EIGENVALUES OF SCHRÖDINGER OPERATORS NEAR THRESHOLDS: TWO TERM APPROXIMATION

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ABSTRACT. We consider one dimensional Schrödinger operators

$$H_\lambda = -\frac{d^2}{dx^2} + U + \lambda V_\lambda$$

with nonlinear dependence on the parameter λ and study the small λ behavior of eigenvalues. Potentials U and V_λ are real-valued bounded functions of compact support. Under some assumptions on U and V_λ, we prove the existence of a negative eigenvalue that is absorbed at the bottom of the continuous spectrum as $\lambda \to 0$. We also construct two-term asymptotic formulas for the threshold eigenvalues.

1. INTRODUCTION

About forty years ago, Simon and Klaus [1–4] started studying the low energy behavior of the so-called weakly coupled Hamiltonians $-\Delta + \lambda V$. The considerable interest has been in the study of negative-energy bound states and their small λ behavior, as well as in the study of the absorption of the eigenvalues by the continuous spectrum. The main results here have been concerned with Schrödinger operators in one and two dimensions, because in three dimensions the weakly coupled Hamiltonians have no bound state if λ is small enough, i.e., if potential λV is a sufficiently shallow well. For the case of 1D Hamiltonians $\mathcal{H}_\lambda = -\frac{d^2}{dx^2} + \lambda V$, a suitable short-range potential V can produce a bound state for all small λ. Assuming that V is different from zero and $\int_\mathbb{R}(1 + |x|^2)|V(x)|\,dx < \infty$, Simon [1] proved that the operator $\mathcal{H}_\lambda$ has a negative-energy bound state e_λ for all small positive λ if and only if $\int_\mathbb{R} V(x)\,dx \le 0$. If $\mathcal{H}_\lambda$ does have an eigenvalue, then it is unique and simple, and obeys

$$(1) \qquad \sqrt{-e_\lambda} = -\frac{\lambda}{2}\int_\mathbb{R} V(x)\,dx - \frac{\lambda^2}{4}\iint_{\mathbb{R}^2} V(x)\,|x - y|\,V(y)\,dx\,dy + o(\lambda^2)$$

as $\lambda \to 0$. This asymptotic formula is due to Abarbanel, Callan and Goldberger, but it was not published by them; (1) was firstly announced by Simon [1]. The eigenvalue e_λ approaches zero as λ goes to zero and it is absorbed in the limit at the bottom of the continuous spectrum $[0, +\infty)$. Then we say that $\lambda = 0$ is a coupling constant threshold for $\mathcal{H}_\lambda$. Klaus [2] has extended this result to the class of potentials V obeying the condition $\int_\mathbb{R}(1 + |x|)|V(x)|\,dx < \infty$.

In [5,6], the threshold behavior has been studied as a general perturbation phenomenon and some general results on existence and asymptotic behavior of eigenvalues for self-adjoint operators $A + \lambda B$ have been obtained. The main tool was the so-called Birman-Schwinger principle. Klaus [6] has also applied these results to several special cases. One of them has been concerned with the Hamiltonian $-\frac{d^2}{dx^2} + U + \lambda V$. If a certain relation

between the potentials U and V holds, then the operator has a small negative-energy bound state (not necessarily a unique one) in the limit of weak coupling. Namely, it has been proved that the operator has the coupling constant threshold $\lambda = 0$, if the unperturbed operator $-\frac{d^2}{dx^2} + U$ possesses a zero-energy resonance with a half-bound state u and $\int_{\mathbb{R}} Vu^2\,dx < 0$. Among the negative eigenvalues there exists only one that is absorbed by the continuous spectrum as $\lambda \to 0$. A unique threshold eigenvalue e_λ is analytic at $\lambda = 0$ and obeys

$$(2) \qquad \sqrt{-e_\lambda} = -\frac{\lambda}{u_-^2 + u_+^2} \int_{\mathbb{R}} Vu^2\,dx + O(\lambda^2)$$

as $\lambda \to 0$, where $u_\pm = \lim_{x\to\pm\infty} u(x)$. If $\int_{\mathbb{R}} Vu^2\,dx = 0$ and the support of V lies between two consecutive zeros of u, then there exists a bound state near zero for all small enough λ (positive and negative). Finally, if $\int_{\mathbb{R}} Vu^2\,dx > 0$, then the operator has no bound state and therefore $\lambda = 0$ is not a coupling constant threshold. We will give the precise definitions of the zero-energy resonances, half-bound states, and coupling constant threshold in the next section.

One of the motivations for writing this article was the desire to improve approximation (2). As another motivation for investigating the threshold behavior of eigenvalues, we mention applications of this phenomenon to the study of the stability of solutions for the Korteweg-de Vries equation [21] and the existence of 'breathers' (the localized periodic solutions) for discrete nonlinear Schrödinger systems [22, 23].

In this paper, we consider a more general class of Schrodinger operators

$$(3) \qquad H_\lambda = -\frac{d^2}{dx^2} + U + \lambda V_\lambda, \quad \operatorname{dom} H_\lambda = W_2^2(\mathbb{R})$$

with nonlinear dependence on the positive parameter λ. We analyze the existence of negative eigenvalues and their threshold behavior. Here U and V_λ are functions of compact support and $V_\lambda = V + \lambda V_1 + o(\lambda)$ as $\lambda \to 0$. The spectrum of H_λ consists of the essential spectrum $[0, \infty)$ and possibly a finite number of negative eigenvalues. Under certain conditions on the potentials U, V and V_1 the operator H_λ has a negative eigenvalue e_λ that is absorbed at the bottom of the essential spectrum as λ goes to zero. The threshold eigenvalue may or may not be the ground state. We examine the asymptotic behavior of e_λ as $\lambda \to 0$ and compute the two term asymptotic formula which in particular improves the approximation (2). For the case $U = 0$ and $V_\lambda = V$, our asymptotics turns into the Abarbanel-Callan-Goldberger formula.

The threshold behavior of eigenvalues for operators $-\frac{d^2}{dx^2} + U + \lambda\alpha_\lambda V(\alpha_\lambda \cdot)$, where the positive sequence α_λ converges to a finite or infinite limit as $\lambda \to 0$, has recently been studied in [7]. These results gives us an example of the non-analytic threshold behavior of negative eigenvalues.

The question of how negative eigenvalues are absorbed in the bottom of the essential spectrum has been discussed by many authors [8–18]. The Hamiltonians with periodic potentials perturbed by short range ones and the threshold phenomena in gaps of the continuous spectrum were studied in [6, 19, 20].

2. Main Results

We start with some definitions. Let A and B_λ be self-adjoint operators and B_λ be relatively A-compact for all $\lambda > 0$; then $\sigma_{ess}(A + B_\lambda) = \sigma_{ess}(A)$. Suppose that the interval (a, b) is a gap in the spectrum of A. If we can find an eigenvalue e_λ of $A + B_\lambda$ in (a, b) for all $\lambda > 0$ with the property that $e_\lambda \to a$ or $e_\lambda \to b$ as $\lambda \to 0$, then we call $\lambda = 0$ the *coupling constant threshold*. So the eigenvalue e_λ is absorbed by the continuous spectrum at "time" $\lambda = 0$.

We say operator $-\frac{d^2}{dx^2} + U$ possesses a *zero-energy resonance* if there exists a non trivial solution u of the equation

(4) $$-u'' + Uu = 0$$

that is bounded on the whole line. We then call u the *half-bound state*. Any half-bound state u possesses finite limits $\lim_{x\to\pm\infty} u(x)$, because u is constant outside the support of U; both the limits are different from zero. Since a half-bound state is defined up to a scalar multiplier, we say a half-bound state u is *normalized* if $\lim_{x\to-\infty} u(x) = 1$. Let θ hereafter denote the limit of the normalized half-bound state as $x \to +\infty$, i.e., $\theta := \lim_{x\to+\infty} u(x)$. We also introduce the function

$$\Theta(x) = \begin{cases} 1 & \text{if } x < 0, \\ \theta & \text{if } x > 0. \end{cases}$$

Assume u_1 is a solution of (4) such that $u_1(x) = x$ to the left of the support of U. Then u and u_1 are linearly independent solutions of (4) and we will show below that there exists a constant θ_1 such that $u_1(x) = \theta^{-1}x + \theta_1$ for all x large enough (see Fig. 1). Let v_* be a solution of $-v'' + Uv = -Vu$ which vanishes to the left of the supports of U and V.

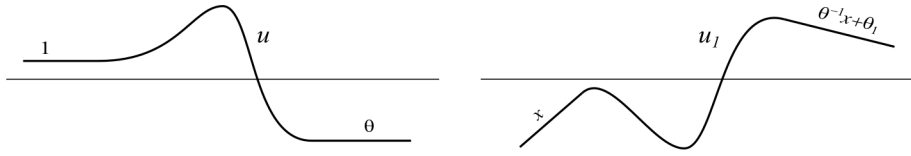FIGURE 1. Plots of normalized half-bound state u and solution u_1.

Here and subsequently, $\|\cdot\|$ stands for the norm in $L_2(\mathbb{R})$.

Theorem 1. *Suppose that U, V and V_1 are functions of compact support belonging to $L^\infty(\mathbb{R})$, and $\|V_\lambda - V - \lambda V_1\| = o(\lambda)$ as $\lambda \to 0$. Assume operator $-\frac{d^2}{dx^2} + U$ has a zero-energy resonance with normalized half-bound state u. If*

(5) $$\int_{\mathbb{R}} V u^2 \, dx < 0,$$

then operator $H_\lambda = -\frac{d^2}{dx^2} + U + \lambda V_\lambda$ possesses the coupling constant threshold $\lambda = 0$, i.e., for all small positive λ there exists a negative eigenvalue e_λ of H_λ such that $e_\lambda \to 0$ as $\lambda \to 0$. Moreover the threshold eigenvalue e_λ has the asymptotic expansion $e_\lambda = -\lambda^2 \left(\omega_0 + \omega_1 \lambda + o(\lambda)\right)^2$ as $\lambda \to 0$, where

(6) $$\omega_0 = \frac{1}{\theta^2 + 1} \int_{\mathbb{R}} V u^2 \, dx,$$

(7) $$\omega_1 = \frac{1}{\theta^2 + 1} \left(\int_{\mathbb{R}} V\left(v_* + \omega_0(\theta^2 - 1)u_1\right)u \, dx \right.$$
$$\left. + \omega_0^2 \int_{\mathbb{R}} (u^2 - \Theta^2) \, dx - \omega_0^2 \theta^3 \theta_1 + \int_{\mathbb{R}} V_1 u^2 \, dx \right).$$

The threshold phenomenon is also possible if inequality (5) turns into the equality. In this case the absorption of the eigenvalue at the bottom of $\sigma_{ess}(H_\lambda)$ occurs with the rate $O(\lambda^4)$ as $\lambda \to 0$.

Theorem 2. *Under the assumptions of Theorem 1, we suppose that*

$$(8) \qquad \int_{\mathbb{R}} V u^2 \, dx = 0.$$

Then the operator H_λ has the coupling constant threshold $\lambda = 0$, if

$$(9) \qquad \int_{\mathbb{R}} (V v_* + V_1 u) u \, dx < 0.$$

Moreover the threshold eigenvalue e_λ admits the asymptotics

$$e_\lambda = -\frac{\lambda^4}{(\theta^2 + 1)^2} \left(\int_{\mathbb{R}} V v_* u \, dx + \int_{\mathbb{R}} V_1 u^2 \, dx \right)^2 + o(\lambda^4) \quad as \quad \lambda \to 0.$$

Return now to operator family $-\frac{d^2}{dx^2} + U + \lambda V$ studied in [6].

Corollary 1. *Assume the operator $-\frac{d^2}{dx^2} + U$ has a zero-energy resonance with half-bound state u. If*

$$\int_{\mathbb{R}} V u^2 \, dx < 0,$$

then $-\frac{d^2}{dx^2} + U + \lambda V$ possesses the coupling constant threshold $\lambda = 0$ and a negative eigenvalue e_λ admits the asymptotics

$$(10) \qquad e_\lambda = -\lambda^2 (\omega_0 + \lambda \omega_1 + o(\lambda))^2,$$

where ω_0 is given by (6) and

$$\omega_1 = \frac{1}{\theta^2 + 1} \left(\int_{\mathbb{R}} V \left(v_* + \omega_0 (\theta^2 - 1) u_1 \right) u \, dx + \omega_0^2 \int_{\mathbb{R}} (u^2 - \Theta^2) \, dx - \omega_0^2 \theta^3 \theta_1 \right).$$

If V is different from zero and

$$(11) \qquad \int_{\mathbb{R}} V u^2 \, dx = 0,$$

then the operator $-\frac{d^2}{dx^2} + U + \lambda V$ has a negative eigenvalue e_λ with the asymptotics

$$(12) \qquad e_\lambda = -\frac{\lambda^4}{(u_-^2 + u_+^2)^2} \left(\iint_{\mathbb{R}^2} V(x) u(x) \mathcal{E}_U (x - y) V(y) u(y) \, dx \, dy + o(1) \right)^2,$$

where $\mathcal{E}_U$ is the fundamental solution for $\frac{d^2}{dx^2} - U$ which vanishes to the left of $\operatorname{supp} U$.

Proof. Most of the proof follows from the previous theorems, assuming $V_\lambda = V$ for all λ. We are left with the task of deriving (12). If (11) holds, then $\omega_0 = 0$ and

$$(13) \qquad \omega_1 = \frac{1}{\theta^2 + 1} \int_{\mathbb{R}} V v_* u \, dx.$$

Recall that v_* solves equation $v_*'' - U v_* = V u$ and vanishes to the left of the supports of U and V. Then v_* can be represented as the convolution $\mathcal{E}_U * (V u)$. Hence

$$(14) \qquad \begin{aligned} \int_{\mathbb{R}} V v_* u \, dx &= \int_{\mathbb{R}} V(x) u(x) (\mathcal{E}_U * V u)(x) \, dx \\ &= \iint_{\mathbb{R}^2} V(x) u(x) \mathcal{E}_U (x - y) V(y) u(y) \, dx \, dy. \end{aligned}$$

Substituting (14) into (13) finishes up the proof. $\qquad \square$

Remark 1. Klaus did not use the notion of a normalized half-bound state. To agree the asymptotic formulas, we rewrite ω_0 and ω_1 in (10) in terms of an arbitrary half-bound state u for which $\lim\limits_{x \to \pm\infty} u(x) = u_\pm$. Then in notation of [6] we obtain

$$\omega_0 = \frac{1}{u_-^2 + u_+^2} \int_{\mathbb{R}} V u^2 \, dx,$$

$$\omega_1 = \frac{1}{u_-^2 + u_+^2} \left(u_- \int_{\mathbb{R}} V \left(v_* + \frac{\omega_0(u_+^2 - u_-^2)}{u_-^2} u_1 \right) u \, dx \right.$$

$$\left. + \omega_0^2 \int_{\mathbb{R}} (u^2 - \Xi^2) \, dx - \omega_0^2 \theta_1 \frac{u_+^3}{u_-} \right),$$

where $\Xi(x) = u_-$ for $x < 0$ and $\Xi(x) = u_+$ for $x > 0$.

Let us compare our results with those of Simon when the unperturbed operator is the free Schrödinger operator.

Corollary 2. *Assume that $U = 0$. If the mean value of V is negative, i.e.,*

$$(15) \qquad\qquad \int_{\mathbb{R}} V \, dx < 0,$$

then $H_\lambda = -\frac{d^2}{dx^2} + \lambda V_\lambda$ has a negative eigenvalue of the form

$$e_\lambda = -\lambda^2 (\omega_0 + \lambda\omega_1 + o(\lambda))^2$$

as λ tends to zero, where

$$(16) \qquad \omega_0 = \frac{1}{2} \int_{\mathbb{R}} V \, dx, \quad \omega_1 = \frac{1}{4} \iint_{\mathbb{R}^2} V(x) \, |x - y| \, V(y) \, dx \, dy + \frac{1}{2} \int_{\mathbb{R}} V_1 \, dx.$$

In the case $V_\lambda = V$, this asymptotic formula coincides with (1).

Proof. The trivial potential $U = 0$ has a zero-energy resonance with half-bound state $u = 1$; then $\theta = 1$ and $\Theta(x) = 1$ for all $x \in \mathbb{R}$. In addition, we have $\theta_1 = 0$, because equation $u'' = 0$ possesses the solution $u_1 = x$. Therefore condition (5) becomes (15), and (6), (7) simplify to read

$$\omega_0 = \frac{1}{2} \int_{\mathbb{R}} V \, dx, \quad \omega_1 = \frac{1}{2} \int_{\mathbb{R}} V v_* \, dx + \frac{1}{2} \int_{\mathbb{R}} V_1 \, dx.$$

The fundamental solution $\mathcal{E}_0(x) = \frac{1}{2}(|x| + x)$ for the differential operator $\frac{d^2}{dx^2}$ vanishes for $x < 0$. As in Corollary 1, we derive

$$\int_{\mathbb{R}} V(x) v_*(x) \, dx = \int_{\mathbb{R}} V(x)(\mathcal{E}_0 * V)(x) \, dx$$

$$= \frac{1}{2} \iint_{\mathbb{R}^2} V(x) \, |x - y| \, V(y) \, dx \, dy + \frac{1}{2} \iint_{\mathbb{R}^2} V(x) \, (x - y) \, V(y) \, dx \, dy$$

$$= \frac{1}{2} \iint_{\mathbb{R}^2} V(x) \, |x - y| \, V(y) \, dx \, dy,$$

because $\iint_{\mathbb{R}^2} f(x) \, (x - y) \, f(y) \, dx \, dy = 0$ for any f, for which the integral exists. This gives the second equality in (16), and the proof is complete. $\square$

Corollary 3. *Assume that $U = 0$ and V is different from zero. If*

$$\int_{\mathbb{R}} V \, dx = 0,$$

then for all nonzero λ, positive or negative, the operator $H_\lambda = -\frac{d^2}{dx^2} + \lambda V_\lambda$ possesses an eigenvalue e_λ having the asymptotics

$$(17) \qquad e_\lambda = -\frac{\lambda^4}{16} \left(\iint_{\mathbb{R}^2} V(x) \, |x - y| \, V(y) \, dx \, dy + o(1) \right)^2$$

as $\lambda \to 0$. This asymptotic formula can be also written in the form

$$(18) \qquad e_\lambda = -\frac{\lambda^4}{4} \left(\int_{\mathbb{R}} \left(\int_{-\infty}^{x} V(y) \, dy \right)^2 dx + o(1) \right)^2.$$

This assertion will be proved in Section 4.

3. Preliminaries

We first record some technical facts. Assume, without loss of generality, the supports of potentials U and V_λ lie within $\mathcal{I} = (-\ell, \ell)$ for λ small enough. Then a half-bound state of operator $-\frac{d^2}{dx^2} + U$ is constant outside $\mathcal{I}$ and its restriction to $\mathcal{I}$ is a non-trivial solution of the problem

$$-u'' + Uu = 0, \quad t \in \mathcal{I}, \qquad u'(-\ell) = 0, \quad u'(\ell) = 0.$$

Moreover, if u is the normalized half-bound state, then $u(-\ell) = 1$ and $u(\ell) = \theta$.

Proposition 1. *Assume that h belongs to $L_2(\mathcal{I})$ and γ is a real number. Let w be a solution of the Cauchy problem*

$$(19) \qquad -w'' + Uw = h, \quad t \in \mathcal{I}, \qquad w(-\ell) = 0, \quad w'(-\ell) = \gamma.$$

If $-\frac{d^2}{dx^2} + U$ has a zero-energy resonance with normalized half-bound state u, then

$$(20) \qquad \theta w'(\ell) = \gamma - \int_{-\ell}^{\ell} hu \, dx.$$

In addition, this solution obeys the estimate

$$\|w\|_{C^1(\mathcal{I})} \le C(|\gamma| + \|h\|_{L_2(\mathcal{I})})$$

for some positive C being independent of γ and h.

Proof. Since $u(-\ell) = 1$ and $u(\ell) = \theta$, (20) can be easily obtained by multiplying the equation in (19) by u and integrating by parts. Next, application of the variation of parameters method yields

$$(21) \qquad w(x) = \gamma(u_1(x) + \ell u(x)) + \int_{-\ell}^{x} k(x, s) h(s) \, ds,$$

where $k(x, s) = u(x) u_1(s) - u(s) u_1(x)$. Under the assumptions made on potential U, u and u_1 belong to $W_2^2(\mathcal{I})$; consequently $u, u_1 \in C^1(\mathcal{I})$ by the Sobolev embedding theorem. From this and the representation of the first derivative

$$w'(x) = \gamma(u_1'(x) + \ell u'(x)) + \int_{-\ell}^{x} \frac{\partial k}{\partial x}(x, s) h(s) \, ds$$

we have $|w(x)| + |w'(x)| \le |\gamma|(\|u_1\|_{C^1(\mathcal{I})} + |\ell| \|u\|_{C^1(\mathcal{I})}) + c_1 \|k\|_{C^1(\mathcal{I} \times \mathcal{I})} \|h\|_{L_2(\mathcal{I})} \le C(|\gamma| + \|h\|_{L_2(\mathcal{I})})$ for $x \in \mathcal{I}$, which completes the proof. $\square$

Proposition 2. *Let u_1 be the solution of (4) as described in Section 2. Then for some constant θ_1 we have $u_1(x) = \theta^{-1} x + \theta_1$ for all $x > \ell$.*

Proof. The function $v = u_1 + \ell u$ solves the Cauchy problem

$$-v'' + Uv = 0, \quad t \in \mathcal{I}, \qquad v(-\ell) = 0, \quad v'(-\ell) = 1$$

and therefore $u_1'(\ell) = \theta^{-1}$ by (20). Hence $u_1(x) = \theta^{-1}x + \theta_1$ for some θ_1 and all $x > \ell$, which is the desired conclusion. $\qquad\square$

Our method is different from that of Simon and Klaus. We don't use the Birman-Schwinger principle. To prove the main results, we use the asymptotic method of quasimodes or in other words of "almost" eigenvalues and eigenfunctions. Let A be a self-adjoint operator in a Hilbert space L. We say a pair $(\mu, \phi) \in \mathbb{R} \times \operatorname{dom} A$ is a *quasimode* of A with accuracy δ, if $\|\phi\|_L = 1$ and $\|(A - \mu I)\phi\|_L \leq \delta$.

Lemma 1 ([24, p. 139]). *Assume (μ, ϕ) is a quasimode of A with accuracy $\delta > 0$ and the spectrum of A is discrete in the interval $[\mu - \delta, \mu + \delta]$. Then there exists an eigenvalue λ of A such that $|\lambda - \mu| \leq \delta$.*

Proof. If $\mu \in \sigma(A)$, then $\lambda = \mu$. Otherwise the distance d_μ from μ to the spectrum of A can be computed as

$$d_\mu = \|(A - \mu I)^{-1}\|^{-1} = \inf_{\psi \neq 0} \frac{\|\psi\|_L}{\|(A - \mu I)^{-1}\psi\|_L},$$

where ψ is an arbitrary vector of L. Taking $\psi = (A - \mu I)\phi$, we deduce

$$d_\mu \leq \frac{\|(A - \mu I)\phi\|_L}{\|\phi\|_L} \leq \delta,$$

from which the assertion follows. $\qquad\square$

4. Proof of Main Results

4.1. **Proof of Theorem 1.** In order to prove the existence of a negative eigenvalue for H_λ, we will construct a quasimode $(-\omega_\lambda^2, \phi_\lambda)$ of H_λ as follows. Suppose that $-\frac{d^2}{dx^2} + U$ has a zero-energy resonance with normalized half-bound state u. We assume $\omega_\lambda = \lambda(\omega_0 + \lambda\omega_{1,\lambda} + \lambda^2\omega_{2,\lambda})$ and $\phi_\lambda = \psi_\lambda/\|\psi_\lambda\|$, where

$$(22) \qquad \psi_\lambda(x) = \begin{cases} e^{-\omega_\lambda(x+\ell)} & \text{for } x < -\ell, \\ u(x) + \lambda v_1(x) + \lambda^2 v_{2,\lambda}(x) + \lambda^3 v_{3,\lambda}(x) & \text{for } |x| < \ell, \\ a_\lambda\, e^{\omega_\lambda(x-\ell)} + b_\lambda \rho(x - \ell) & \text{for } x > \ell. \end{cases}$$

The functions v_1, $v_{2,\lambda}$ and $v_{3,\lambda}$ are solutions of the problems

$$(23) \qquad -v_1'' + Uv_1 = -Vu, \qquad v_1(-\ell) = 0, \quad v_1'(-\ell) = -\omega_0;$$

$$(24) \qquad \begin{cases} -v_2'' + Uv_2 = -Vv_1 - (V_1 + g_\lambda)u, \\ v_2(-\ell) = 0, \;\; v_2'(-\ell) = -\omega_{1,\lambda}; \end{cases}$$

$$(25) \qquad -v_3'' + Uv_3 = -f_{3,\lambda}, \qquad v_3(-\ell) = 0, \quad v_3'(-\ell) = -\omega_{2,\lambda}$$

respectively. Here we set $g_\lambda = \lambda^{-1}(V_\lambda - V - \lambda V_1)$ and

$$f_{3,\lambda} = Vv_{2,\lambda} + (V_1 + \omega_0^2 + g_\lambda)v_1 + 2\omega_0\omega_{1,\lambda}u.$$

We also presume that $\omega_{1,\lambda}$ and $\omega_{2,\lambda}$ have finite limits as $\lambda \to 0$. The function ρ is smooth in $\mathbb{R} \setminus \{0\}$, $\rho(x) = 0$ for $x \leq 0$ and $x \geq 1$, and $\rho'(+0) = 1$. In addition, ρ'' is bounded in $[0, 1]$. Hence ρ is continuous at $x = 0$, but the first derivative ρ' has the unit jump at this point. This function corrects the discontinuity of ψ_λ' at $x = \ell$.

Let us first show that constants ω_0, $\omega_{1,\lambda}$, $\omega_{2,\lambda}$, a_λ and b_λ in (22) can be chosen so that ψ_λ will belong to $\operatorname{dom} H_\lambda$. First of all, the $L_2(\mathbb{R})$-norm of ψ_λ is finite if and only if $\omega_\lambda < 0$; therefore we must impose the conditions $\omega_0 < 0$ (the case $\omega_0 = 0$ will be treated in Theorem 2). Note that u and v_k belong to the Sobolev space $W_2^2(\mathcal{I})$ as solutions of

the equation $-y'' + Uy = f$ with $f \in L_2(\mathcal{I})$. By construction, ψ_λ and its first derivative are continuous at $x = -\ell$, then it is enough to ensure the continuous differentiability of ψ_λ at $x = \ell$.

Since $\psi_\lambda(\ell + 0) - \psi_\lambda(\ell - 0) = \theta + \lambda v_1(\ell) + \lambda^2 v_{2,\lambda}(\ell) + \lambda^3 v_{3,\lambda}(\ell) - a_\lambda$, we set

$$
(26) \qquad a_\lambda = \theta + \lambda v_1(\ell) + \lambda^2 v_{2,\lambda}(\ell) + \lambda^3 v_{3,\lambda}(\ell).
$$

To see this, we calculate

$$
\begin{aligned}
\psi'_\lambda(\ell + 0) - \psi'_\lambda(\ell - 0) &= \omega_\lambda a_\lambda + b_\lambda \rho'(0) - \lambda v'_1(\ell) - \lambda^2 v'_{2,\lambda}(\ell) - \lambda^3 v'_{3,\lambda}(\ell) \\
&= \lambda(\omega_0 + \lambda\omega_{1,\lambda} + \lambda^2\omega_{2,\lambda})\big(\theta + \lambda v_1(\ell) + \lambda^2 v_{2,\lambda}(\ell) + \lambda^3 v_{3,\lambda}(\ell)\big) \\
&\quad + b_\lambda - \lambda v'_1(\ell) - \lambda^2 v'_{2,\lambda}(\ell) - \lambda^3 v'_{3,\lambda}(\ell) \\
&= \lambda\big(\omega_0\theta - v'_1(\ell)\big) + \lambda^2\big(\omega_{1,\lambda}\theta + \omega_0 v_1(\ell) - v'_{2,\lambda}(\ell)\big) \\
&\quad + \lambda^3\big(\omega_{2,\lambda}\theta + \omega_{1,\lambda}v_1(\ell) + \omega_0 v_{2,\lambda}(\ell) - v'_{3,\lambda}(\ell)\big) + b_\lambda \\
&\quad + \lambda^4\Big(\omega_0 v_{3,\lambda}(\ell) + \omega_{1,\lambda}(v_{2,\lambda}(\ell) + \lambda v_{3,\lambda}(\ell)) \\
&\quad + \omega_{2,\lambda}(v_1(\ell) + \lambda v_{2,\lambda}(\ell) + \lambda^2 v_{3,\lambda}(\ell))\Big).
\end{aligned}
$$

In order to achieve $\psi'_\lambda(\ell + 0) = \psi'_\lambda(\ell - 0)$, we assume

$$
(27) \qquad \omega_0 = \theta^{-1}v'_1(\ell), \quad \omega_{1,\lambda} = \theta^{-1}(v'_{2,\lambda}(\ell) - \omega_0 v_1(\ell)),
$$

$$
(28) \qquad \omega_{2,\lambda} = \theta^{-1}(v'_{3,\lambda}(\ell) - \omega_0 v_{2,\lambda}(\ell) - \omega_{1,\lambda}v_1(\ell)),
$$

$$
(29) \qquad b_\lambda = -\lambda^4\big(\omega_0 v_{3,\lambda} + \omega_{1,\lambda}(v_{2,\lambda} + \lambda v_{3,\lambda}) + \omega_{2,\lambda}(v_1 + \lambda v_{2,\lambda} + \lambda^2 v_{3,\lambda})\big)|_{x=\ell}.
$$

On the other hand, applying Proposition 1 to problems (23)–(25), we deduce

$$
(30) \qquad \theta v'_1(\ell) = -\omega_0 + \int_{-\ell}^{\ell} V u^2\, dx, \quad \theta v'_{3,\lambda}(\ell) = -\omega_{2,\lambda} + \int_{-\ell}^{\ell} f_{3,\lambda} u\, dx,
$$

$$
(31) \qquad \theta v'_{2,\lambda}(\ell) = -\omega_{1,\lambda} + \int_{-\ell}^{\ell} V v_1 u\, dx + \int_{-\ell}^{\ell} (V_1 + \omega_0^2 + g_\lambda)u^2\, dx.
$$

Then combining (27), (28), (30) and (31) yields

$$
\omega_0 = \frac{1}{\theta^2 + 1}\int_{-\ell}^{\ell} V u^2\, dx,
$$

$$
(32) \qquad \omega_{1,\lambda} = \frac{1}{\theta^2 + 1}\Big(\int_{-\ell}^{\ell} V v_1 u\, dx - \theta\omega_0 v_1(\ell) + \int_{-\ell}^{\ell}(V_1 + \omega_0^2 + g_\lambda)u^2\, dx\Big),
$$

$$
\omega_{2,\lambda} = \frac{1}{\theta^2 + 1}\Big(\int_{-\ell}^{\ell} f_{3,\lambda} u\, dx - \theta(\omega_0 v_{2,\lambda}(\ell) + \omega_{1,\lambda}v_1(\ell))\Big).
$$

Since V has a compact support, ω_0 does not depend on ℓ and can be finally written in the form

$$
(33) \qquad \omega_0 = \frac{1}{\theta^2 + 1}\int_{\mathbb{R}} V u^2\, dx.
$$

Moreover ω_0 is negative if condition (5) holds; then ω_λ is negative for all λ small enough and therefore $\psi_\lambda \in L_2(\mathbb{R})$.

The function g_λ has an infinitely small $L_2(\mathbb{R})$-norm as $\lambda \to 0$, since

$$
\|V_\lambda - V - \lambda V_1\| = o(\lambda) \quad \text{as} \quad \lambda \to 0.
$$

Consequently there exists limit $\omega_1 = \lim_{\lambda \to 0} \omega_{1,\lambda}$, where

$$(34) \qquad \omega_1 = \frac{1}{\theta^2 + 1} \left(\int_{-\ell}^{\ell} V v_1 u \, dx - \theta \omega_0 v_1(\ell) + \int_{-\ell}^{\ell} (V_1 + \omega_0^2) u^2 \, dx \right).$$

But it is not obvious that ω_1 does not depend on ℓ, because the right hand side of (34) contains the integrand $\omega_0^2 u^2$ without a compact support as well as the solution v_1 of (23) which depends on ℓ. We first note that $u^2 - \Theta^2$ is a function of compact support. Then we have

$$(35) \qquad \int_{-\ell}^{\ell} u^2 \, dx = \int_{-\ell}^{\ell} (u^2 - \Theta^2) \, dx + \int_{-\ell}^{\ell} \Theta^2 \, dx = \int_{\mathbb{R}} (u^2 - \Theta^2) \, dx + \ell(\theta^2 + 1).$$

Next, v_1 can be written as $v_1 = v_* - \omega_0(u_1 + \ell u)$, where v_* is the solution of the Cauchy problem $-v_*'' + U v_* = -V u$, $v_*(-\ell) = 0$, $v_*'(-\ell) = 0$. Invoking (33), we derive

$$(36) \qquad \begin{aligned} \int_{-\ell}^{\ell} V v_1 u \, dx &= \int_{-\ell}^{\ell} V(v_* - \omega_0 u_1) u \, dx - \omega_0 \ell \int_{-\ell}^{\ell} V u^2 \, dx \\ &= \int_{\mathbb{R}} V(v_* - \omega_0 u_1) u \, dx - \omega_0^2 \ell(\theta^2 + 1). \end{aligned}$$

In order to compute $v_1(\ell)$, we multiply the equation in (23) by u_1 and integrate by parts twice $(v_1' u_1 - v_1 u_1')\big|_{-\ell}^{\ell} = \int_{-\ell}^{\ell} V u u_1 \, dx$. Since $u_1(-\ell) = -\ell$, $u_1(\ell) = \ell\theta^{-1} + \theta_1$, $u_1'(\ell) = \theta^{-1}$ and $v_1'(\ell) = \omega_0 \theta$, we obtain

$$(37) \qquad v_1(\ell) = \omega_0 \theta^2 \theta_1 - \theta \int_{\mathbb{R}} V u_1 u \, dx.$$

Substitute (35)–(37) into (34), to find

$$(38) \qquad \omega_1 = \frac{1}{\theta^2 + 1} \left(\int_{\mathbb{R}} V v_0 u \, dx + \int_{\mathbb{R}} V_1 u^2 \, dx + \omega_0^2 \int_{\mathbb{R}} (u^2 - \Theta^2) \, dx - \omega_0^2 \theta^3 \theta_1 \right),$$

where $v_0 = v_* + \omega_0(\theta^2 - 1) u_1$. Hence ω_1 does not depend on ℓ either. A similar arguments can be applied to $\omega_{2,\lambda}$. Therefore ψ_λ belongs to $\mathrm{dom}\, H_\lambda$ by our choice of ω_0, $\omega_{1,\lambda}$, $\omega_{2,\lambda}$, a_λ and b_λ.

Proposition 3. *There exist constants c and C such that*

$$c \omega_\lambda^{-1/2} \le \|\psi_\lambda\| \le C \omega_\lambda^{-1/2}.$$

Proof. We first note that the solutions $v_{2,\lambda}$ and $v_{3,\lambda}$ are bounded in $L_2(\mathbb{R})$ uniformly on λ. In addition, by Proposition 1 we have

$$\|v_{2,\lambda}\|_{C^1(\mathcal{I})} \le C(|\omega_{1,\lambda}| + \|V v_1 + (V_1 + g_\lambda) u\|_{L_2(\mathcal{I})}) \le c_1,$$
$$\|v_{3,\lambda}\|_{C^1(\mathcal{I})} \le C(|\omega_{2,\lambda}| + \|f_{3,\lambda}\|_{L_2(\mathcal{I})}) \le c_2,$$

where c_1 and c_2 are independent of λ. Combining these bounds with (26) and (29) yields

$$(39) \qquad |a_\lambda| \le c_3, \quad |b_\lambda| \le c_3 \lambda^4.$$

Therefore the main contribution as $\lambda \to 0$ to the norm of ψ_λ is given by the exponents $e^{\pm \omega_\lambda(x \mp \ell)}$. A direct calculation verifies $\|e^{-\omega_\lambda(x+\ell)}\|_{L_2(-\infty, -\ell)} = (2\omega_\lambda)^{-1/2}$ and $\|e^{\omega_\lambda(x-\ell)}\|_{L_2(\ell, +\infty)} = (2\omega_\lambda)^{-1/2}$. Hence $\|\psi_\lambda\| \sim a\omega_\lambda^{-1/2}$ as $\lambda \to 0$. In particular, $\|\psi_\lambda\| \sim a_0 \lambda^{-1/2}$ if $\omega_0 \ne 0$ and $\|\psi_\lambda\| \sim a_1 \lambda^{-1}$ if $\omega_0 = 0$. $\qquad \square$

Lemma 2. *The pair $(-\omega_\lambda^2, \phi_\lambda)$ is a quasimode of H_λ with the accuracy $o(\lambda^{9/2})$ as $\lambda \to 0$.*

Proof. Let $r_\lambda = (H_\lambda + \omega_\lambda^2 I)\psi_\lambda$. Then $(H_\lambda + \omega_\lambda^2 I)\phi_\lambda = \|\psi_\lambda\|^{-1} r_\lambda$. We must estimate the L_2-norm of r_λ. Since $e^{\pm\omega_\lambda(x\mp\ell)}$ are exact solutions of $-\psi'' + \omega_\lambda^2 \psi = 0$ and $\operatorname{supp}\rho = [0,1]$, we have

$$(40) \qquad r_\lambda(x) = -b_\lambda(\rho''(x-\ell) - \omega_\lambda^2 \rho(x-\ell)) \quad \text{for} \quad \ell \le x \le \ell + 1$$

and $r_\lambda(x) = 0$ for other x from set $\{x\colon |x| > \ell\}$. In view of (39), we have the bound

$$(41) \qquad |r_\lambda(x)| \le c_1 \lambda^4 \quad \text{for} \quad |x| \ge \ell,$$

because ρ and ρ'' are bounded on $[0,1]$. Next, we calculate r_λ for $|x| < \ell$. Recalling (4) and (23)–(25), we derive

$$\begin{aligned}
r_\lambda &= \left(-\tfrac{d^2}{dx^2} + U + \lambda V_\lambda + \omega_\lambda^2\right)\psi_\lambda \\
&= \left(-\tfrac{d^2}{dx^2} + U + \lambda V + \lambda^2 V_1 + \lambda^2 g_\lambda + \omega_\lambda^2\right)(u + \lambda v_1 + \lambda^2 v_2 + \lambda^3 v_3) \\
&= (-u'' + Uu) + \lambda(-v_1'' + Uv_1 + Vu) + \lambda^2\big(-v_2'' + Uv_2 + Vv_1 \\
&\quad + V_1 u + \omega_0^2 u + g_\lambda u\big) + \lambda^3\big(-v_3'' + Uv_3 + f_{3,\lambda}\big) + \lambda^4 R_\lambda = \lambda^4 R_\lambda,
\end{aligned}$$

where the norm $\|R_\lambda\|_{L_2(\mathcal{I})}$ is bounded uniformly with respect to λ. From this we conclude that $\|r_\lambda\|_{L_2(\mathcal{I})} = O(\lambda^4)$, and hence that $\|r_\lambda\| = O(\lambda^4)$ as $\lambda \to 0$, in view of (41). Finally we have

$$(42) \qquad \|(H_\lambda + \omega_\lambda^2 I)\phi_\lambda\| = \|\psi_\lambda\|^{-1}\|r_\lambda\| \le c\lambda^{9/2}$$

as $\lambda \to 0$, by Proposition 3. $\qquad\square$

Owing to Lemmas 1 and 2, the operator H_λ possesses a negative eigenvalue e_λ satisfying the bound $|e_\lambda + \lambda^2(\omega_0 + \lambda\omega_{1,\lambda} + \lambda^2\omega_{2,\lambda})^2| \le c\lambda^{9/2}$. Since

$$(\omega_0 + \lambda\omega_{1,\lambda} + \lambda^2\omega_{2,\lambda})^2 - (\omega_0 + \lambda\omega_1)^2 \sim 2\omega_0\lambda(\omega_{1,\lambda} - \omega_1)$$

and $\omega_{1,\lambda} - \omega_1 = o(1)$ as $\lambda \to 0$, we derive the asymptotic formula

$$e_\lambda + \lambda^2(\omega_0 + \lambda\omega_1)^2 = o(\lambda^3),$$

which we rewrite as $(\lambda^{-1}\sqrt{-e_\lambda})^2 - (\omega_0 + \lambda\omega_1)^2 = o(\lambda)$. From this we immediately deduce that $\lambda^{-1}\sqrt{-e_\lambda} + \omega_0 + \lambda\omega_1 = o(\lambda)$, and hence that

$$\sqrt{-e_\lambda} = -\lambda(\omega_0 + \lambda\omega_1 + o(\lambda))$$

as $\lambda \to 0$, which completes the proof of Theorem 1. $\qquad\square$

4.2. **Proof of Theorem 2.** Now we consider the critical case when inequality (5) turns into the equality

$$(43) \qquad \int_{\mathbb{R}} V u^2\, dx = 0.$$

Hence $\omega_0 = 0$ in view of (4.1). Therefore $v_0 = v_*$ and (38) becomes

$$\omega_1 = \frac{1}{\theta^2 + 1}\left(\int_{\mathbb{R}} V v_* u\, dx + \int_{\mathbb{R}} V_1 u^2\, dx\right).$$

We must prove that a negative eigenvalue of H_λ exists, the key point being that this existence assertion follows from inequality $\omega_1 < 0$. Indeed, almost eigenfunction ψ_λ given by (22) belongs to $L_2(\mathbb{R})$ if $\omega_\lambda = \lambda^2(\omega_{1,\lambda} + \lambda\omega_{2,\lambda})$ is negative, at least for small λ. Condition (9) ensures $\omega_{1,\lambda} < 0$ and thereby $\omega_\lambda < 0$ for λ small enough.

By Proposition 3, we have $\|\psi_\lambda\| \sim a\lambda^{-1}$ as $\lambda \to 0$, provided $\omega_0 = 0$. Therefore estimate (42) can be improved $\|(H_\lambda + \omega_\lambda^2)\phi_\lambda\| \le c\lambda^5$ and then

$$|e_\lambda + \lambda^4(\omega_{1,\lambda} + \lambda\omega_{2,\lambda})^2| \le c_1 \lambda^5.$$

As in the proof of Theorem 1, if we rewrite this bound in the form

$$(\lambda^{-2}\sqrt{-e_\lambda})^2 - (\omega_{1,\lambda} + \lambda\omega_{2,\lambda})^2 = O(\lambda),$$

then we derive $\lambda^{-2}\sqrt{-e_\lambda} = -\omega_{1,\lambda} + O(\lambda) = -\omega_1 + o(1)$. Finally, we have

$$\sqrt{-e_\lambda} = -\lambda^2(\omega_1 + o(1)) \quad \text{as} \quad \lambda \to 0,$$

and this is precisely the assertion of Theorem 2.   □

4.3. **Proof of Corollary 3.** This statement differs from all earlier proved by the fact that here the threshold eigenvalue exists for both positive and negative λ small enough. For the case $V_\lambda = V$ this result has been proved by Simon [1].

Since $\int_{\mathbb{R}} V \, dx = 0$, from (16) we observe $\omega_0 = 0$ and

$$\omega_1 = \frac{1}{4} \iint_{\mathbb{R}^2} V(x) \, |x - y| \, V(y) \, dx \, dy.$$

Proposition 4. *If V is a function of zero mean, then*

$$\iint_{\mathbb{R}^2} V(x) \, |x - y| \, V(y) \, dx \, dy = -2 \int_{\mathbb{R}} \left(\int_{-\infty}^{x} V(y) \, dy \right)^2 dx.$$

Proof. From $\int_{\mathbb{R}} V \, dx = 0$ we immediately deduce

$$\int_{-\infty}^{x} V(y) \, dy = -\int_{x}^{+\infty} V(y) \, dy,$$

$$\int_{\mathbb{R}} V(x) \int_{-\infty}^{x} yV(y) \, dy \, dx = -\int_{\mathbb{R}} V(x) \int_{x}^{+\infty} yV(y) \, dy \, dx.$$

Therefore

$$\iint_{\mathbb{R}^2} V(x) \, |x - y| \, V(y) \, dx \, dy = \int_{\mathbb{R}} V(x) \int_{-\infty}^{x} (x - y)V(y) \, dy \, dx$$

$$+ \int_{\mathbb{R}} V(x) \int_{x}^{+\infty} (y - x)V(y) \, dy \, dx = 2 \int_{\mathbb{R}} xV(x) \int_{-\infty}^{x} V(y) \, dy \, dx$$

$$- 2 \int_{\mathbb{R}} V(x) \int_{-\infty}^{x} yV(y) \, dy \, dx = 4 \int_{\mathbb{R}} xV(x) \int_{-\infty}^{x} V(y) \, dy \, dx,$$

because integrating by parts yields

$$\int_{\mathbb{R}} V(x) \int_{-\infty}^{x} yV(y) \, dy dx = -\int_{\mathbb{R}} xV(x) \int_{-\infty}^{x} V(y) \, dy \, dx.$$

The proof is completed by showing that

$$\int_{\mathbb{R}} xV(x) \int_{-\infty}^{x} V(y) \, dy \, dx = -\frac{1}{2} \int_{\mathbb{R}} \left(\int_{-\infty}^{x} V(y) \, dy \right)^2 dx.$$

                                                                                                □

In view of this proposition, if potential V is different from zero, then $\omega_1 < 0$. Hence $\omega_\lambda = \lambda^2 \omega_{1,\lambda} + \lambda^3 \omega_{2,\lambda}$ is negative for λ small enough, positive or negative.

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