Methods of Functional Analysis and Topology
Vol. 26 (2020), no. 2, pp. 126–140
https://doi.org/10.31392/MFAT-npu26_2.2020.05

WHEN UNIVERSAL SEPARATED GRAPH C^*-ALGEBRAS ARE EXACT

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Abstract. We consider when the universal C^*-algebras associated to separated graphs are exact. Specifically, for finite separated graphs we show that the universal C^*-algebra is exact if and only if the C^*-algebra is isomorphic to a graph C^*-algebra which occurs precisely when the universal and reduced C^*-algebras of the separated graph are isomorphic.

1. Introduction and preliminaries

The idea of considering a separated graph and the associated C^*-algebras was considered in [1, 3, 4] and from a different perspective (as directed graphs with edge-colorings) was considered in [10]. It is natural, given the study of directed graph algebras (for a survey of this study see [13]), to consider the algebras associated to separated graphs (called edge-colored directed graphs in [10]). The focus is then on understanding the C^*-algebras associated to a separated graph.

In [10] the universal C^*-algebra for a separated graph was studied by considering these algebras as universal free products of graph algebras. This perspective allowed a natural extension of many results about graph algebras to the separated graph context. In this paper we return to this subject to investigate further some problems left open in [10]. In [4], it was shown that the reduced C^*-algebra associated to a separated graph is nuclear. However, there are examples of separated graphs whose universal C^*-algebras are not exact and one is left with the question of when this is the case.

Since the universal C^*-algebras of edge-colored directed graphs are considered as universal free products, one approach to these questions is to consider exactness of free products. An analysis of exactness for free products of finite dimensional algebras in [9] led us to reconsider the question in the context of separated graph C^*-algebras. In [9] one quickly realizes the role played by the amalgamating subalgebra in exactness. This proved to be a useful point of view for separated graph C^*-algebras as well.

We now explain the main results of the paper. We first introduce an operation on a separated graph, reversing edges, which produces a new graph with isomorphic associated C^*-algebras. We then, proceeding in cases, consider a set of algorithms that allow us to either show that the associated universal C^*-algebra is not exact, or eventually turn the graph into a directed graph, with no separation. As directed graph C^*-algebras are nuclear this completely answers the question concerning exactness and nuclearity for the universal C^*-algebra of a separated graph.

Putting our results together with the known results on nuclearity of the reduced C^*-algebra of a separated graph we are also able to completely determine when the universal C^*-algebra of a finite separated graph is isomorphic to the reduced C^*-algebra of the separated graph. Giving an answer to [4, Problem 7.2], this occurs precisely when the universal C^*-algebra is nuclear.

2010 *Mathematics Subject Classification.* 46L05, 46L09.

Key words and phrases. Edge-colored directed graph, separated graph, C^*-algebra, exact.

Some of the results of this paper will certainly apply in the context of arbitrary separated graphs, and for row-finite graphs direct limits can partly extend these results. However, for arbitrary graphs the extension is not obvious, since the algorithms we construct will not necessarily terminate for infinite graphs. Extending these results in a natural way to infinite graphs will likely require other tools.

2. Separated graph C^*-algebras

By a separated graph we mean a countable directed graph $G = (V, E, r, s)$ together with a set of partitions $C = \{C_v : v \in V\}$ where C_v is a partition of the edges with range equal to v. In this context we will mean, by countable, that both the edge and vertex sets are countable. Many of the results in this paper extend to non-countable graphs, however it is not clear that the algorithms we construct will apply in that context.

A separated Cuntz-Krieger family for a separated graph (G, C) consists of collections of orthogonal projections $\{P_v : v \in V\}$ and partial isometries $\{S_e : e \in E\}$ that satisfy the following properties:

(1) $S_e^* S_e = P_{s(e)}$ for all $e \in E$.
(2) $P_{r(e)} S_e S_e^* = S_e S_e^*$ for all $e \in E$.
(3) for every v and every set $E' \in C_v$ the set $\{S_e S_e^* : e \in E'\}$ consists of mutually orthogonal projections.
(4) for every v and every set $E' \in C_v$ we have $\sum_{\{e:r(e)=v,\chi(e)=i\}} S_e S_e^* = P_v$, when $\{e : r(e) = v, e \in E'\}$ is finite.

There is a universal C^*-algebra for separated CK-families associated to a separated graph and we denote it by $C^*(G, C)$; we will write $C = \{C_v\}$ so that for any fixed v, C_v is a partition of the edges with range equal to v, a set in the partition we will write as c_v. There is a standard construction of the algebra using free products as follows (see [10, Theorem 2] for more on this particular decomposition). Let G_{c_v} denote the subgraph given by $(V, c_v, r|_{c_v}, s_{x_v})$ then G_{c_v} is a directed graph and $C^*(G, C) = \mathop{*}_{P} \{C^*(G_{c_v}) : c_v \in C_v, v \in V\}$, where P is the subalgebra generated by the set of projections, $\{P_v\}$ which all of the algebras share. We will refer to the G_{c_v} as the decomposed subgraphs of G.

An important point to note is that if no two edges share the same range in (G, C) then the CK-family for the separated graph will be a CK-family for the directed graph G and hence we will often treat such a situation as if (G, C) is a directed graph. We refer the reader to [10] for more details and other relevant results concerning the universal C^*-algebras of separated graphs.

In what follows (G, C) will denote a fixed separated graph. Given a directed graph $G = (V, E, r, s)$ we can consider the underlying undirected graph. This graph is not a graph in the traditional sense (i.e. ordered pairs indicating the presence of an edge between two vertices) since we will allow multiple edges between two vertices. We will assume that the undirected graph is connected (i.e there is a path in the undirected graph between any two vertices in the graph). Without this assumption we can just consider connected components of the undirected graph and see that the C^*-algebra of the separated graph is the direct sum of the separated graph for each connected component.

3. Some non-exact algebras

We start with some (known) examples involving exactness of free products which will be helpful in analyzing the general situation.

Proposition 1. *The following universal free products are not exact:*

(1) $C(\mathbb{T}) \mathop{*}_{\mathbb{C}} C(\mathbb{T})$.

(2) $C(\mathbb{T})\underset{\mathbb{C}}{*}\mathbb{C}^k$ *with* $k \geq 2$.
(3) $M_n\underset{\mathbb{C}}{*}M_m$, *if* $n, m \geq 2$.
(4) $M_r\underset{\mathbb{C}}{*}M_s\underset{\mathbb{C}}{*}M_t$, *if* $r, s, t \geq 2$.

Proof. We notice first that $C(\mathbb{T}) \cong C^*(\mathbb{Z})$ and that $\mathbb{C}^k = C^*(\mathbb{Z}_k)$ for all finite k.

(1) We have that $C(\mathbb{T})\underset{\mathbb{C}}{*}C(\mathbb{T}) \cong C^*(\mathbb{Z})\underset{\mathbb{C}}{*}C^*(\mathbb{Z})$ which is isomorphic to (see for example [7, II.8.3]) $C^*(\mathbb{Z}*\mathbb{Z}) \cong C^*(\mathbb{F}_2)$ which is known to be not exact [16].
(2) Next $C(\mathbb{T})\underset{\mathbb{C}}{*}\mathbb{C}^k \cong C^*(\mathbb{Z})\underset{\mathbb{C}}{*}C^*(\mathbb{Z}_k) \cong C^*(\mathbb{Z}*\mathbb{Z}_k)$. Notice that if the generators of $\mathbb{Z}_k$ are $a_1, a_2, \ldots, a_k$ then consider $A = \mathbb{Z}$ and $B = a_1 A a_1^{-1}$ inside $\mathbb{Z}*\mathbb{Z}_k$ and let C be the subgroup generated by A and B. Then A and B are both isomorphic to $\mathbb{Z}$ and $C = A*B \subseteq \mathbb{Z}*\mathbb{Z}_k$. Hence the associated C^*-algebra contains a subalgebra of the previous type and hence is not exact.
(3) If $n = m = 2$ then this is [9, Proposition 1]. If $m \geq 3$ and $n \geq 2$ then there are subalgebras $A \subseteq M_m$, $B \subseteq M_n$ both isomorphic to M_2 and then there is a quotient from the subalgebra generated by A and B onto $M_2\underset{\mathbb{C}}{*}M_2$ which is the case $n = m = 2$.
(4) This is [9, Proposition 5].

□

Note that in [9, Theorem 2] it is shown that $M_2\underset{\mathbb{C}^2}{*}M_2$ is exact as it is isomorphic to a graph C^*-algebra. This is the motivating example for the idea of reversing edges, that we introduce later. Finally we have that $\mathbb{C}\underset{\mathbb{C}}{*}\mathbb{C}^k \cong \mathbb{C}^k$ and hence any algebra of this form is also nuclear.

We will in what follows do a case by case analysis which will allow us to completely answer the question of exactness for $C^*(G, C)$ when the graph G is finite. For the most part this analysis rests on excluding subalgebras of the type in Proposition 1 using the projections associated to vertices in the directed graph. We consider how such projections can "propagate" through the graph and then consider the added complications that cycles in the undirected graph give rise to.

4. Reversing edges and subgraphs

Let (G, C) be a countable separated graph. If H is a directed subgraph of G then consider (H, C) the separated graph with partitions just being the restrictions to $E(H)$ of the partitions from G. We will call (H, C) a separated subgraph of (G, C). Notice that (H, C) is a separated graph in its own right and we will denote this graph with the same symbol. We have the following results connecting the C^*-algebra of a separated subgraph to a C^*-subalgebra of the original graph C^*-algebra.

Theorem 1. *Let* $(H, C) \subseteq (G, C)$ *and let* A *be the subalgebra of* $C^*(G, C)$ *generated by* $\{S_e : e \in E(H)\}$ *and* $\{P_v : v \in V(H)\}$, *then there is a surjection* $\pi : A \to C^*(H, C)$.

Proof. We first prove this for directed graphs and then use the free product decomposition of a separated graph to extend to the general case.

So assume that H is a subgraph of G and G has trivial separation (i.e. C_v is the single set $\{e : r(e) = v\}$ for every v).Now notice that the subalgebra A will be generated by partial isometries satisfying the following relations:

(1) The P_v are mutually orthogonal nonzero projections.
(2) $S_e^* S_e = P_{s(e)}$.
(3) $P_{r(e)} S_e S_e^* = S_e S_e^*$, and

(4) $\sum_{r(e)=v} S_e S_e^* \leq P_v$ if $\{e : r(e) = v\}$ is finite.

In addition the gauge action on $C^*(G)$ will reduce to a gauge action on the subalgebra A, and hence the family $\{P_v, S_e\}$ generating A is a gauge-invariant Toeplitz-Cuntz-Krieger family. Since the graph algebra $C^*(H)$ is co-universal for gauge-invariant Toeplitz-Cuntz-Kreiger families (see [11], [14], and [15] for a general discussion of co-universality of graph algebras) it follows that there is a $*$-representation $\pi : A \to C^*(H)$ which is onto.

Now for (G, C) we consider the graphs G_{c_v} so that $C^*(G, C) = \underset{P}{*} C^*(G_{c_v})$. Consider $A_{c_v} = A \cap C^*(G_{C_v})$ which is generated by $P' = \{P_v : v \in V(H)\}$ and $\{S_e : e \in E(H) \cap E(G_{c_v})\}$. Then $A = \underset{P'}{*} A_{c_v}$ by [10, Theorem 2] (see also [12, Theorem 4.2]). Now for each c_v there is $\pi_{c_v} : A_{c_v} \to C^*(H_{c_v})$, the latter of which is the subalgebra of $C^*(H, C)$ generated by $\{S_e, e \in c_v\}$ which when restricted to the subalgebra P' all coincide. Thus there exists a $*$-representation $*\,\pi_{c_v} : \underset{P'}{*} A_{c_v} \to \underset{P'}{*} C^*(H_{c_v})$ which is onto. The former algebra is of course A and the latter is $C^*(H, C)$. ☐

Definition 1. We say that (H, C) is a full separated subgraph of (G, C) if the map in the previous theorem is an injection.

For a directed graph the full subgraphs of G are given by those subgraphs such that $\{e \in E(H) : r(e) = v\} = \{e \in E(G) : r(e) = v\}$ for every vertex in H (This is just an application of the gauge-invariant uniqueness theorem for arbitrary graphs, see [6, Theorem 2.1]). A similar result is true of a separated graph.

Proposition 2. *Let $\{e \in G : r(e) = v\}$ be partitioned into disjoint sets. If for each such set, call it c, either $c \subset V(H)$ or $c \cap V(H) = \emptyset$ then the separated subgraph (H, C) is a full subgraph of the separated graph (G, χ).*

Proof. This is just the fact that a Toeplitz-Cuntz-Krieger family gives a Cuntz-Krieger family if and only if the TCK-inequality is an equality for any vertex receiving finitely many edges. It follows that the subalgebra A in $C^*(G, \chi)$ generated by the edge partial isometries and edge projections corresponding to the edges and vertices in (H, χ) gives a Cunt-Krieger family for (H, χ). Hence there is a surjection from $C^*(H, \chi)$ onto A which gives an inverse for the map π from Theorem 1. ☐

In this section we will assume that $G = (V, E, r, s)$ with both E and V being countable. In this case the partition c_v will be consist of at most countably many disjoint sets and we will hence label them with natural numbers to simplify the notation.

We start with the fact that $C^*(G, C)$ is generated by the sets $\mathcal{P} := \{P_v, v \in V\}$ and $\mathcal{S} := \{S_e : e \in E\}$, and $\mathcal{S}^* := \{S_e^* : e \in E\}$ with the following relations:

$$P_v S_e = \begin{cases} S_e, & \text{if } r(e) = v \\ 0 & \text{otherwise.} \end{cases}$$

$$S_e P_v = \begin{cases} S_e, & \text{if } s(e) = v \\ 0 & \text{otherwise.} \end{cases}$$

$S_e S_f = 0,$ if $r(f) \neq s(e)$, otherwise it is nonzero.

$S_e^* S_f = 0,$ if $r(e) \neq r(f)$ or if $r(e) = r(f)$, and e and f are in the same partition in $c_{r(e)}$, otherwise it is nonzero. Also if $e = f$ then this is $P_{s(e)}$.

$S_e S_f^* = 0$ if $s(e) \neq s(f)$ otherwise it is nonzero. If $e = f$ then this is a subprojection of $P_{r(e)}$.

To see that the "nonzero" products above are non-zero one can consider the fact that the products are non-zero in the Leavitt path algebra associated to (G, C) [3] and that the Leavitt path algebra injects into $C^*(G, C)$ applying [4, Theorem 3.8]. However, those products of S_e and S_f^* which are nonzero in the preceding list need not give rise to a partial isometry, since we don't know a priori that the range projections of the S_e commute (in fact they often do not, see [2] where they consider a variant of the separated graph algebras where they quotient by the commutators of non-commuting range projections). An important fact we will use, however, is that when we know that the family $\{S_eS_e^* : e \in E\}$ consists of mutually commuting projections any such product will be a partial isometry [2].

For a vertex v we define the vertex degree of the vertex to be equal to an tuple $d_v = (a_1, a_2, \ldots, a_n)$ where a_i is the number of edges in the i-th partition of c_v. We will allow $a_i = \infty$ in the case that the partition includes a countably infinite set. We also allow d_v to be a sequence (in the case that c_v consists of infinitely many sets).

Remark 1. Notice that by [10, Theorem 2] and by considering the defining relations for the C^*-algebra of an separated graph C^*-algebra we can see that the partition is local in the sense that as long as the local picture at a vertex is unchanged the C^*-algebra is unchanged. Specifically, if for every vertex v there is an injective map $\sigma_v : \mathbb{N} \to \mathbb{N}$ and we define a new partition by applying a permutation to the index set for c_v, calling the new graph $(G, \sigma(C))$. then $C^*(G, \sigma(C))$ is isomorphic to $C^*(G, C)$. It follows that we can without loss of generality assume that $a_1 \geq a_2 \cdots \geq a_n$. We will do so implicitly throughout (with two exceptions that occur inside proofs to simplify the arguments and will be noted explicitly).

We now introduce a construction which will allow us to "reverse" certain edges in the graph without affecting the associated C^*-algebra.

Let (G, C) be an edge-colored directed graph with $e \in E(G)$. Construct a new graph by reversing the edge e, call it G_e. Formally we have $V(G) = V(G_e)$, $E(G_e) = (E(G) \setminus \{e\}) \cup \{\overline{e}\}$, $r_{G_e}(f) = r(f)$ and $s_{G_e}(f) = s(f)$ for all $f \in E(G) \setminus \{e\}$, and $r_{G_e}(\overline{e}) = s(e)$ and $s_{G_e}(\overline{e}) = r(e)$. Next define a new partition C' where $\overline{e}$ is in its own distinct, the partitions that did not include e are left alone and the set $X \in C$ that contained e is either $\emptyset$ in C' if $X = \{e\} \in C$ or $X \setminus \{e\} \in C'$. We say that (G_e, C') is the graph obtained from (G, C) by reversing the edge e. Note that the new graph changes the partitions and hence reversing an edge e and then reversing the reversed edge may not yield the original graph.

Example. Consider the directed graph

$$\begin{tikzcd} \bullet \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & \bullet \end{tikzcd}$$

Reversing the edge g gives a graph which is equivalent to

$$\begin{tikzcd} \bullet \arrow[r, bend left, "f"] & \bullet \arrow[l, bend left, "\widehat{g}"] \end{tikzcd}$$

and then reversing the edge $\widehat{g}$ yields the graph

$$\begin{tikzcd} \bullet \arrow[r, bend left, "f"] \arrow[r, bend right, dotted, "\widehat{\widehat{g}}"'] & \bullet \end{tikzcd}$$

In the first case the associated C^*-algebra is M_3 and the third graph gives rise to the C^*-algebra $M_2 \underset{\mathbb{C}^2}{*} M_2$ which is not finite dimensional.

The next proposition now yields information about the new C^*-algebra when an edge is reversed.

Proposition 3. *Let (G, C) be a separated graph and e an edge in G. If in the partition C, the set X that contains e is a singleton then $C^*(G, C)$ is isomorphic to $C^*(G_e, C')$.*

Proof. For the proof we let $(G \setminus e, C)$ denote the graph obtained from G be removing the edge e, and let (H, C) be the separated graph $(V(G), \{e\}, r, s)$ with $C = \{\{e\}\}$. Then given the construction of the separated graph as a free product we see that $C^*(G, C) = C^*(G_e, C) \underset{P}{*} C^*(H, C)$. The map $s_e \mapsto s_{\bar{e}}^*$, $p_v \mapsto p_v$ yields a separated CK-family for the graph the (H, C) and the reverse map gives a separated CK-family for the graph (H_e, C') and hence using universal properties we have that $C^*(H_e), C') = C^*(H, C)$.

Putting this all together we get the series of isomorphisms

$$\begin{aligned} C^*(G, C) &= C^*(G \setminus e, C) \underset{P}{*} C^*(H, C) \\ &= C^*(G \setminus e, C) \underset{P}{*} C^*(H_e, C') \\ &= C^*(G_e, C'). \end{aligned}$$

□

The same proof applies to the reduced separated graph C^*-algebras using the reduced free product rather than the universal free product (as constructed in [4, Definition 3.5]) since the universal property for the reduced separated graph C^*-algebras will still be preserved by the construction.

Proposition 4. *Let (G, C) be a separated graph and e an edge in G. If in the partition C, the set X that contains e is a singleton then $C_r^*(G, C)$ is isomorphic to $C_r^*(G_e, C')$.*

In the case that $r(e) = s(e)$ this theorem just gives rise to a trivial change and hence doesn't provide any useful difference in the graph, however if $r(e) \neq s(e)$ then this operation can be used to simplify some graphs.

We will say that an edge e in a separated graph (G, C) is reversible if the set $\{e\}$ is in the partition C. In other words, these are the edges for which the previous theorem applies. We denote by E_{rev} the set of all reversible edges in the graph. Similarly we say that a vertex v supports an irreversible edge if there is an edge $e \notin E_{\text{rev}}$ with $r(e) = v$. We denote by V_{irr} the set of vertices which support an irreversible edge.

5. For finite graphs when is $C^*(G, \chi)$ exact/nuclear

We are now in a position to consider exactness/nuclearity of the C^*-algebra of a separated graph. We proceed in cases depending on the set V_{irr}. In each case we will construct an algorithm that either ends when the algebra is not exact, or exhausts the possible graphs.

5.1. The set V_{irr} is empty.

Let (G, C) be a graph in which every edge is reversible (this is equivalent to V_{irr} is empty). In this case $S_e S_e^* = P_{r(e)}$ for every edge e and hence any product of the generating partial isometries will be a partial isometry. We construct an algorithm in which the vertex set is unchanged throughout, but in each iteration of the algorithm we will replace (G, C) with a separated graph (with isomorphic C^*-algebra) with some edges reversed.

Algorithm. *Algorithm for Reversible Edges*

Base step:

Fix a vertex $v \in V$. Let $V_0 = \{v\}$, $E_0^{\text{loop}} = \{e \in E : r(e) = s(e) = v\}$ and let $E_0 = \{e \in E \setminus E_0^{\text{loop}} : r(e) \in V_0\}$. Also let $m_1 = |E_0^{\text{loop}}|$.

Step 1:

Let (G_1, C_1) be the edge-colored directed graph formed by reversing each of the edges in E_0. Let $V_1 = \{w : r(e) = w, s(e) = v$ for some $e \in E(G_1)\} \setminus V_0$. Now $E_1^{\text{loop}} = \{(e, f) \in$

$E(G_1)^2 : e \neq f, s(e), s(f) \in V_0, r(e) = r(f) \in V_1\}$. Now let $E_{1'}^{loop}$ be the set of all edges $e \in E_1$ such that $s(e) \in V_1, r(e) \in V_1$. Let $E_1 = \{e \in E(G_1) : r(e) \in V_1, s(e) \notin V_0 \cup V_1\}$. Finally we let $m_2 = |E_1^{\text{loop}}| + |E_{1'}^{\text{loop}}|$.

Step i:

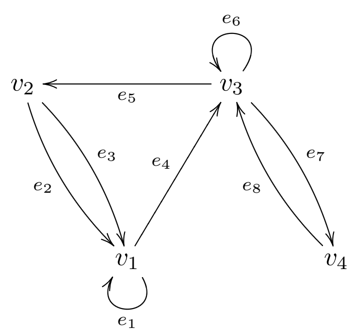
Let (G_i, C_i) be the separated graph formed by reversing each of the edges in E_{i-1}. Let $V_i = \{w :$ there is an $e \in G_i$ with $r(e) = w, s(e) \in V_{i-1}\} \setminus \cup_{n=0}^{i-1} V_n$. Let $E_i^{\text{loop}}\{(e, f) \in E(G_i)^2 : e \neq f$ with $s(e), s(f) \in V_{i-1}, r(e) = s(e) \in V_i\}$ and $E_{i'}^{loop}$ be the set of all edges e such that $s(e) \in V_{i-1}, r(e) \in V_{i-1}$. Let $E_i = \{e \in E(G_i) : r(e) \in V_i, s(e) \notin \cup_{n=i}^{i} V_i\}\}$. Now let $m_i = |E_i^{\text{loop}}|/2 + |E_{i'}^{\text{loop}}|$.

As the original underlying directed graph is connected for any vertex v there is some k such that v is in V_k. As a final step we let $M(G, C) = \sum m_i$.

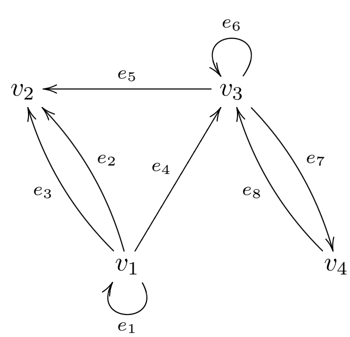
Note that this algorithm is guaranteed to stop since the set of vertices is finite, and at each stage the size of $\cup_{n=1}^{i-1} V_k$ is strictly increasing until the graph is exhausted.

We provide an example to illustrate the algorithm.

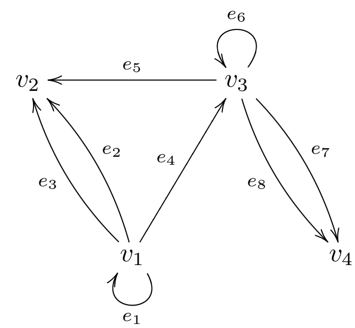
Example. Let G be the directed graph below and assume that each element of C is a singleton, hence (G, C) has no irreversible edges.



In the first step we $V_0 = \{v_1\}$. Then $E_0^{\text{loop}} = \{e_1\}, E_0 = \{e_2, e_3\}$, and $m_1 = 1$. Now we reverse the edges in E_0 to get the graph



Next $V_1 = \{v_2, v_3\}, E_1^{\text{loop}} = \{(e_2, e_3), (e_3, e_2)\}, E_{1'}^{\text{loop}} = \{e_5, e_6\}, E_1 = \{e_8\}$, and $m_2 = 3$. We now reverse the edges in E_1 to get the graph



Now $V_2 = \{v_4\}, E_2^{\text{loop}} = \{(e_7, e_8), (e_8, e_7)\}, E_{2'}^{\text{loop}} = \emptyset, E_2 = \emptyset$, and $m_3 = 1$. We finish by noting that $M(G, C) = 6$.

We have the following theorem which completely describes when the universal separated graph algebras are exact/nuclear in this context.

Theorem 2. *Let (G, C) be a finite separated graph in which every edge is reversible. Then $C^*(G, C)$ is nuclear if $M_{(G,C)} \leq 1$ and $C^*(G, C)$ is not exact if $M_{(G,C)} \geq 2$.*

Proof. We begin with the case in which $M_{(G,C)} = 0$. In this case at the end of the algorithm we are left with a graph (G', C') in which no two edges share a common range. It follows that the graph (G', C') is a directed graph and hence the C^*-algebra is isomorphic to the C^*-algebra of a directed graph. Such C^*-algebras are always nuclear [13, Remark 4.3].

If $M_{(G,\chi)} = 1$ then either there is an edge in $E_{i'}^{\text{loop}}$ or a pair of edges $(e, f) \in (E_i^{\text{loop}})^2$. In either case, after the algorithm is complete we are left with the situation that either e is a loop or there are two edges e, f with $r(e) = r(f)$.

If e is itself a loop then redo the algorithm choosing the initial vertex to be the range of e. Since there are no other "loop" edges we will end up with a separated graph with no two edges sharing a common range, and hence, with an algebra isomorphic to that of a directed graph. Then the C^*-algebra will be isomorphic to a nuclear graph C^*-algebra.

If instead we have that there are two edges e and f with $r(e) = r(f) = w$ then no other edges share a common range. There are then two finite directed paths from the fixed vertex v to w one that ends with the edge e and the other that ends with the edge f. If one then reverses the edges in one of these two paths then one is left with a separated graph (G', C') in which now two edges share the same range. Again this means that we are left with a C^*-algebra isomorphic to a nuclear graph C^*-algebra.

On the other hand, if $M_{(G,\chi)} \geq 2$ then we have at least one of the following graph possibilities:

(1) at least two edges which are loops based at the vertex v;
(2) one edge which is a loop based at the vertex v and there is a pair of edges e and f with $r(e) = r(f)$;
(3) there are three edges e, f, and g with $r(e) = r(f) = r(g)$;
(4) or there are four edges e, f, g, and h with $r(e) = r(f)$ and $r(g) = r(h)$.

In each of these cases we will identify a subalgebra which maps onto a copy of $C^*(\mathbb{F}_2)$. To do this let U and V denote the unitary generators of $C^*(\mathbb{F}_2)$. If $n = |V(G)|$ we will first identify a separated Cuntz-Krieger family in $\mathcal{A} = M_n(C^*(\mathbb{F}_2))$. The identification depends on which of the cases we have above. For notation sake we will write $E_{i,j}$ to be the matrix in $\mathcal{A}$ with a $1_{C^*(\mathbb{F}_2))}$ in the $i - j$ entry and zeroes everywhere else. Similarly we will write $U_{i,j}$ and $V_{i,j}$ for the matrix with U (V respectively) in the i, j entry and zeroes everywhere else.

For the first graph possibility let e be the first loop based at v and g the second loop based at v. We will write $A = S_e$ and $B = S_g$ inside $C^*(G, \chi)$.

In the second situation there is a loop edge g based at v and two directed paths $\mu = e_1 e_2 \cdots e_n e$ and $\nu = f_1 f_2 \cdots f_m f$. Then let

$$A = S_{e_1} S_{e_2} \cdots S_{e_n} S_e S_f^* S_{f_m}^* \cdots S_{f_1}^*$$

and $B = S_g$ in the C^*-algebra $C^*(G, \chi)$.

In the third situation we have paths $\mu = e_1 e_2 \cdots e_n e$, $\nu = f_1 f_2, \cdots f_m f$ and $\tau = g_1 g_2 \cdots g_k g$ all of which begin at v and end at the common vertex $w = r(e) = r(f) = r(g)$. And consider the two elements of $C^*(G, \chi)$ given by

$$A = S_{e_1} S_{e_2} \cdots S_{e_n} S_e S_f^* S_{f_m}^* \cdots S_{f_1}^*$$

and

$$B = S_{g_1} S_{g_2} \cdots S_{g_k} S_g S_f^* S_{f_m}^* \cdots S_{f_1}^*.$$

In the fourth situation we have paths $\mu = e_1 e_2 \cdots e_n e$, $\nu = f_1 f_2, \cdots f_m f$, $\tau = g_1 g_2 \cdots g_k g$, and $\sigma = h_1 h_2 \cdots h_l h$ all of which begin at v and two of which end at the common vertex $w = r(e) = r(f)$ and two of which end at the common vertex $u = r(g) = r(h)$. Here we let

$$A = S_{e_1} S_{e_2} \cdots S_{e_n} S_e S_f^* S_{f_m}^* \cdots S_{f_1}^*$$

and

$$B = S_{g_1} S_{g_2} \cdots S_{g_k} S_g S_h^* S_{h_l}^* \cdots S_{h_1}^*.$$

Now to each vertex in G we assign a unique element $E_{i,i}$ in $M_n(C^*(\mathbb{F}_2))$. If x is the vertex we will write $E_{x,x}$ for the assigned element. Next for any edge d not equal to e or g we assign the partial isometry $d \mapsto E_{s(d),r(d)}$. Finally we map e to $U_{s(e),r(e)}$ and f to $V_{s(f),r(f)}$. It is straightforward to see that these assignments give rise to an edge-colored CK-family associated to the graph and that hence there is a $*$-homomorphism $\pi : C^*(G, \chi) \to M_n(C^*(\mathbb{F}_2))$. Notice that under this homomorphism A is sent to $U_{v,v}$ and B is sent to $V_{v,v}$. It follows that the range of π contains a copy of $C^*(\mathbb{F}_2)$ and hence $C^*(G, \chi)$ is not exact.

$\square$

Remark 2. Although it is not obvious the value of $M_{(G,C)}$ is equal to the topological genus of the underlying undirected graph. We investigate this situation, and are able to completely describe the associated C^*-algebras in [8].

Restating this result we have the following:

Corollary 1. *Let (G, C) be a finite separated graph consisting of only reversible edges. Then $C^*(G, C)$ is nuclear if and only if there is a directed graph which is obtained by reversing edges in (G, C). Otherwise $C^*(G, C)$ is not exact.*

5.2. **Facts about V_{irr}.** Assume that $v \in V_{\text{irr}}$ and let $d_v = (a_1, a_2, \ldots, a_k)$ be the vertex degree of v, and as in Remark 1 we have $a_1 \geq a_2 \geq \cdots \geq a_k$. Since exactness is preserved by subalgebras we will often fix a vertex v and focus on the subalgebra $P_v C^*(G, \chi) P_v$ which if we can show this is non-exact will tell us about the overall algebra. This allows us to rule out a lot of graphs.

Proposition 5. *If $C^*(G, C)$ is exact and $v \in V_{\text{irr}}$ has vertex degree*

$$d_v = (a_1, a_2, \ldots, a_k)$$

then we have that $a_2 \leq 1$.

Proof. Assume that the partition C_v consists of k sets $X_1, X_2, \ldots, X_k$ where $X_i = a_i$ and $a_2 \geq 2$. Let $G_i = (V, X_i, r, s)$ be the directed subgraph of (G, C). Notice that $P_vC^*(G_i)P_v$ is a subalgebra of $C^*(G_i)$ for each i and by [5] the algebra $P_vC^*(G,C)P_v = P_v \underset{i}{*} (C^*(G_i))\, P_v$ contains

$$P_vC^*(G_1)P_v \underset{\mathbb{C}}{*} C^*(G_2) \underset{\mathbb{C}}{*} \cdots \underset{\mathbb{C}}{*} P_vC^*(G_n)P_v,$$

where $\mathbb{C}$ in the free products is the subalgebra generated by P_v. We proceed in cases:

If $s(e) \neq v$ for any $e \in X_i$ for all i then $P_vS^*(G_i)P_v$ is isomorphic to M_{a_i}. It follows that $C^*(G,C)$ contains a subalgebra of the form $M_{a_1} \underset{\mathbb{C}}{*} M_{a_2} \underset{\mathbb{C}}{*} \cdots \underset{\mathbb{C}}{*} M_{a_n}$ which, since $a_1 \geq a_2 \geq 2$ is not exact by Proposition 1.

Consider on the other hand edges e with $s(e) = v = r(e)$ and notice that if $e \in X_i$ then $P_vC^*(G_i)P_v$ contains an algebra which has quotient equal to the Toeplitz algebra (consider the algebra generated by S_e). The subalgebra $P_vC^*(G_i)P_v$ then contains an algebra which has a quotient isomorphic to $C(\mathbb{T})$. Then $P_vC^*(G,C)P_v$ contains a copy of either $M_2 \underset{\mathbb{C}}{*} A$ (if there is just one i containing such an edge e) or $A \underset{\mathbb{C}}{*} A$ where A surjects onto $C(\mathbb{T})$ (if there are two such $X_i, X_j, i \neq j$ containing such edges). Since exactness is preserved by surjections combined with Proposition 1 it follows that $P_vC^*(G,C)P_v$ is not exact. $\square$

We will say that a vertex whose vertex degree is of the form $(a_1, a_2, \ldots, a_n)$ with $a_1 \geq a_2 \geq \cdots \geq a_n$ and $a_2 = 1$ is *vertex-exact*. Then another way of phrasing these propositions is that if the vertex is not vertex-exact then $C^*(G,C)$ is not exact. We will see in what follows that although this is a necessary condition it is not sufficient. Keep in mind though, that if the graph is vertex-exact then the set of range projections $\{S_eS_e^* : e \in E(G)\}$ is commutative and hence we know (in this case) that any finite product of the generating partial isometries is a partial isometry.

We let (G,C) be a separated graph with $V_{\text{irr}} = \{v_1, v_2, \ldots, v_k\}$ and assume that $d_{v_i} = (a_{i,1}, a_{i,2}, \ldots, a_{i,m})$ with $a_1 \geq 2$ and $a_2 \leq 1$ (i.e. each of the vertices is vertex exact but none of the vertices are reversible). We let E_{v_i} consist of those edges e with $r(e) = v_i$ which are in the non-trivial first partition set in the partition C_{v_i}. We let $E_v = \cup E_{v_i}$.

Algorithm. Algorithm Based at Irreducible Vertices

Base step:

Set $V^1 = \{v_1\}$.

Base sub-step: Set $V_0^1 = \{w : w = s(e), r(e) \in V^1, e \notin E_v\}$. Let $E_0^{1\text{loop}} = \{e : s(e) = r(e) \in V^1 \text{ and } e \notin E_v\}$. Let $E_0^1 = \{e : r(e) \in V_0^1, \text{ and } e \notin \left(E_v \cup E_0^{1\text{loop}}\right)\}$.

1st sub-step: Let (G_1^1, C_1^1) be the edge-colored directed graph formed by reversing all of the edges in E_0^1. Let $V_1^1 = \{v \in G_1^1 : r(e) = v, e \in E_0^1\}$. Let $E_1^{1\text{loop}} = \{(e,f) \in E(G_1^1)^2 : e \neq f, r(e) = r(f) \in V_1^1, s(e), s(f) \in V_0^1\}$ and $E_{1'}^{1\text{loop}} = \{e : r(e), s(e) \in V_1^1\}$. Then let $E_1^1 = \{e : r(e) \in V_1^1, e \notin \left(E_v \cup E_1^{1\text{loop}} \cup E_{1'}^{1\text{loop}}\right)\}$.

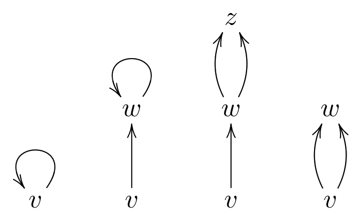
ith sub-step: Let (G_j^1, C_j^1) be the edge-colored directed graph formed by reversing all of the edges in E_{j-1}^1. Let $V_j^1 = \{v \in G_j^1 : r(e) = v, s(e) \in E_{j-1}^1\}$. Let $E_j^{1\text{loop}} = \{(e,f) \in E(G_j^1)^2 \setminus E_v : e \neq f, r(e) = r(f), s(e), s(f) \in V_{j-1}^1\}$ and $E_{i'}^{1\text{loop}} = \{e : r(e), s(e) \in V_j^1\}$. Then let $E_j^1 = \{e : s(e) \in V_j^1, e \notin E_v \cup E_j^{1\text{loop}} \cup E_{j'}^{1\text{loop}}\}$.

Continue repeating sub-steps until you get to a point where $E_j^1 = \emptyset$.

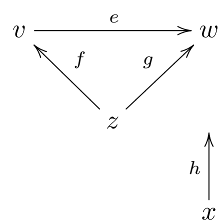
i-th step:

Set $V^i = \{v_i\}$. If $v_i \in V_j^{k-1}$ for any j, k then skip the i-th step. Otherwise, repeat the construction in the base step, replacing all of the 1s with is.

We call the set of loops identified by this algorithm the set of *reversible loops.* Such a loop actually consists of a series of reversible edges starting at some $v \in V_{\text{irr}}$ and (treating concatenated edges traveling in a single direction as a single arrow) looks like one of the following possibilities



Notice that appropriate choices of the partial isometries associated to the reversible edges in the pictures above will yield a partial isometry S such that $S^*S = P_v = SS^*$. For example, if we have the following graph



then the partial isometry $S_hS_fS_eS_g^*S_h^*$ is a partial isometry as described.

Notice that this algorithm stops when V is finite, since we will eventually exhaust all of V_{irr}. However, it is possible after completing this algorithm that there are edges not in E_v that have not appeared in any of the E_i sets. However, since the graph is connected then the set of edges which are not in E_v and have not appeared in any E_i must connect to one of the vertices in V_{irr}. This connection must occur through (at least) one of the elements of E_v, and not through a sequence of reversible edges that ends in V_{irr}, otherwise it would have been swept up in the previous algorithm. We call the set of such edges the set of edges avoiding V_{irr}, and denote it E_a. We proceed with a second algorithm for these edges; it is essentially the same algorithm for the case where V_{irr} is empty with some slight modifications.

Algorithm. Algorithm for Elements of E_v

Base step:

Fix a vertex v which is not in V_i^j for any i, j and such that $v = r(e)$ for some $e \in E_a$. Let $V_0 = \{v\}$, $E_0^{\text{loop}} = \{e : r(e) = s(e) = v\}$ and let $E_0 = \{e \in E \setminus E_0^{\text{loop}} : r(e) \in V_0\}$. Also let $m_1 = |E_0^{\text{loop}}|$.

Step 1:

Let (G_1, C_1) be the separated graph formed by reversing each of the edges in E_0. Now let $V_1 = \{w : r(e) = w, s(e) = v$ for some $e \in E_a\} \setminus V_0$. Now $E_1^{\text{loop}} = \{(e,f) \in (E(G_1))^2 :$ $e\ neqg, s(e), s(f) \in V_0, r(e) = s(e) \in V_1\}$ and let $E_{1'}^{loop}$ be the set of all edges $e \in E_1$ such that $s(e) \in V_1, r(e) \in V_1$. Let $E_1 = \{e \in E(G_1) : r(e) \in V_1, s(e) \notin (V_0 \cup V_1)\}$. Finally we let $m_2 = \frac{|E_1^{\text{loop}}|}{2} + |E_{1'}^{\text{loop}}|$.

Step i:

Let (G_i, C_i) be the separated graph formed by reversing each of the edges in E_{i-1}. Let $V_i = \{w : r(e) = w \notin V_{i-1}, s(e) \in V_{i-1}$ for some $e \in \setminus E_a\}$. Let $E_i^{\text{loop}}\{(e,f) \in (E(G_i))^2 :$ $e \neq g, s(e), s(f) \in V_{i-1}, r(e) = s(e) \in V_i\}$ and $E_{i'}^{loop}$ be the set of all edges $e \in E_{i-1}$ such that $s(e) \in V_{i-1}, r(e) \in V_{i-1}$. Let $E_i = \{e \in E(G_i) : r(e) \in V_i, s(e) \notin \left(\cup_{n=1}^{i} V_i\right)\}$. Now let $m_i = \frac{|E_i^{\text{loop}}|}{2} + |E_{i'}^{\text{loop}}|$.

For the vertex v we define the sum $M_v = \sum m_i$. We then repeat the algorithm for every vertex w which is a range of an edge in E_a that has not already appeared in one of the vertex sets in either algorithm, each time finding a value M_w. As the original underlying directed graph is connected, and the graph is finite then these algorithms will eventually terminate and include every edge and vertex in the original graph.

5.3. **The set V_{irr} is a singleton.** We now have the following proposition which completely describes exactness and nuclearity in the context where V_{irr} is a singleton.

Theorem 3. *Let (G, C) be a finite separated graph in which $V_{\text{irr}} = \{v\}$ and assume that v is vertex-exact. Then $C^*(G, C)$ is not exact if*

- *(G, C) contains a reversible loop.*
- *There is an edge e in E_a such that $M_{r(e)} \geq 2$.*

otherwise $C^*(G, \chi)$ is nuclear.

Proof. Since $v \in V_{\text{irr}}$ we know that there are at least two edges with range v which are in the same partition at v, call these edges e and f. It follows that $P_v C^*(G) P_v$ contains a copy of $\mathbb{C}^k$ where k is the number of edges in said partition.

If (G, C) contains a reversible loop then there is a nontrivial path in the underlying undirected graph which begins and ends at v and does not include any edge in E_v. Let Z be the partial isometry in $C^*(G, \chi)$ which traces out this path. The the subalgebra generated by this partial isometry will be contained in $P_v C^*(G) P_v$. And since all of the edges that give rise to this partial isometry are reversible none of these edges are contained in the partition that contains e.Now $P_v C^*(G) P_v$, as in the proof of the second case of 2 contains a copy of $C(\mathbb{T})$ generated by Z. We now have that $P_v C^*(G, \chi) P_v$ contains a copy of $\mathbb{C}^k \underset{\mathbb{C}}{*} C^*(Z)$ [5] which contains a copy of $\mathbb{C}^k \underset{\mathbb{C}}{*} C(\mathbb{T})$ which is not exact, and hence $C^*(G, C)$ is not exact.

Now if (G, χ) does not contain a reversible loop then after completing Algorithm 5.2 we are left with a graph which is essentially a directed graph except potentially for those edges in E_a (i.e. the edges that are dealt with in Algorithm 5.2). However for each of these subgraphs we use the same arguments as in Theorem 2 to see that if there is e with $M_{s(e)} \geq 2$ then the algebra is not exact and otherwise we have a collection of reversing of edges which turns (G, C) into a directed graph and hence the C^*-algebra is nuclear. $\square$

A variation of the previous then applies to graphs where V_{irr} is not a singleton. The extra complication is presented by the fact that two elements of V_{irr} may be connected via a path. Assume that there are edges $e, f \in E_a$. If after completing the algorithms there is a directed path $\mu = e_1 e_2 \cdots e_n$ such that $s(\mu) = s(e)$ and $r(\mu) = s(f)$, then we say that there is a *path connecting e and f*.

5.4. **The set V_{irr} contains two or more elements.** As in the previous subsection if any element of V_{irr} is not vertex-exact then $C^*(G, C)$ is not exact, so for our purposes we will assume throughout this subsection that every element of V_{irr} is vertex-exact and hence any finite sequence of generating partial isometries is a partial isometry itself.

We will assume the algorithms as described have already been applied to the graph (G, C). Here we have to be a little more careful about paths connecting pairs of edge e and f both elements of E_a since such paths may "connect" two distinct elements of V_{irr}. We also have to deal with directed paths that connect two vertices in V_{irr}. We will treat these cases as one. Let $\mu = e_1 e_2 \cdots, e_n$ be a path in the undirected graph which connects two vertices $v, w \in V_{\text{irr}}$. Notice that, without loss of generality $r(e_1) = v$ or $s(e_1) = v$ and $r(e_n) = w$ or $s(e_n) = w$. (it is possible in this construction that $v = w$).

We will say that μ is reduced if $s(e_1) \neq r(e_1), s(e_n) \neq r(e_n)$, and $e_2, e_3, \ldots, e_{n-1}$ are reversible. Essentially, we don't want v_1 or v_n to be loops and we don't want μ to pass through any elements of V_{irr} except at the start and end of the path.

Now we say that μ has a *reversible end* if v_1 or v_n is reversible. Without loss of generality we will assume v_1 is reversible and in this case we can reverse any of the edges in μ so that there is a directed path from v to w. On the other hand, if neither e_1 nor e_n are reversible then we can form a directed path in the graph by reversing the other edges (as needed) so that $v_2 v_3 \cdots v_n$ is a directed path from $s(v_1)$ to w, or alternatively a directed path from $s(v_n)$ to v.

Assume that (G, C) is vertex-exact and that μ has a reversible end which gives rise to a directed path from v to w (i.e. e_1 is reversible) with e_n in the k-th partition of $C_{s(e_n)}$. If v and w have vertex degree $(a_1, a_2, \ldots, a_n)$ and $(b_1, \ldots, b_m)$, respectively then we define the propagated vertex-degree along μ to equal $(a_1 + b_1 - 1, b_2, b_3, \ldots, b_m)$ if $k = 1$ and $(a_1, b_2, \ldots, b_{k-1}, a_1, b_{k+1}, \ldots, b_m)$ otherwise. In the latter case we will say that the propagated vertex degree along μ is *mixed.*

Proposition 6. *Assume that μ has a reversible end which gives rise to a directed path from v to w (i.e. e_1 is reversible). If $C^*(G, C)$ is exact then the propagated vertex degree along μ is not mixed.*

Proof. Assume that the propagated vertex degree along μ is mixed. For the proof we will relabel the partitions so that $k = 2$. Notice that after appropriate reversing of edges we have $e_1, e_2, \ldots, e_{n-1}$ are reversible and $s(e_n) = r(e_{n-1})$ it follows that $S_\mu := S_{e_n} S_{e_{n-1}} \cdots S_{e_1}$ is a partial isometry with $S_\mu^* S_\mu = P_v$ and $S_\mu S_\mu^* = S_{e_n} S_{e_n}^* = P_w$.

We know that

$$\begin{aligned} S_\mu P_v C^*(G_2) P_v S_\mu^* &= S_\mu S_\mu^* S_\mu C^*(G_2) S_\mu^* S_\mu S_\mu^* \\ &= P_w S_\mu C^*(G_2) S_\mu P_w \\ &\in P_w C^*(G_2) P_w. \end{aligned}$$

It follows that $P_w C^*(G, \chi) P_w$ contains a copy of

$$P_w C^*(G_1) P_w \mathop{*}_{\mathbb{C}} S_\mu P_v C^*(G_2) P_v S_\mu^*$$

(here $\mathbb{C}$ is the subalgebra generated by P_w).

Now we have two cases:

Case 1: ($a_2 \geq 2$, $b_1 \geq 2$ and there is an edge e such that $s(e) = r(e) \in \{v, w\}$.) Without loss of generality assume that $s(e) = v$ and with $k = 2$ we can see that $P_{r(e)} C^*(G_{\chi(e)}) P_{r(e)}$ contains a subalgebra (generated by S_e, an isometry when restricted to $P_{r(e)} C^*(G_{\chi(e)}) P_{r(e)}$) which is an extension of $C(\mathbb{T})$. Then $S_\mu P_v C^*(G_2) P_v S_\mu^*$ contains a copy of M_{b_1} for which P_w is the identity and we are left with a subalgebra of $P_w C^*(G_2) P_w$ of the form $A \mathop{*}_{\mathbb{C}} M_{b_1}$ which has a quotient of the form $C(\mathbb{T}) \mathop{*}_{\mathbb{C}} M_{b_1}$ which is not exact by Proposition 1.

Case 2: ($a_2 \geq 2$, $b_1 \geq 2$ and there are no loops based at v and w.) Just as in the proof of Proposition 1 we can see that $P_v C^*(G_2) P_v$ contains a copy of M_{a_2} and $P_w C^*(G_1) P_w$ contains a copy of M_{b_1} where P_v and P_w are the identities in the associated copies of the matrix algebras. Then $S_\mu P_v C^*(G_2) P_v S_\mu^*$ contains a copy of M_{b_1} for which P_w is the identity and we are left with a subalgebra of $P_w C^*(G_2) P_w$ of the form $M_{a_2} \mathop{*}_{\mathbb{C}} M_{b_1}$ which is not exact by Proposition 1.

In either case we are left with a non-exact algebra contradicting the assumption, hence the propagation along μ is not mixed. $\square$

If every possible propagation along paths is not mixed then we will say the graph is *propagation-exact.*

We are now in a position to prove a theorem about finite separated graph C^*-algebras in the case that V_{irr} contains at least two elements.

Theorem 4. *Let (G, χ) be a finite separated graph in which V_{irr} consists of at least two vertices and is vertex-exact. then $C^*(G, \chi)$ is not exact if*

- *(G, C) contains a reversible loop.*
- *There is an edge e in E_a such that $M_{s(e)} \geq 2$.*
- *The graph is not propagation-exact.*

otherwise $C^*(G, C)$ is nuclear.

Proof. The arguments in the proof of Theorem 3 will apply to show that if (G, C) contains a reversible loop or there is an edge $e \in E_a$ with $M_{s(e)} \geq 2$ then $C^*(G, \chi)$ is not exact. Similarly if the graph is not propagation-exact then the preceding proposition tells us that $C^*(G, C)$ is not exact.

The only difference in the proof comes when we assume that (G, C) does not contain a reversible loop, every edge in $M_{s(e)} \leq 1$ and the graph is propagation-exact. In this case the only technicality may be if $u, v \in V_{\text{irr}}$ and there is a path μ that propagates u to v and μ contains a cycle. However since (G, C) does not contain a reversible loop then the path μ must connect v to w only through two edges e and f both in E_a. Now, since $M_{s(e)} \leq 1$ and $M_{s(e)} \leq 1$ we know that there is at most one cycle in μ. If h is an edge in μ such that μ is not in the cycle then reverse the edge so that it points away from the cycle. After doing this to all edges in μ that are not in the cycle then μ will consist of a directed subgraph of (G, C).

It follows that if (G, C) does not contain a reversible loop, there are no edges in E_a with $M_{s(e)} \geq 2$ and (G, χ) is propagation-exact then again there is a collection of reversing of edges which results in (G, C) turning into a directed graph, and hence the C^*-algebra $C^*(G, C)$ is isomorphic to a nuclear C^*-algebra. □

In effect, when considering the algorithms we have the following corollary which characterizes completely the situation.

Corollary 2. *Let (G, C) be a finite separated graph then $C^*(G, C)$ is nuclear if and only if there is a collection of reversals of reversible edges which results in a directed graph. If no such collection of reversals exists then $C^*(G, C)$ is not exact.*

Of course once we have that a graph is a directed graph we know that the universal separated graph C^*-algebra is isomorphic to the reduced C^*-algebra of the separated graph. We can now conclude the following. (This gives a complete solution to Problem 7.2 of [4] in the case of a finite graph G).

Theorem 5. *For a finite separated graph (G, C), $C^*(G, C)$ is nuclear if and only if $C^*_r(G, C) = C^*(G, C)$.*

Proof. If $C^*(G, C)$ is nuclear then it is isomorphic to $C^*(H)$ where H is a directed graph constructed by reversing reversible edges in (G, C). But by [4, Theorem 3.8] $C^*(H) = C^*_r(H)$ and then applying Proposition 4 we have that $C^*_r(H) \cong C^*_r(G, C)$ and the forward direction follows.

On the other hand if $C^*(G, C) = C^*_r(G, C)$ then $C^*(G, C)$ is a nuclear C^*-algebra since the reduced C^*-algebras are always nuclear. □

Acknowledgments. The author wishes to thank two anonymous referees who found significant errors in previous versions, as well as providing suggestions which vastly improved both the results of the paper and the exposition throughout.

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Received 17.01.2018; Revised 21.01.2020