1^{-1}v_1 \\ wu_2^*R_1(w^2)H_2^{-1}u_2 & I - w^2u_1^*R_1(w^2)H_1^{-1}v_1 \end{pmatrix}, \\ 2i(\bar{w}_0 - w_0)v^*R(w_0)H^{-1}R^*(w_0)v &= v_1^*R_1(w_0^2)H_1^{-1}R_1^*(w_0^2)v_1 \\ &\quad + w_0\bar{w}_0v_1^*R_1(w_0^2)H_2^{-1}R_1^*(w_0^2)v_1. \end{aligned}$$

Proof. Using (15), (16), and (17), we get

$$\begin{aligned} \alpha(w) &= I + wv^*R(w)P' \begin{pmatrix} H_1^{-1} & O_{(n+1)m \times (n+1)m} \\ O_{(n+1)m \times (n+1)m} & H_2^{-1} \end{pmatrix} Pu \\ &= I + w(v_1^*R_1(w^2) \quad wv_1^*R_1(w^2)) \begin{pmatrix} H_1^{-1} & O \\ O & H_2^{-1} \end{pmatrix} \begin{pmatrix} O \\ u_2 \end{pmatrix} \\ &= I + w^2v_1^*R_1(w^2)H_2^{-1}u_2. \end{aligned}$$

Formulas

$$\begin{aligned} \beta(w) &= -wv_1^*R_1(w^2)H_1^{-1}v_1, \\ \gamma(w) &= wu_2^*R_1(w^2)H_2^{-1}u_2, \\ \delta(w) &= I - w^2u_1^*R_1(w^2)H_1^{-1}v_1, \end{aligned}$$

and (19) are proved by analogy. This completes the proof of theorem 3. \square

Lemma 1. *Let $U(z)$ be a resolvent matrix for the Hamburger symmetric matrix moment problem that corresponds to sequence (12). Then*

$$(20) \quad U(iy)J_\pi U^*(iy) - J_\pi = O_{2m \times 2m}, \quad U^*(iy)J_\pi U(iy) - J_\pi = O_{2m \times 2m} \quad \forall y > 0.$$

Proof. We have

$$\begin{aligned} U(iy)J_\pi U^*(iy) - J_\pi &= \begin{pmatrix} -iyv_1^*R_1(-y^2)H_1^{-1}v_1 & I - y^2v_1^*R_1(-y^2)H_2^{-1}u_2 \\ I + y^2u_1^*R_1(-y^2)H_1^{-1}v_1 & iyu_2^*R_1(-y^2)H_2^{-1}u_2 \end{pmatrix} \\ &\times \begin{pmatrix} I - y^2u_2^*H_2^{-1}R_1^*(-y^2)v_1 & -iyu_2^*H_2^{-1}R_1^*(-y^2)u_2 \\ iyv_1^*H_1^{-1}R_1^*(-y^2)v_1 & I + y^2v_1^*H_1^{-1}R_1^*(-y^2)u_1 \end{pmatrix} - J_\pi = \begin{pmatrix} A_{11}(y) & A_{12}(y) \\ A_{21}(y) & A_{22}(y) \end{pmatrix}. \end{aligned}$$

Using (14), we get

$$\begin{aligned}
A_{11}(y) &= (-iyv_1^*R_1(-y^2)H_1^{-1}v_1)(I - y^2u_2^*H_2^{-1}R_1^*(-y^2)v_1) \\
&+ (I - y^2v_1^*R_1(-y^2)H_2^{-1}u_2)(iyv_1^*H_1^{-1}R_1^*(-y^2)v_1) \\
&= -iyv_1^*R_1(-y^2)H_1^{-1}v_1 + iy^3v_1^*R_1(-y^2)H_1^{-1}v_1u_2^*H_2^{-1}R_1^*(-y^2)v_1 \\
&+ iyv_1^*H_1^{-1}R_1^*(-y^2)v_1 - iy^3v_1^*R_1(-y^2)H_2^{-1}u_2v_1^*H_1^{-1}R_1^*(-y^2)v_1 \\
&= -iyv_1^*R_1(-y^2)H_1^{-1}v_1 + iy^3v_1^*R_1(-y^2)H_1^{-1}(T_1^*H_2 - H_1)H_2^{-1}R_1^*(-y^2)v_1 \\
&+ iyv_1^*H_1^{-1}R_1^*(-y^2)v_1 - iy^3v_1^*R_1(-y^2)H_2^{-1}(H_2T_1 - H_1)H_1^{-1}R_1^*(-y^2)v_1 \\
&= iyv_1^*R_1(-y^2)H_1^{-1}(-R_1^*(-y^2) + y^2(T_1^*H_2 - H_1)H_2^{-1})R_1^*(-y^2)v_1 \\
&+ iyv_1^*R_1(-y^2)(R_1^{-1}(-y^2) - y^2H_2^{-1}(H_2T_1 - H_1))H_1^{-1}R_1^*(-y^2)v_1 \\
&= iyv_1^*R_1(-y^2)H_1^{-1}(-I - y^2T_1^* + y^2T_1^* - y^2H_1H_2^{-1})R_1^*(-y^2)v_1 \\
&+ iyv_1^*R_1(-y^2)(I + y^2T_1 - y^2T_1 + y^2H_2^{-1}H_1)H_1^{-1}R_1^*(-y^2)v_1 \\
&= -iyv_1^*R_1(-y^2)(H_1^{-1} + y^2H_2^{-1})R_1^*(-y^2)v_1 \\
&+ iyv_1^*R_1(-y^2)(H_1^{-1} + y^2H_2^{-1})R_1^*(-y^2)v_1 = O_{m \times m}.
\end{aligned}$$

Thus we have $A_{11}(y) = O_{m \times m}$. The equalities $A_{12}(y) = A_{21}(y) = A_{22}(y) = O_{m \times m}$ are proved in a similar way. Finally, we obtain $U(iy)J_\pi U^*(iy) - J_\pi = O_{2m \times 2m}$. It follows (see, for example, [1]) that $U^*(iy)J_\pi U(iy) - J_\pi = O_{2m \times 2m}$. Lemma 1 is proved. \square

A Nevanlinna pair $\text{col}(p(w) \ q(w))$ is called *symmetric* if

$$(p^*(iy) \ q^*(iy))J_\pi \begin{pmatrix} p(iy) \\ q(iy) \end{pmatrix} = O, \quad y > 0.$$

If some Nevanlinna pair is symmetric then all equivalent Nevanlinna pairs is also symmetric. The corresponding equivalence class of Nevanlinna pairs is called symmetric. The set of equivalence classes of Nevanlinna symmetric pairs will be denoted by $\tilde{\mathcal{R}}_\infty$.

The Nevanlinna MF $f(w) \in \mathcal{R}$ is called *symmetric* if Nevanlinna pair $\text{col}(I \ f(w))$ is symmetric, i.e.,

$$f(iy) + f^*(iy) = O, \quad y > 0.$$

The set of Nevanlinna symmetric MF will be denoted by $\tilde{\mathcal{R}}$.

Consider the symmetric Hamburger moment problem. By definition, put

$$\tilde{\mathcal{F}} = \{f(w) \in \mathcal{F} \mid f(w) \in \tilde{\mathcal{R}}\}.$$

Theorem 4. *Let*

$$(21) \quad U(w) = \begin{pmatrix} \alpha(w) & \beta(w) \\ \gamma(w) & \delta(w) \end{pmatrix} = \begin{pmatrix} I + w^2v_1^*R_1(w^2)H_2^{-1}u_2 & -wv_1^*R_1(w^2)H_1^{-1}v_1 \\ wu_2^*R_1(w^2)H_2^{-1}u_2 & I - w^2u_1^*R_1(w^2)H_1^{-1}v_1 \end{pmatrix}$$

be a resolvent matrix for the Hamburger symmetric matrix moment problem that corresponds to sequence (12). Then the formula

$$f(w) = (\gamma(w)p(w) + \delta(w)q(w))(\alpha(w)p(w) + \beta(w)q(w))^{-1}$$

establishes a bijective correspondence between $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{R}}_\infty$.

Proof. By $f, p, q, \alpha, \beta, \gamma, \delta$ denote $f(iy), p(iy), q(iy), \alpha(iy), \beta(iy), \gamma(iy), \delta(iy)$, $y > 0$ respectively. We have

$$\begin{aligned}
f + f^* &= \begin{pmatrix} I & f^* \end{pmatrix} J_\pi \begin{pmatrix} I \\ f \end{pmatrix} \\
&= \begin{pmatrix} I & (\alpha p + \beta q)^{-*} (p^* \gamma^* + q^* \delta^*) \end{pmatrix} J_\pi \begin{pmatrix} I \\ (\gamma p + \delta q) (\alpha p + \beta q)^{-1} \end{pmatrix} \\
&= (\alpha p + \beta q)^{-*} \begin{pmatrix} p^* \alpha^* + q^* \beta^* & p^* \gamma^* + q^* \delta^* \end{pmatrix} J_\pi \begin{pmatrix} \alpha p + \beta q \\ \gamma p + \delta q \end{pmatrix} (\alpha p + \beta q)^{-1} \\
&= (\alpha p + \beta q)^{-*} \begin{pmatrix} p^* & q^* \end{pmatrix} U^* J_\pi U \begin{pmatrix} p \\ q \end{pmatrix} (\alpha p + \beta q)^{-1} \\
&= (\alpha p + \beta q)^{-*} \begin{pmatrix} p^* & q^* \end{pmatrix} J_\pi \begin{pmatrix} p \\ q \end{pmatrix} (\alpha p + \beta q)^{-1}.
\end{aligned}$$

Thus

$$\begin{aligned}
f(iy) + f^*(iy) &= (\alpha(iy)p(iy) + \beta(iy)q(iy))^{-*} \\
&\quad \times \begin{pmatrix} p^*(iy) & q^*(iy) \end{pmatrix} J_\pi \begin{pmatrix} p(iy) \\ q(iy) \end{pmatrix} (\alpha(iy)p(iy) + \beta(iy)q(iy))^{-1}, \quad y > 0.
\end{aligned}$$

This implies that

$$f \in \tilde{\mathcal{F}} \Leftrightarrow \text{col} \begin{pmatrix} p & q \end{pmatrix} Q \in \tilde{\mathcal{R}}_\infty.$$

Theorem 4 is proved. \square

Denote by $\tilde{\mathcal{I}}(f; iy)$, $y > 0$ the restriction of the entropy functional (10) to $\tilde{\mathcal{F}} \subset \mathcal{F}$.

Theorem 5. *The entropy functional has an upper bound*

$$\begin{aligned}
(22) \quad \tilde{\mathcal{I}}(f; iy) &\leq \ln \det \left(4y \left(v_1^* R_1(-y^2) H_1^{-1} R_1^*(-y^2) v_1 \right. \right. \\
&\quad \left. \left. + y^2 v_1^* R_1(-y^2) H_2^{-1} R_1^*(-y^2) v_1 \right) \right)^{-1}
\end{aligned}$$

for all $f \in \tilde{\mathcal{F}}$ with equality if and only if

$$\tilde{f}(w) = (\gamma(w)\tilde{p}(w) + \delta(w)\tilde{q}(w)) (\alpha(w)\tilde{p}(w) + \beta(w)\tilde{q}(w))^{-1} \in \tilde{\mathcal{F}},$$

$$\begin{pmatrix} \tilde{p}(w) \\ \tilde{q}(w) \end{pmatrix} = \begin{pmatrix} -\beta^*(iy) \\ \alpha^*(iy) \end{pmatrix} Q(w) \in \tilde{\mathcal{R}}_\infty.$$

Proof. Using (11), (19), we get (22). Next we have

$$\begin{aligned}
\begin{pmatrix} \tilde{p}^*(w) & \tilde{q}^*(w) \end{pmatrix} J_\pi \begin{pmatrix} \tilde{p}(w) \\ \tilde{q}(w) \end{pmatrix} &= Q^*(w) \begin{pmatrix} -\beta(iy) & \alpha(iy) \end{pmatrix} J_\pi \begin{pmatrix} -\beta^*(iy) \\ \alpha^*(iy) \end{pmatrix} Q(w) \\
&= Q^*(w) (-\beta(iy)\alpha^*(iy) - \alpha(iy)\beta^*(iy)) Q(w) = 0.
\end{aligned}$$

Here we used the formula (20). Thus

$$(23) \quad \begin{pmatrix} -\beta^*(iy) \\ \alpha^*(iy) \end{pmatrix} Q(w) \in \tilde{\mathcal{R}}_\infty.$$

Using (23), Theorem 4, and Theorem 2, we get Theorem 5. \square

4. ENTROPY FUNCTIONALS FOR THE STIELTJES MOMENT PROBLEM

Let

$$(24) \quad s_0, O, s_1, O, s_2, O, \dots, O, s_{2n+1}.$$

be a sequence of $m \times m$ matrices such that block matrix H (see (13)) is positive. Consider the Hamburger symmetric matrix moment problem that corresponds to sequence (24) and denote by $\tilde{\mathcal{F}}$ the set of symmetric associate MF.

Consider the sequence

$$s_0, s_1, \dots, s_{2n+1}.$$

Using (18), we get $H_1 > O$, $H_2 > O$. Denote by \mathcal{F}_+ the set MF that are associated with the Stieltjes matrix moment problem

$$s_j = \int_{\mathbb{R}_+} t^j \sigma(dt), \quad 0 \leq j \leq 2n, \quad s_{2n+1} \geq \int_{\mathbb{R}_+} t^{2n+1} \sigma(dt).$$

Lemma 2. *The formula*

$$\left(\begin{array}{c} p_1(w) \\ \frac{1}{w} g_1(w) \end{array} \right) \Big|_{w^2=z} = \left(\begin{array}{c} p(z) \\ g(z) \end{array} \right)$$

establishes a bijective correspondence between $\tilde{\mathcal{R}}_\infty$ and \mathcal{S}_∞ .

Moreover, the formula

$$\frac{1}{w} f(w) \Big|_{w^2=z} = s(z)$$

establishes a bijective correspondence between $\tilde{\mathcal{F}}$ and \mathcal{F}_+ .

Proof. The first statement of the lemma is obvious (see [12], [14]). Let us prove the second statement of the lemma. If $f \in \tilde{\mathcal{F}}$, then there exists a pair $\text{col} \left(\begin{array}{c} p_1(w) \\ g_1(w) \end{array} \right) \in \tilde{\mathcal{R}}_\infty$ such that $f(w) = (\gamma(w)p_1(w) + \delta(w)g_1(w)) (\alpha(w)p_1(w) + \beta(w)g_1(w))^{-1}$.

Consequently

$$\begin{aligned} \frac{1}{w} f(w) \Big|_{w^2=z} &= \frac{1}{w} \left(\gamma(w)p_1(w) + \delta(w)g_1(w) \right) \left(\alpha(w)p_1(w) + \beta(w)g_1(w) \right)^{-1} \Big|_{w^2=z} \\ &= \left(\frac{\gamma(w)}{w} p_1(w) + \delta(w) \frac{g_1(w)}{w} \right) \left(\alpha(w)p_1(w) + w\beta(w) \frac{g_1(w)}{w} \right)^{-1} \Big|_{w^2=z} \\ &= \left(\frac{wu_2^* R_1(w^2) H_2^{-1} u_2}{w} p_1(w) + (I - w^2 u_1^* R_1(w^2) H_1^{-1} v_1) \frac{g_1(w)}{w} \right) \\ &\quad \times \left((I + w^2 v_1^* R_1(w^2) H_2^{-1} u_2) p_1(w) + (-w^2 v_1^* R_1(w^2) H_1^{-1} v_1) \frac{g_1(w)}{w} \right)^{-1} \Big|_{w^2=z} \\ &= \left(\gamma_1(z)p(z) + \delta_1(z)q(z) \right) \left(\alpha_1(z)p(z) + \beta_1(z)q(z) \right)^{-1} = s(z) \in \mathcal{F}_+. \end{aligned}$$

Lemma 2 is proved. \square

Proof of Theorem 1. The proof is divided into steps.

Step 1. If $f(w)$ and $s(z)$ are as in Lemma 2 and

$$\tilde{\mathcal{I}}(f; iy_0) = \int_{\mathbb{R}} \ln \left(\det \frac{f(x) - f^*(x)}{2i} \right) \frac{y_0}{\pi(x^2 + y_0^2)} dx, \quad y_0 > 0,$$

then

$$(25) \quad \tilde{\mathcal{I}}(f; iy_0) = \frac{m}{2} \ln(-t_0) + \int_0^\infty \ln \det \left\{ \frac{s(t) - s^*(t)}{2i} \right\} \cdot \frac{\sqrt{-t_0}}{\pi(t - t_0)} \frac{dt}{\sqrt{t}}, \quad t_0 = -y_0^2.$$

Proof of step 1. We have

$$\begin{aligned}\tilde{\mathcal{I}}(f; iy_0) &= \int_{\mathbb{R}} \ln \left(\det \frac{f(x) - f^*(x)}{2i} \right) \frac{y_0}{\pi(x^2 + y_0^2)} dx \\ &= 2 \int_{\mathbb{R}_+} \ln \left(\det x \frac{f(x) - f^*(x)}{2i} \right) \frac{y_0}{\pi(x^2 + y_0^2)} dx.\end{aligned}$$

Substituting x^2 for t in the last integral, we get

$$\tilde{\mathcal{I}}(f; iy_0) = \frac{m}{2} \int_0^\infty \ln(t) \cdot \frac{\sqrt{-t_0}}{\pi(t-t_0)} \frac{dt}{\sqrt{t}} + \int_0^\infty \ln \det \left\{ \frac{s(t) - s(t)^*}{2i} \right\} \cdot \frac{\sqrt{-t_0}}{\pi(t-t_0)} \frac{dt}{\sqrt{t}}.$$

But

$$\frac{m}{2} \int_0^\infty \ln(t) \cdot \frac{\sqrt{-t_0}}{\pi(t-t_0)} \frac{dt}{\sqrt{t}} = \frac{m}{2} \ln(-t_0).$$

Indeed, we get this result if substituting \sqrt{t} for x in the last integral and using the formula (see [11], p. 546)

$$\int_0^\infty \frac{\ln x dx}{x^2 + a^2} = \frac{\pi}{2a} \ln a, \quad a > 0.$$

Combining last formulas, we get (25).

Step 2. There is an inequality

$$(26) \quad \int_0^\infty \ln \det \left\{ \frac{s(t) - s(t)^*}{2i} \right\} \cdot \frac{\sqrt{-t_0}}{\pi(t-t_0)} \frac{dt}{\sqrt{t}} \leq \ln \det \frac{s_K(t_0) - s_F(t_0)}{4} \quad \forall s \in \mathcal{F}_+.$$

Proof of step 2. Using Theorem 5 and (25), we get

$$\begin{aligned}\frac{m}{2} \ln(-t_0) + \int_0^\infty \ln \det \left\{ \frac{s(t) - s^*(t)}{2i} \right\} \cdot \frac{\sqrt{-t_0}}{\pi(t-t_0)} \frac{dt}{\sqrt{t}} \\ \leq \ln \det \left(4\sqrt{-t_0} \left(v_1^* R_1(t_0) H_1^{-1} R_1^*(t_0) v_1 - t_0 v_1^* R_1(t_0) H_2^{-1} R_1^*(t_0) v_1 \right) \right)^{-1}.\end{aligned}$$

From [6], we get the following formula:

$$\left(v_1^* R_1(t_0) H_1^{-1} R_1^*(t_0) v_1 - t_0 \bar{w}_0 v_1^* R_1(t_0) H_2^{-1} R_1^*(t_0) v_1 \right)^{-1} = -t_0 (s_K(t_0) - s_F(t_0)).$$

This yields that

$$\begin{aligned}\frac{m}{2} \ln(-t_0) + \int_0^\infty \ln \det \left\{ \frac{s(t) - s^*(t)}{2i} \right\} \cdot \frac{\sqrt{-t_0}}{\pi(t-t_0)} \frac{dt}{\sqrt{t}} \\ \leq \ln \det \left((4\sqrt{-t_0})^{-1} (-t_0) (s_K(t_0) - s_F(t_0)) \right) = \ln \det \left(\sqrt{-t_0} \frac{s_K(t_0) - s_F(t_0)}{4} \right) \\ = \ln \left((-t_0)^{\frac{m}{2}} \det \frac{s_K(t_0) - s_F(t_0)}{4} \right) = \frac{m}{2} \ln(-t_0) + \ln \det \frac{s_K(t_0) - s_F(t_0)}{4}.\end{aligned}$$

It immediately follows (26).

Step 3. In inequality (26) we have equality if and only if

$$\begin{aligned}\tilde{s}(z) &= \left(\gamma_1(z) \tilde{p}(z) + \delta_1(z) \tilde{q}(z) \right) \left(\alpha_1(z) \tilde{p}(z) + \beta_1(z) \tilde{q}(z) \right)^{-1}, \\ \begin{pmatrix} \tilde{p}(z) \\ \tilde{q}(z) \end{pmatrix} &= \begin{pmatrix} \beta_1^*(t_0)/\sqrt{t_0} \\ \alpha_1^*(t_0)/\sqrt{z} \end{pmatrix} Q(z).\end{aligned}$$

Proof of step 3. Using (5) and (21), we get

$$\begin{aligned}\alpha(iy_0) &= I + (-y_0^2) v_1^* R_1(-y_0^2) H_2^{-1} u_2 = I + t_0 v_1^* R_1(t_0) H_2^{-1} u_2 = \alpha_1(t_0), \\ -\beta(iy_0) &= -iy_0 v_1^* R_1(-y_0^2) H_1^{-1} v_1 = -\sqrt{t_0} v_1^* R_1(t_0) H_1^{-1} v_1 = \frac{\beta_1(t_0)}{\sqrt{t_0}}.\end{aligned}$$

Using Theorem 5, we get

$$\begin{aligned}
\tilde{s}(z) &= \frac{1}{w} \tilde{f}(w) \Big|_{w^2=z} \\
&= \frac{1}{w} \left(\gamma(w)(-\beta^*(iy_0)) + \delta(w)\alpha^*(iy_0) \right) \left(\alpha(w)(-\beta^*(iy_0)) + \beta(w)\alpha^*(iy_0) \right)^{-1} \Big|_{w^2=z} \\
&= \left(\frac{\gamma(w)}{w}(-\beta^*(iy_0)) + \delta(w)\frac{\alpha^*(iy_0)}{w} \right) \left(\alpha(w)(-\beta^*(iy_0)) + w\beta(w)\frac{\alpha^*(iy_0)}{w} \right)^{-1} \Big|_{w^2=z} \\
&= \left(\frac{wu_2^*R_1(w^2)H_2^{-1}u_2}{w}(-\beta^*(iy_0)) + (I - w^2u_1^*R_1(w^2)H_1^{-1}v_1)\frac{\alpha^*(iy_0)}{w} \right) \\
&\quad \times \left((I + w^2v_1^*R_1(w^2)H_2^{-1}u_2)(-\beta^*(iy_0)) + (-w^2v_1^*R_1(w^2)H_1^{-1}v_1)\frac{\alpha^*(iy_0)}{w} \right)^{-1} \Big|_{w^2=z} \\
&= \left(\gamma_1(z)\frac{\beta_1^*(t_0)}{\sqrt{t_0}} + \delta_1(z)\frac{\alpha_1^*(t_0)}{\sqrt{z}} \right) \left(\alpha_1(z)\frac{\beta_1^*(t_0)}{\sqrt{t_0}} + \beta_1(z)\frac{\alpha_1^*(t_0)}{\sqrt{z}} \right)^{-1}.
\end{aligned}$$

Thus

$$\begin{aligned}
\tilde{s}(z) &= \left(\gamma_1(z)\tilde{p}(z) + \delta_1(z)\tilde{q}(z) \right) \left(\alpha_1(z)\tilde{p}(z) + \beta_1(z)\tilde{q}(z) \right)^{-1}, \\
\begin{pmatrix} \tilde{p}(z) \\ \tilde{q}(z) \end{pmatrix} &= \begin{pmatrix} \beta_1^*(t_0)/\sqrt{t_0} \\ \alpha_1^*(t_0)/\sqrt{z} \end{pmatrix} Q(z).
\end{aligned}$$

Step 4 completes the proof.

By [6], it follows that

$$\begin{aligned}
&\begin{pmatrix} \alpha_1(t_0) & \beta_1(t_0) \\ \gamma_1(t_0) & \delta_1(t_0) \end{pmatrix} \begin{pmatrix} O & I \\ I & O \end{pmatrix} \begin{pmatrix} \alpha_1(t_0) & \beta_1(t_0) \\ \gamma_1(t_0) & \delta_1(t_0) \end{pmatrix}^* - \begin{pmatrix} O & I \\ I & O \end{pmatrix} \\
&= 2 \begin{pmatrix} (s_K(t_0) - s_F(t_0))^{-1} & (s_K(t_0) - s_F(t_0))^{-1}s_F(t_0) \\ s_F(t_0)(s_K(t_0) - s_F(t_0))^{-1} & (s_F^{-1}(t_0) - s_K^{-1}(t_0))^{-1} \end{pmatrix}.
\end{aligned}$$

Hence

$$\begin{aligned}
\alpha_1(t_0)\beta_1^*(t_0) + \beta_1(t_0)\alpha_1^*(t_0) &= 2(s_K(t_0) - s_F(t_0))^{-1}, \\
\gamma_1(t_0)\beta_1^*(t_0) + \delta_1(t_0)\alpha_1^*(t_0) &= 2s_F(t_0)(s_K(t_0) - s_F(t_0))^{-1} + I \\
&= (2s_F(t_0) + (s_K(t_0) - s_F(t_0)))(s_K(t_0) - s_F(t_0))^{-1} \\
&= (s_K(t_0) + s_F(t_0))(s_K(t_0) - s_F(t_0))^{-1}.
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
\tilde{s}(t_0) &= \left(\gamma_1(t_0)\beta_1^*(t_0) + \delta_1(t_0)\alpha_1^*(t_0) \right) \left(\alpha_1(t_0)\beta_1^*(t_0) + \beta_1(t_0)\alpha_1^*(t_0) \right)^{-1} \\
&= (s_K(t_0) + s_F(t_0))(s_K(t_0) - s_F(t_0))^{-1}(s_K(t_0) - s_F(t_0))/2 = (s_K(t_0) + s_F(t_0))/2.
\end{aligned}$$

Theorem 1 is proved. \square

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