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UPPER SEMI-FREDHOLM AND KATO SPECTRUM OF AN α-TIMES INTEGRATED SEMIGROUP

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Abstract. Let $(T(t))_{t\geq 0}$ be an α-times integrated semigroup with generator A on a Banach space X. In this paper, we show that the spectral mapping theorem holds for upper semi-Fredholm spectrum and we will give some consequences of this result. We also give an application on the Schrödinger operator.

1. Introduction

Let X be a complex Banach space, $\mathcal{B}(X)$ denote the algebra of all bounded linear operators on X and $\mathcal{C}(X)$ the set of all closed linear operators from X to X. We denote by $D(A)$, $R(A)$, $N(A)$, $\sigma(A)$, $\sigma_p(A)$, $\sigma_r(A)$ and $\sigma_{ap}(A)$, respectively, the domain, the range, the kernel, the spectrum, the point spectrum, the residual spectrum and the approximate point spectrum of an operator $A \in \mathcal{C}(X)$, (see [6]).

Next, $A \in \mathcal{C}(X)$ is called an upper semi-Fredholm if the range $R(A)$ is closed and $\dim N(A) < \infty$. The upper semi-Fredholm spectrum of A is defined by (see e.g. [12])

$$\sigma_{uf}(A) = \{\lambda \in \mathbb{C} : \lambda - A \text{ is not upper semi-Fredholm}\}.$$

Recall that a closed operator A is said to be a Kato operator or semi-regular if $R(A)$ is closed and $N(A) \subseteq R^\infty(A)$. The Kato spectrum of A is defined by

$$\sigma_k(A) = \{\lambda \in \mathbb{C} : A - \lambda \text{ is Kato}\}.$$

We say that an operator $A \in \mathcal{C}(X)$ is essentially Kato or essentially semi-regular if $R(A)$ is closed and there exists a finite-dimensional subspace $F \subseteq X$ such that $N(A) \subseteq R^\infty(A) + F$. The essentially Kato spectrum of A is defined by

$$\sigma_{ek}(A) = \{\lambda \in \mathbb{C} : A - \lambda \text{ is essentially Kato}\}.$$

For more information see [1] and [4].

An operator A is said to have a single valued extension property at $\lambda_0 \in \mathbb{C}$ (SVEP) if for every open disc $D_{\lambda_0} \subseteq \mathbb{C}$ centered at λ_0, the only analytic function $f : D_{\lambda_0} \longrightarrow D(A)$ which satisfies the equation $(A - zI)f(z) = 0$ for all $z \in D_{\lambda_0}$ is the function $f \equiv 0$. A is said to have the SVEP if A has the SVEP for every $\lambda \in \mathbb{C}$ (see [2]). Denote by

$$\mathcal{S}(A) = \{\lambda \in \mathbb{C} : A \text{ has not the SVEP at } \lambda\}.$$

Note that, $\mu \in S(A)$ if and only if there exists a sequence $(x_i)_{i\geq 0} \subseteq D(A)$ not all of them equal to zero such that $(A - \mu)x_{i+1} = x_i$ with $x_0 = 0$ and $\sup\limits_{i\geq 1} ||x_i||^{\frac{1}{i}} < \infty$, (see [7]).

Let $(S(t))_{t\geq 0}$ be a strongly continuous, uniformly exponentially bounded family of bounded linear operators on a Banach space X. Let $A : D(A) \to X$ be a closed linear

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operator with domain $D(A) \subseteq X$. If there are $\omega, \alpha \geq 0$ such that λ is in the resolvent set of A for all λ with $\mathrm{Re}(\lambda) > \omega$ and the resolvent of A is given by

$$R(\lambda, A)x = \lambda^\alpha \int_0^\infty e^{-\lambda s} S(s)x ds \quad \text{for all} \quad x \in X,$$

then A generates $(S(t))_{t\geq 0}$ as an α-times integrated semigroup on X (see e.g. [5]).

W. Arendt showed that if $A \in \mathcal{C}(X)$ generates $T(t)$ as an n-times integrated semigroup, with $n \in \mathbb{N}$, then the Abstract Cauchy Problem $u'(t) = Au(t)$, $u(0) = x$ has a classical solution for all $x \in D(A^{n+1})$. The notion of α-times integrated semigroup for arbitrary $\alpha > 0$, was introduced by M. Hieber in [11]. He showed in [10] and [11] that the functional analytic framework of α-times integrated semigroups can be used to describe Abstract Cauchy Problems of certain differential operators on $L^p(\mathbb{R}^n)$.

In [5], C. R. Day studied the point, the approximate point and residual spectra of the α-times integrated semigroups. In order to understand the behavior of the solutions in terms of the data concerning A, one seeks information about spectra of $T(t)$ in terms of spectra of A (see [5, 9]). In the same direction, we continue the development of the spectral theory of integrated semigroups.

In this paper, we show that if $(T(t))_{t\geq 0} \subset \mathcal{B}(X)$ is an α-times integrated semigroup with generator A, then

$$\sigma_{uf}(T(t)) \cup \{0\} = \left\{ \int_0^t \frac{s^{\alpha-1}}{\Gamma(\alpha)} e^{\lambda s} ds, \lambda \in \sigma_{uf}(A) \right\} \cup \{0\}.$$

We also prove that

$$\left\{ \int_0^t \frac{s^{\alpha-1}}{\Gamma(\alpha)} e^{\lambda s} ds, \lambda \in \sigma_\star(A) \right\} \cup \{0\} \subseteq \sigma_\star(T(t)) \cup \{0\},$$

where $\star \in \{k, ek\}$. Moreover, we have equality if A has the SVEP.

2. Main results

We begin by the following lemmas.

Lemma 1. *Let $(T(t))_{t\geq 0}$ be an α-times integrated semigroup with generator A on Banach space X. Then, for all $\lambda \in \mathbb{C}$, $t \geq 0$ and $n \in \mathbb{N}$,*

1. $\left(\int_0^t \frac{s^{\alpha-1}}{\Gamma(\alpha)} e^{\lambda s} ds - T(t) \right)^n x = (\lambda - A)^n D_\lambda^n(t)x$, *for all $x \in X$.*
2. $\left(\int_0^t \frac{s^{\alpha-1}}{\Gamma(\alpha)} e^{\lambda s} ds - T(t) \right)^n x = D_\lambda^n(t)(\lambda - A)^n x$, *for all $x \in D(A^n)$.*
3. $R^\infty \left(\int_0^t \frac{s^{\alpha-1}}{\Gamma(\alpha)} e^{\lambda s} ds - T(t) \right) \subseteq R^\infty(\lambda - A)$.
4. $N(\lambda - A)^n \subseteq N \left(\int_0^t \frac{s^{\alpha-1}}{\Gamma(\alpha)} e^{\lambda s} ds - T(t) \right)^n$.

Proof. Clearly, $D_\lambda(t)$ is a bounded linear operator on X. Moreover (see [5]), we have

$$\begin{aligned} \left(\int_0^t \frac{s^{\alpha-1}}{\Gamma(\alpha)} e^{\lambda s} ds - T(t) \right) x &= (\lambda - A) D_\lambda(t)x, \text{ for all } x \in X \\ &= D_\lambda(t)(\lambda - A)x, \text{ for all } x \in D(A). \end{aligned}$$

Proceeding by induction, we get the desired result. The assertions 3 and 4 follow easily from 1 and 2. □

Lemma 2. [5, Lemma 2.2] *Let $(T(t))_{t\geq 0}$ be an α-times integrated semigroup with generator A. For $\mu \neq 0$, A is bounded on* $\ker(T(t) - \mu)$ *for any $t \geq 0$.*

Lemma 3. [5, Theorem 2.6] *Let $(T(t))_{t\geq 0}$ be an α-times integrated semigroup with generator A. Then, for all $t \geq 0$*

$$\sigma_{ap}(T(t)) \cup \{0\} = \left\{ \int_0^t \frac{s^{\alpha-1}}{\Gamma(\alpha)} e^{\lambda s} ds, \lambda \in \sigma_{ap}(A) \right\} \cup \{0\}.$$

We now state the general spectral mapping theorem for upper semi-Fredholm spectrum.

Theorem 1. *Let $(T(t))_{t\geq 0}$ be an α-times integrated semigroup with generator A. Then*

$$\sigma_{uf}(T(t)) \cup \{0\} = \left\{ \int_0^t \frac{s^{\alpha-1}}{\Gamma(\alpha)} e^{\lambda s} ds, \lambda \in \sigma_{uf}(A) \right\} \cup \{0\}.$$

Proof. Let $\mu = \int_0^t \frac{s^{\alpha-1}}{\Gamma(\alpha)} e^{\lambda s} ds \notin \sigma_{uf}(T(t))$, then $\dim N(\mu - T(t)) < \infty$ and $R(\mu - T(t))$ is closed. From Lemma 1, we have $\dim N(A - \lambda) < \infty$. Now, we prove that $R(A - \lambda)$ is closed. Since $N(\mu - T(t))$ is finite dimensional, there exists a closed subspace Y of X such that $N(\mu - T(t)) \oplus Y = X$. But, $(\lambda - A)(N(\mu - T(t)) \cap D(A))$ is finite dimensional and therefore closed. Hence, we need only to show that $(\lambda - A)(Y \cap D(A))$ is closed.

Indeed, from the closed-graph theorem and the closedness of $R(\mu - T(t))$, it follows that there is a constant $C > 0$ such that $\|(\mu - T(t))x\| \geq C\|x\|$ for all $x \in Y$. From Lemma 1, we obtain that for every $x \in D(A)$, $\|(\mu - T(t))x\| \leq M\|(\lambda - A)x\|$ for some $M > 0$.

The combination of the last two inequalities gives us

$$\|(\lambda - A)x\| \geq \frac{C}{M}\|x\| \quad \text{for all} \quad x \in Y \cap D(A).$$

So that $R(\lambda - A)$ is closed. Consequently, $\lambda - A$ is upper semi Fredholm.

Conversely, let $\mu \in \sigma_{uf}(T(t))\backslash\{0\}$ and $F = N(\mu - T(t))$. Then, F is A-invariant and $T(t)$-invariant closed subspace of X. By Lemma 2, $A_{|F}$ is a bounded operator generates the α-times integrated semigroup $(T(t)_{|F})_{t\geq 0}$, which yields $\sigma_{uf}(A_{|F}) \neq \emptyset$. Now, let $\lambda_0 \in \sigma_{uf}(A_{|F})$, according to the first implication of this theorem, we obtain

$$\int_0^t \frac{s^{\alpha-1}}{\Gamma(\alpha)} e^{\lambda_0 s} ds \in \sigma_{uf}(T(t)_{|F}) = \sigma(T(t)_{|F}) = \{\mu\}.$$

Thus $\mu = \int_0^t \frac{s^{\alpha-1}}{\Gamma(\alpha)} e^{\lambda_0 s} ds$. We prove that $\lambda_0 \in \sigma_{uf}(A)$. Indeed, suppose that $A - \lambda_0$ is upper semi Fredholm, then $\dim N(\lambda_0 - A) < \infty$ and $R(A - \lambda_0)$ is closed. Since $N(\lambda_0 - A_{|F}) = D(A) \cap N(\lambda_0 - A) \subseteq N(\lambda_0 - A)$, then $\dim N(\lambda_0 - A_{|F}) < \infty$.

Next, let $(x_n)_{n\geq 0}$ be a sequence of elements of $D(A) \cap F$ such that $(\lambda_0 - A_{|F})x_n \longrightarrow y$. Since $R(\lambda_0 - A)$ is closed, there exists $x \in D(A)$ such that $y = (\lambda_0 - A)x$. As $D_{\lambda_0}(t)$ is bounded, then $D_{\lambda_0}(t)(\lambda_0 - A_{|F})x_n \longrightarrow D_{\lambda_0}(t)(\lambda_0 - A)x$, which give $(\mu - T(t))x_n \longrightarrow (\mu - T(t))x$. Hence $(\mu - T(t))x = 0$. Consequently, $x \in F$ and $y \in R(\lambda_0 - A_{|F})$, which is absurd. □

Theorem 2. *Let $(T(t))_{t\geq 0}$ be an α-times integrated semigroup with generator A. Then*

$$\left\{ \int_0^t \frac{s^{\alpha-1}}{\Gamma(\alpha)} e^{\lambda s} ds, \lambda \in \sigma_\star(A) \right\} \cup \{0\} \subseteq \sigma_\star(T(t)) \cup \{0\} \quad \textit{for all} \quad t \geq 0,$$

with $\star \in \{k, ek\}$.

Proof. Let $\mu \in \sigma_{ap}(T(t)) \cup \{0\}\backslash\sigma_\star(T(t)) \cup \{0\}$. From Lemma 3, there exists $\lambda \in \sigma_{ap}(A)$ such that $\mu = \int_0^t \frac{s^{\alpha-1}}{\Gamma(\alpha)} e^{\lambda s} ds$. Hence, $M = R^\infty(\mu - T(t))$ is closed and A-invariant.

Moreover, [5, Proposition 1.5], implies that $\tilde{T}(t) = T(t)/M$ is an α-times integrated semigroup with generator $\tilde{A} = A/M$ on X/M. If $\mu - T(t)$ is semi-regular, it follows from Lemma 1 that

$$N(\lambda - A) \subseteq N(\mu - T(t)) \subseteq R^\infty(\mu - T(t)) \subseteq R^\infty(\lambda - A).$$

Furthermore, from [8, Theorem 12.21], $\mu - \tilde{T}(t)$ is bounded below, which means that $\lambda - \tilde{A}$ is bounded below by Lemma 3. Now, if $\mu - T(t)$ is essentially semi-regular, then there exists a finite-dimensional subspace $F \subseteq X$ such that $N(\mu - T(t)) \subseteq R^\infty(\mu - T(t)) + F$. Hence

$$N(\lambda - A) \subseteq N(\mu - T(t)) \subseteq R^\infty(\mu - T(t)) + F \subseteq R^\infty(\lambda - A) + F.$$

Furthermore, by [8, Theorem 21.7], $\mu - \tilde{T}(t)$ is upper semi Fredholm and by Theorem 1 $\lambda - \tilde{A}$ is upper semi Fredholm. In both cases $R(\lambda - \tilde{A})$ is closed. So it is sufficient to prove that $R(\lambda - A)$ is closed. Now, let $(\lambda - A)x_n \longrightarrow y$, there exists $\hat{a} \in X/M$ such that $(\lambda - \tilde{A})\hat{x}_n \longrightarrow \hat{y} = (\lambda - \hat{A})\hat{a}$. Therefore $y - (\lambda - A)a \in M \subseteq R(\lambda - A)$, this implies that $y \in R(\lambda - A)$. Consequently $\lambda \notin \sigma_\star(A)$, which ends the proof. □

The following question arises in a natural way from our result.

Question 1. *Do the inclusions of this Theorem are strict ?*

Proposition 1. *Let $(T(t))_{t\geq 0}$ be an α-times integrated semigroup with generator A. Then*

$$\mathcal{S}(T(t)) \subseteq \left\{ \int_0^t \frac{s^{\alpha-1}}{\Gamma(\alpha)} e^{\lambda s} ds, \lambda \in \mathcal{S}(A) \right\}.$$

In particular, if A has the SVEP, then $T(t)$ has the SVEP for all $t \geq 0$.

Proof. Let $\mu \in \mathcal{S}(T(t))$, by [5, Theorem 3.9], there exists $\lambda_0 \in \sigma_{ap}(A)$ such that $\mu = \int_0^t \frac{s^{\alpha-1}}{\Gamma(\alpha)} e^{\lambda_0 s} ds$. This implies there exists a sequence $(x_i)_{i\geq 0} \subseteq X$ such that $x_i = (\mu - T(t))x_{i+1}$, where $x_0 = 0$ and $\sup\limits_{i\geq 1} \|x_i\|^{\frac{1}{i}} < \infty$. We put $y_0 = 0$ and $y_i = D^i_{\lambda_0}(t)x_i$ for all $i \geq 0$, then $(y_i)_{i\geq 0} \subseteq D(A)$ and we have

$$\begin{aligned}(\lambda_0 - A)y_i &= (\lambda_0 - A)D^i_{\lambda_0}(t)x_i \\ &= D^{i-1}_{\lambda_0}(t)x_{i-1} \\ &= y_{i-1}.\end{aligned}$$

Then $\|y_i\| = \|D^i_{\lambda_0}(t)x_i\| \leq \|D^i_{\lambda_0}(t)\|\|x_i\| \leq M^i\|x_i\|$ and $\sup\limits_{i\geq 1} \|y_i\|^{\frac{1}{i}} \leq M \sup\limits_{i\geq 1} \|x_i\|^{\frac{1}{i}} < \infty$, which implies that $\lambda_0 \in \mathcal{S}(A)$. □

Remark 1. P. Aiena proved that if a bounded linear operator A has the SVEP, then $\sigma_k(A) = \sigma_{ap}(A)$ and $\sigma_{ek}(A) = \sigma_{uf}(A)$. By using the same proof in [1, Corollary 2.45] and [1, Corollary 3.53], we can easily shows that our results remain true in the case of a closed linear operator. Therefore, the inclusion of the Theorem 2 will become equality when the generator A has the SVEP.

Example 1. Take $Af = f'$ on $X = C[0,1]$ with domain $D(A) = \{f \in C^1([0,1]) : f(1) = 0\}$. The once integrated semigroup generated by A is given by the operators $S_t f(x) = \int_x^m f(s)ds$, where $m = \min\{1, x+t\}$. We can easily show that $\sigma(A) = \emptyset$. Then $\sigma_\star(S_t) = \{0\}$ with $\star \in \{uf, k, ek\}$.

Example 2. Consider the Schrödinger operator $A = i.\Delta$ on $L^p(\mathbb{R})$ for $p \geq 1$. A generates an integrated semigroup $(T(t))_{t\geq 0}$ given by $f \mapsto \mathcal{F}^{-1}(u_t \mathcal{F} f)$, with $u_t(\xi) = \int_0^t e^{-is|\xi|^2} ds$.

It is well known that $\sigma(A) = i\mathbb{R}^-$. Then A has the SVEP and from Proposition 1, $T(t)$ has the SVEP for all $t > 0$. According to the previous results, we obtain

$$\sigma_\star\big(T(t)\big)\backslash\{0\} = \big\{\frac{e^{irt}-1}{ir} : r \leq 0\big\}$$

with $\star \in \{uf, k, ek\}$.

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