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ON A NEW CLASS OF OPERATORS RELATED TO QUASI-FREDHOLM OPERATORS

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Abstract. In this paper, we introduce a generalization of quasi-Fredholm operators [7] to k-quasi-Fredholm operators on Hilbert spaces for nonnegative integer k. The case when $k = 0$, represents the set of quasi-Fredholm operators and the meeting of the classes of k-quasi-Fredholm operators is called the class of pseudo-quasi-Fredholm operators. We present some fundamental properties of the operators belonging to these classes and, as applications, we prove some spectral theorem and finite-dimensional perturbations results for these classes. Also, the notion of new index of a pseudo-quasi-Fredholm operator called pq-index is introduced and the stability of this index by finite-dimensional perturbations is proved. This paper extends some results proved in [5] to closed unbounded operators.

1. Introduction and terminology

Let H be a Hilbert space and let $T\ :\mathsf{D}(T)\subseteq\mathsf{H}\longrightarrow\mathsf{H}$ be an unbounded operator with domain $\mathsf{D}(T)$. We denote by $\mathsf{ker}(T)$ the kernel of T, $\alpha(T)=\dim\mathsf{ker}(T)$ the nullity of T, $\mathsf{Im}(T)=T(\mathsf{H})$ the range of T and $\beta(T)=\dim\mathsf{H}/\mathsf{Im}(T)$ its defect. By $\varphi(\mathsf{H})$ (resp. $\mathscr{B}(\mathsf{H})$) we denote the set of all closed (resp. bounded) linear operators on H. Recall that an operator $T\in\varphi(\mathsf{H})$ is said to be s-regular (semi-regular) if $\mathsf{Im}(T)$ is closed and $\mathsf{ker}(T^n)\subseteq\mathsf{Im}(T)$, for all $n\geq 0$. Let $T\in\varphi(\mathsf{H})$, if $\mathsf{Im}(T)$ is closed and $\alpha(T)<+\infty$ (resp. $\beta(T)<+\infty$), then T is called an upper semi-Fredholm (resp. a lower semi-Fredholm) operator. A semi-Fredholm operator is upper or lower semi-Fredholm. Let $\Phi_+(\mathsf{H})$ (resp. $\Phi_-(\mathsf{H})$) denote the set of upper (resp. lower) semi-Fredholm operators. If both $\alpha(T)$ and $\beta(T)$ are finite then T is called a Fredholm operator. This class of operators is denoted by $\Phi(\mathsf{H})$. The index of a semi-Fredholm operator T is defined by

$$\mathrm{ind}(T)=\alpha(T)-\beta(T)\in\mathbb{Z}\cup\{+\infty,\ -\infty\},$$

with the usual convention : $n-\infty=-\infty$ and $+\infty-n=+\infty$, for all $n\in\mathbb{N}$. Let $\sigma(T)$ (resp. $\varrho(T)$) denote the spectrum (resp. the resolvent set) of T.

An operator T is called a Kato type operator if we can write $T=A\oplus S$ where A is a nilpotent operator and S is a s-regular one. In 1958, Kato proved that a closed semi-Fredholm operator is of Kato type. J. P. Labrousse [7] studied and characterized a new class of operators named quasi-Fredholm operators, in the case of Hilbert spaces and he proved that this class coincide with the set of Kato type operators and the Kato decomposition becomes a characterization of the quasi-Fredholm operators. But in the case of Banach spaces the Kato type operator is also quasi-Fredholm, the converse is not true. A bounded operator T on a Banach space is called has a topological uniform descent for $n\geq d$ if $\mathsf{Im}(T)+\mathsf{ker}(T^k)=\mathsf{Im}(T)+\mathsf{ker}(T^d)$, for all $k\geq d$ and $\mathsf{Im}(T)+\mathsf{ker}(T^d)$ is closed [5, Definition 2.5, Theorem 3.2]. This class contains the bounded operators

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belonging to the class of quasi-Fredholm operators. We can find some examples and basic properties of topological uniform descent of bounded operators in [5].

In this paper we introduce two new classes of closed operators in Hilbert spaces, namely, k-quasi-Fredholm and pseudo-quasi-Fredholm operators. The first class is an extension of the class quasi-Fredholm operators, and the second class is the meeting of the classes of k-quasi-Fredholm operators. The study of first (resp. second) class of operators gives a new important part of the ordinary spectrum called the k-quasi-Fredholm (resp. pseudo-quasi-Fredholm) spectrum $\sigma^k_{q\Phi}(T)$ (resp. $\sigma^\infty_{q\Phi}(T)$) which is the set of all complex λ such that $\lambda I - T$ is not k-quasi-Fredholm (resp. pseudo-quasi-Fredholm). Several properties like, spectrum, topological uniform descent, pq-index, and finite perturbation are investigated. Our paper is organized as follows :

In Section 2, we are interested to know the relationship of pseudo-quasi-Fredholm operators and operators having topological uniform descent. We show that the class of pseudo-quasi-Fredholm operators is not stable by the adjoint.

In Sections 3 and 4, we are interested in the spectral theory of k-quasi-Fredholm and pseudo-quasi-Fredholm. We show that they are closed subsets of the spectrum, and that for $T \in \mathscr{B}(\mathsf{H})$, $\sigma^\infty_{q\Phi}(T)$ (resp. $\sigma^k_{q\Phi}(T)$) is empty precisely when T is algebraic. We also show a spectral mapping theorem for pseudo-quasi-Fredholm operators, more precisely in Theorem 4.12, for $T \in \Gamma(\mathsf{H})$ (see page 149) and P is a non-constant complex polynomial, we prove that $P(\sigma^\infty_{q\Phi}(T)) = \sigma^\infty_{q\Phi}(P(T))$ and $\sigma^k_{q\Phi}(P(T)) \subseteq P(\sigma^k_{q\Phi}(T))$, for $k \in \mathbb{N}$. Furthermore, in Theorem 4.16, we prove that if $T \in \mathscr{B}(\mathsf{H})$ and f is an analytic function in a neighborhood of the usual spectrum $\sigma(T)$ and not locally constant in $\sigma(T)$, then $f(\sigma^\infty_{q\Phi}(T)) = \sigma^\infty_{q\Phi}(f(T))$ and $\sigma^k_{q\Phi}(f(T)) \subseteq f(\sigma^k_{q\Phi}(T))$, for $k \in \mathbb{N}$ (in particular, the topological uniform descent spectrum of a bounded operator on a Hilbert space satisfies the spectral mapping theorem).

In Section 5, we are concerned with the stability of the pseudo-quasi-Fredholm spectrum and the k-quasi-Fredholm spectrum under commuting finite rank perturbations. We show that the class of pseudo-quasi-Fredholm operators is not stable under commuting quasi-nilpotent perturbations. We also show that the set of all k-quasi-Fredholm (resp. pseudo-quasi-Fredholm) operators on a Hilbert space H is not open in $\mathscr{B}(\mathsf{H})$.

In Section 6, we introduce, $\mathrm{ind}_{pq}(T)$, the pq-index of a k-quasi-Fredholm operator which coincide with the usual index in the case of a semi-Fredholm operator. The aim of this section is to show that if T possesses pq-index, then T^n (resp. $T+F$) is also a k-quasi-Fredholm operator possesses pq-index and $\mathrm{ind}_{pq}(T^n) = n\,\mathrm{ind}_{pq}(T)$ (resp. $\mathrm{ind}_{pq}(T+F) = \mathrm{ind}_{pq}(T)$), where $n \in \mathbb{N}\backslash\{0\}$ and T, $F \in \mathscr{B}(\mathsf{H})$ such that $TF = FT$ and $\dim \mathsf{Im}(F) < +\infty$. We also show that if $T \in \mathscr{B}(\mathsf{H})$ is k-quasi-Fredholm and $V \in \mathscr{B}(\mathsf{H})$ commutes with T such that $V - T$ is invertible (resp. V is pseudo-quasi-Fredholm) and that $V - T$ is small in norm, then T possesses pq-index if and only if V is semi-Fredholm (resp. V possesses pq-index). In this case $\mathrm{ind}_{pq}(T) = \mathrm{ind}(V)$ (resp. $\mathrm{ind}_{pq}(T) = \mathrm{ind}_{pq}(V)$).

Finally, in Section 7, as an application, some examples are given to illustrate our theorems.

2. Definitions and first Results

For $T \in \varphi(\mathsf{H})$, we consider the sequence

$$S^k_j(T) = \big(\mathsf{Im}(T^j) \cap \ker(T^{k+1}) + \ker(T^k)\big)/\big(\mathsf{Im}(T^{j+1}) \cap \ker(T^{k+1}) + \ker(T^k)\big),$$

j, $k \in \mathbb{N}$. For $k \in \mathbb{N}$, we denote

$$q_k(T) = \inf\{n \in \mathbb{N} : S^k_j(T) = 0,\ \forall\, j \geq n\},$$

where the infimum over the empty set is taken to be infinite.

We have the following lemma, which will be needed in the sequel.

Lemma 2.1. *Let $k \in \mathbb{N}$ and $T \in \varphi(\mathsf{H})$, then*

$$\begin{aligned} q_k(T) &= \inf\{m \in \mathbb{N} : \mathsf{Im}(T) + \mathsf{ker}(T^{k+n}) = \mathsf{Im}(T) + \mathsf{ker}(T^{k+m}),\ \forall\, n \geq m\} \\ &= \max\{q_0(T) - k,\, 0\}. \end{aligned}$$

Proof. Let $k \in \mathbb{N}$ and $\widetilde{T_k}$ be the operator induced by T on $\mathsf{H}/\mathsf{ker}(T^k)$. It is easy to see that

$$\mathsf{ker}[(\widetilde{T_k})^n] = \mathsf{ker}(T^{k+n})/\mathsf{ker}(T^k),$$

$$\mathsf{Im}[(\widetilde{T_k})^n] = [\mathsf{Im}(T^n) + \mathsf{ker}(T^k)]/\mathsf{ker}(T^k),$$

for all $n \in \mathbb{N}$. This gives that

$$\begin{aligned} \mathsf{ker}(\widetilde{T_k}) \cap \mathsf{Im}(\widetilde{T_k}^n) &= \big([\mathsf{Im}(T^n) + \mathsf{ker}(T^k)] \cap \mathsf{ker}(T^{k+1})\big)/\mathsf{ker}(T^k) \\ &= \big(\mathsf{Im}(T^n) \cap \mathsf{ker}(T^{k+1}) + \mathsf{ker}(T^k)\big)/\mathsf{ker}(T^k), \end{aligned} \tag{1}$$

$$\mathsf{Im}(\widetilde{T_k}) + \mathsf{ker}(\widetilde{T_k}^n) = [\mathsf{Im}(T) + \mathsf{ker}(T^{n+k})]/\mathsf{ker}(T^k). \tag{2}$$

From [4, Lemma 2.3], (1) and (2), it follows that

$$\begin{aligned} q_k(T) &= \inf\{m \in \mathbb{N} : \mathsf{ker}(\widetilde{T_k}) \cap \mathsf{Im}(\widetilde{T_k}^n) = \mathsf{ker}(\widetilde{T_k}) \cap \mathsf{Im}(\widetilde{T_k}^m),\ \forall\, n \geq m\} \\ &= \inf\{m \in \mathbb{N} : \mathsf{Im}(\widetilde{T_k}) + \mathsf{ker}(\widetilde{T_k}^n) = \mathsf{Im}(\widetilde{T_k}) + \mathsf{ker}(\widetilde{T_k}^m),\ \forall\, n \geq m\} \\ &= \inf\{m \in \mathbb{N} : \mathsf{Im}(T) + \mathsf{ker}(T^{k+n}) = \mathsf{Im}(T) + \mathsf{ker}(T^{k+m}),\ \forall\, n \geq m\}. \end{aligned}$$

So we deduce that if $k \geq q_0(T)$, then $q_k(T) = 0$ and if $k < q_0(T)$, then $q_0(T) = q_k(T) + k$. This proves that $q_k(T) = \max\{q_0(T) - k,\, 0\}$. The proof is complete. $\square$

The following definition describes the first class of operators we will study.

Definition 2.2. Let $k \in \mathbb{N}$. An operator $T \in \varphi(\mathsf{H})$ is called k-quasi-Fredholm of degree d ($d \in \mathbb{N}$) if :

(*i*) $q_k(T) = d$;
(*ii*) $\mathsf{Im}(T^d) \cap \mathsf{ker}(T^{k+1}) + \mathsf{ker}(T^k)$ is closed in H;
(*iii*) $\mathsf{Im}(T) + \mathsf{ker}(T^{d+k})$ is closed in H.

In the sequel *k*-$q\Phi(d)(\mathsf{H})$, will denote the set of k-quasi-Fredholm operators of degree d. If there is an integer $d \in \mathbb{N}$ such that $T \in$ k-$q\Phi(d)(\mathsf{H})$, then T is called a k-quasi-Fredholm operator. We will denote by k-$q\Phi(\mathsf{H})$ the set of k-quasi-Fredholm operators.

Remark 2.3. Definition 2.2 generalize the well-known notion of a quasi-Fredholm operator (see [7, Definition 3.1.2]), since a quasi-Fredholm operator is a 0-quasi-Fredholm operator.

The following definition describes the second class of operators we will study.

Definition 2.4. Let $T \in \varphi(\mathsf{H})$. Then T is called a pseudo-quasi-Fredholm operator if there is an integer $k \in \mathbb{N}$ such that $T \in$ *k*-$q\Phi(\mathsf{H})$. By $pq\Phi(\mathsf{H})$ we denote the set of all pseudo-quasi-Fredholm operators.

The following example shows that the class of quasi-Fredholm operators is a proper subclass of pseudo-quasi-Fredholm operators.

Example 2.5.

(i) Let H be a Hilbert space with an orthonormal basis $\{e_{i,j} : i, j \in \mathbb{N}\backslash\{0\}\}$ and let T be the operator defined by

$$Te_{i,j} = \begin{cases} 0 & \text{if } j = 1, \\ \dfrac{e_{i,1}}{i+1} & \text{if } j = 2, \\ e_{i,j-1} & \text{otherwise.} \end{cases}$$

We denote by M (resp. N), the vector subspace generated by $(e_{i,j})_{i\geq 1, j\geq 2}$ (resp. $(e_{i,2})_{i\geq 1}$). It is easy to check that $\mathsf{Im}(T) = \mathsf{M} + T(\mathsf{N})$, $T(\mathsf{M}) = \mathsf{M} + T(\mathsf{N})$ and $T^2(\mathsf{N}) = \{0\}$. Therefore $\mathsf{Im}(T) = \mathsf{Im}(T^2)$. Since for all $i \geq 1$, we have $\|T(e_{i,2})\| = \dfrac{1}{i+1}$, then $\mathsf{Im}(T)$ is not closed. Hence $\mathsf{Im}(T^n)$ is not closed for all $n \geq 1$ and so T is not quasi-Fredholm (see, [7, Corollary 3.3.1]). We have $\mathsf{Im}(T) + \mathsf{ker}(T) = \mathsf{H}$, so by Lemma 2.1, we deduce that $T \in 1\text{-}q\Phi(0)(\mathsf{H})$.

(ii) Let H be a separable Hilbert space and let $K \in \mathscr{B}(\mathsf{H})$ such that $\mathsf{Im}(K)$ is not closed. Consider the bounded operator $T : \bigotimes_{i=0}^{\infty} \mathsf{H} \longrightarrow \bigotimes_{i=0}^{\infty} \mathsf{H}$ defined by $T(h_0, h_1, h_2, \ldots) = (K(h_1), h_2, h_3, \ldots)$. Clearly, $\mathsf{Im}(T^2) = \mathsf{Im}(T)$ is not closed and as in (i), we prove that T is 1-quasi-Fredholm but T is not a quasi-Fredholm operator.

Remark 2.6. For $k \in \mathbb{N}$, we note from Lemma 2.1 that $q_k(T) = 0$ if and only if $q_0(T) \leq k$, and hence a bounded operator has a topological uniform descent for $n \geq k$ is a k-quasi-Fredholm operator of zero degree.

Recall that $P(T) \in \varphi(\mathsf{H})$ for every complex polynomial P whenever $\varrho_e^+(T) = \{\lambda \in \mathbb{C} : \lambda I - T \in \Phi_+(\mathsf{H})\} \neq \emptyset$.

In the following proposition, we establish the link between pseudo-quasi-Fredholm operators and operators having a topological uniform descent.

Proposition 2.7. *Let $T \in \varphi(\mathsf{H})$ such that $\varrho_e^+(T) \neq \emptyset$. The following statements are equivalent :*

(i) $T \in pq\Phi(\mathsf{H})$;

(ii) $q_0(T) < +\infty$ *and* $\mathsf{Im}(T) + \mathsf{ker}(T^{q_0(T)})$ *is closed.*

So the set of bounded operators belonging to the class of pseudo-quasi-Fredholm coincides with the class of bounded operators having topological uniform descent in Hilbert spaces.

Proof. "$(i) \Longrightarrow (ii)$" Let $k, d \in \mathbb{N}$ such that $T \in k\text{-}q\Phi(d)(\mathsf{H})$, then by Lemma 2.1, we have $d + k \geq q_0(T)$ and $\mathsf{Im}(T) + \mathsf{ker}(T^{q_0(T)}) = \mathsf{Im}(T) + \mathsf{ker}(T^{d+k})$ is closed.

"$(ii) \Longrightarrow (i)$" We note first that $\mathsf{ker}(T^n)$ is closed for all $n \in \mathbb{N}$ because $\varrho_e^+(T) \neq \emptyset$. Let $k = q_0(T)$, by Lemma 2.1, we get $q_k(T) = 0$ and hence $T \in k\text{-}q\Phi(0)(\mathsf{H})$. This completes the proof. □

The techniques used in this work are based in the concept of paracomplete subspaces of Hilbert spaces (see, [7, Chapter II]).

Definition 2.8 ([7], Definition 2.1.1, Definition 2.1.2)**.**

(i) A subspace M of H is said to be paracomplete in H, if M is a Banach space and the canonical injection of M in H is continuous. In particular, a closed subspace of a Hilbert space H is a paracomplete subspace of H.

(ii) An operator $T : \mathsf{D}(T) \subseteq \mathsf{H} \longrightarrow \mathsf{H}$ is called paracomplete if its graph is a paracomplete subspace of $\mathsf{H} \times \mathsf{H}$. It is clear that a closed operator in a Hilbert space H is a paracomplete operator in H.

The following lemma follows immediately from [7, Proposition 2.2 page 183] and [7, Proposition 2.1.3, Proposition 2.1.4].

Lemma 2.9. *Let* $T : \mathsf{D}(T) \subseteq \mathsf{H} \longrightarrow \mathsf{H}$ *be a paracomplete operator and let* $k, i, n \in \mathbb{N}$. *Then* $\mathsf{D}(T^k)$, $\mathsf{Im}(T^k)$, $\mathsf{ker}(T^k)$, $\mathsf{ker}(T^k) + \mathsf{Im}(T^n)$ *and* $[\mathsf{ker}(T^k) + \mathsf{Im}(T^n)] \cap \mathsf{ker}(T^i)$ *are paracomplete subspaces in* H.

The ascent and descent of $T \in \varphi(\mathsf{H})$ are defined by

$$\boldsymbol{a}(T) = \inf\{n \in \mathbb{N} : \mathsf{ker}(T^n) = \mathsf{ker}(T^{n+1})\},$$
$$\boldsymbol{d}(T) = \inf\{n \in \mathbb{N} : \mathsf{Im}(T^n) = \mathsf{Im}(T^{n+1})\},$$

respectively, whenever these minima exist. If no such numbers exist the ascent and descent of T are defined to be $+\infty$. The notion of ascent and descent was studied in several articles ([4], [8], [11]). Let d be a positive integer, from [11], we mention the following useful characterizations :

$$\boldsymbol{a}(T) \leq d \Longleftrightarrow \mathsf{Im}(T^d) \cap \mathsf{ker}(T^n) = \{0\} \quad \text{for some (equivalently all) } n \geq 1,$$
$$\boldsymbol{d}(T) \leq d \Longleftrightarrow \mathsf{D}(T^d) \subseteq \mathsf{Im}(T^n) + \mathsf{ker}(T^d) \quad \text{for some (equivalently all) } n \geq 1.$$

Remark 2.10.

(i) An operator $T \in \mathscr{B}(\mathsf{H})$ such that $\boldsymbol{d}(T) < +\infty$ and $\mathsf{Im}(T^{\boldsymbol{d}(T)})$ is not closed is a pseudo-quasi-Fredholm operator but is not a quasi-Fredholm operator (see Example 2.5).

(ii) Let $k \in \mathbb{N}\backslash\{0\}$. We know that if $T \in q\Phi(\mathsf{H})$, then $\mathsf{Im}(T^n)$ is closed for all $n \geq q_0(T)$, but if $T \in k\text{-}q\Phi(\mathsf{H})$, we cannot conclude that $\mathsf{Im}(T^n)$ is closed for some $n > q_k(T)$ (see Example 2.5).

(iii) In operators theory, if T is semi-Fredholm (resp. semi-regular, quasi-Fredholm; ...) and its domain is a dense subset of H, then its adjoint T^* is also semi-Fredholm (resp. semi-regular, quasi-Fredholm; ...). Unfortunately, this is not the case for pseudo-quasi-Fredholm operators. In Example 2.5, the operator T is pseudo-quasi-Fredholm, but its adjoint T^* is not pseudo-quasi-Fredholm. In fact, if T^* is pseudo-quasi-Fredholm, then $T^* \in k\text{-}q\Phi(d)(\mathsf{H})$, for some $k, d \in \mathbb{N}$. Hence $\mathsf{Im}(T^*) + \mathsf{ker}({T^*}^{k+d})$ is closed. Since $\mathsf{Im}(T^2) = \mathsf{Im}(T)$, it follows that $\mathsf{ker}({T^*}^2) = \mathsf{ker}(T^*)$ and so $\boldsymbol{a}(T^*) \leq 1$. Therefore $\mathsf{Im}(T^*) + \mathsf{ker}(T^*) = \mathsf{Im}(T^*) + \mathsf{ker}({T^*}^{k+d})$ is closed and $\mathsf{Im}(T^*) \cap \mathsf{ker}(T^*) = \{0\}$ ($k \geq 1$ because T^* is not quasi-Fredholm). From [7, Proposition 2.1.1] and Lemma 2.9, we can see that $\mathsf{Im}(T^*)$ is closed. Hence $\mathsf{Im}(T)$ is closed, which is a contradiction. Consequently, T^* is not pseudo-quasi-Fredholm.

Let M be a closed subspace of H, then H/M is a Hilbert space with the following scalar product

$$\begin{array}{rcccl} \langle \cdot \, , \, \cdot \rangle_{\mathsf{M}} & : & \mathsf{H}/\mathsf{M} \times \mathsf{H}/\mathsf{M} & \longrightarrow & \mathbb{R} \\ & & (\overline{x} \, , \, \overline{y}) & \longmapsto & \langle P(x) \, , \, P(y) \rangle, \end{array}$$

where P is the orthogonal projection on $\mathsf{M}^{\perp}$ and $\langle \cdot \, , \, \cdot \rangle$ is the scalar product of H. Note that the topology in the Hilbert space $(\mathsf{H}/\mathsf{M}, \langle \cdot \, , \, \cdot \rangle_{\mathsf{M}})$ coincides with the quotient topology in H/M :

$$\|\overline{x}\| = \sqrt{\langle \overline{x}, \overline{x} \rangle_{\mathsf{M}}} = \sqrt{\langle P(x), P(x) \rangle} = \mathrm{dist}(x, \mathsf{M}),$$

where $\mathrm{dist}(x, \mathsf{M})$ is the distance of x to M. In particular, if $T \in \varphi(\mathsf{H})$ such that $\mathsf{ker}(T^k)$ is closed for $k \in \mathbb{N}$, then $\mathsf{H}/\mathsf{ker}(T^k)$ is a Hilbert space. For $k \in \mathbb{N}$, let $\widetilde{T_k}$ denote the following operator

$$\begin{array}{rcccl} \widetilde{T_k} & : & \mathsf{D}(\widetilde{T_k}) \subseteq \mathsf{H}/\mathsf{ker}(T^k) & \longrightarrow & \mathsf{H}/\mathsf{ker}(T^k) \\ & & \overline{x} & \longmapsto & \overline{Tx}. \end{array}$$

By $q\Phi(\mathsf{H})$ (resp. $q\Phi(d)(\mathsf{H})$) we denote the set of all quasi-Fredholm operators (resp. of degree d).

Proposition 2.11. *Let $T\ :\mathsf{D}(T)\subseteq\mathsf{H}\longrightarrow\mathsf{H}$ be a paracomplete operator and $k,\,d\in\mathbb{N}$ such that $\mathsf{ker}(T^k)$ is closed. Then*

$$T\in k\text{-}q\Phi(d)(\mathsf{H})\Longleftrightarrow\widetilde{T_k}\in q\Phi(d)(\mathsf{H}/\mathsf{ker}(T^k)).$$

Proof. Define

$$\begin{array}{rcccl}\pi & : & \mathsf{H}\times\mathsf{H} & \longrightarrow & (\mathsf{H}/\mathsf{ker}(T^k))\times(\mathsf{H}/\mathsf{ker}(T^k))\\ & & (x,\,y) & \longmapsto & (\overline{x},\,\overline{y}).\end{array}$$

Since $\mathsf{G}(\widetilde{T_k})$, the graph of $\widetilde{T_k}$ is equal to $\pi(\mathsf{G}(T))$, we deduce from [7, Proposition 2.1.4], that $\mathsf{G}(\widetilde{T_k})$ is paracomplete. For all $n\in\mathbb{N}$, we have

$$\mathsf{Im}(\widetilde{T_k})+\mathsf{ker}(\widetilde{T_k}^n)=[\mathsf{Im}(T)+\mathsf{ker}(T^{n+k})]/\mathsf{ker}(T^k)\tag{1}$$

and

$$\mathsf{ker}(\widetilde{T_k})\cap\mathsf{Im}(\widetilde{T_k}^n)=\big(\mathsf{Im}(T^n)\cap\mathsf{ker}(T^{k+1})+\mathsf{ker}(T^k)\big)/\mathsf{ker}(T^k).\tag{2}$$

Now by (2) we deduce that $q_k(T)=q_0(\widetilde{T_k})$. If $\widetilde{T_k}\in q\Phi(d)(\mathsf{H}/\mathsf{ker}(T^k))$, from [7, Remark page 205], it follows that $\widetilde{T_k}$ is closed. So, by [9, Lemma 1.4], there exists $\lambda\in\mathbb{C}\backslash\{0\}$ such that $\lambda I-\widetilde{T_k}$ is s-regular. Since $\mathsf{Im}(\lambda I-\widetilde{T_k})=\mathsf{Im}(\lambda I-T)/\mathsf{ker}(T^k)$ and $\mathsf{ker}(\lambda I-\widetilde{T_k})=[\mathsf{ker}(\lambda I-T)+\mathsf{ker}(T^k)]/\mathsf{ker}(T^k)$ are closed, then by Lemma 2.9 and [7, Proposition 2.1.1], we see that $\mathsf{Im}(\lambda I-T)$ and $\mathsf{ker}(\lambda I-T)$ are also closed and consequently $T=\lambda I-(\lambda I-T)$ is closed (see, [7, Proposition 2.2.3]). So by (1) and (2), we get

$$T\in k\text{-}q\Phi(d)(\mathsf{H})\Longleftrightarrow\widetilde{T_k}\in q\Phi(d)(\mathsf{H}/\mathsf{ker}(T^k)).$$

The proof is complete. □

As a direct consequence of Proposition 2.11 and [7, Remark page 205] we obtain the following result :

Corollary 2.12. *Let $k\in\mathbb{N}$ and $T\ :\mathsf{D}(T)\subseteq\mathsf{H}\longrightarrow\mathsf{H}$ be a paracomplete operator such that*

(i) $q_k(T)=d<+\infty$ and $\mathsf{ker}(T^k)$ is closed in H,
(ii) $\mathsf{Im}(T^d)\cap\mathsf{ker}(T^{k+1})+\mathsf{ker}(T^k)$ is closed in H,
(iii) $\mathsf{Im}(T)+\mathsf{ker}(T^{d+k})$ is closed in H,

then T is closed operator i.e., $T\in k\text{-}q\Phi(d)(\mathsf{H})$.

Next we proceed to obtain a necessary condition and a sufficient condition for that a k-quasi-Fredholm operator is a quasi-Fredholm operator.

Theorem 2.13. *Let $k,\,d\in\mathbb{N}$ and $T\in k\text{-}q\Phi(d)(\mathsf{H})$. Then*

$$T\in q\Phi(\mathsf{H})\Longleftrightarrow\mathsf{ker}(T)\cap\mathsf{Im}(T^{d+k})\quad\textit{is closed.}$$

Proof. By Lemma 2.1, we conclude that $q_0(T)\leq d+k$ and

$$\mathsf{Im}(T)+\mathsf{ker}(T^{q_0(T)})=\mathsf{Im}(T)+\mathsf{ker}(T^{d+k})$$

is closed. Hence

$$T\in q\Phi(q_0(T))(\mathsf{H})\Longleftrightarrow\mathsf{ker}(T)\cap\mathsf{Im}(T^{q_0(T)})=\mathsf{ker}(T)\cap\mathsf{Im}(T^{k+d})\quad\text{is closed.}$$

This completes the proof of the theorem. □

3. Pseudo-quasi-Fredholm spectrum and k-quasi-Fredholm spectrum

Throughout the remainder of the paper, for $T \in \varphi(\mathsf{H})$ and $\lambda \in \mathbb{C}$, we denote by T_λ the operator $\lambda I - T$.

For $k \in \mathbb{N}$, the k-quasi-Fredholm resolvent and k-quasi-Fredholm spectrum of an operator $T \in \varphi(\mathsf{H})$ are defined respectively by

$$\varrho^k_{q\Phi}(T) = \{\lambda \in \mathbb{C} \ : T_\lambda \in k\text{-}q\Phi(\mathsf{H})\}$$

and

$$\sigma^k_{q\Phi}(T) = \mathbb{C} \backslash \varrho^k_{q\Phi}(T).$$

We denote by $\sigma_e(T)$ the essential quasi-Fredholm spectrum of T (see [9]). We note that $\sigma_e(T) = \sigma^0_{q\Phi}(T)$. The set $\sigma^\infty_{q\Phi}(T) := \bigcap\limits_{k \geq 0} \sigma^k_{q\Phi}(T)$ is called pseudo-quasi-Fredholm spectrum of T. The complementary set $\varrho^\infty_{q\Phi}(T) = \mathbb{C} \backslash \sigma^\infty_{q\Phi}(T)$ is the pseudo-quasi-Fredholm resolvent. For all $k \in \mathbb{N}$, it is clear that

$$\varrho(T) \subseteq \varrho^k_{q\Phi}(T) \subseteq \varrho^\infty_{q\Phi}(T).$$

If $T \in \mathscr{B}(\mathsf{H})$, it follows from Proposition 2.7 that

$$\varrho^\infty_{q\Phi}(T) = \{\lambda \in \mathbb{C} \ : T_\lambda \text{ has topological uniform descent}\}.$$

Throughout this section we assume that $\varrho^+_e(T) \neq \emptyset$.

Now, we are ready to state our main result of this section, which represents an improvement of [9, Lemma 1.4] to the class of k-quasi-Fredholm operators.

Lemma 3.1. *Let d, $k \in \mathbb{N}$ and $T \in k\text{-}q\Phi(d)(\mathsf{H})$, then there exists $\varepsilon > 0$ such that for all $\lambda \in \mathbb{C}$, $0 < |\lambda| < \varepsilon$:*

(i) *T_λ is a s-regular operator,*
(ii) $\alpha(T_\lambda) = \dim \ker(T) \cap \operatorname{Im}(T^{d+k})$,
(iii) $\beta(T_\lambda) = \dim \mathsf{H}/[\operatorname{Im}(T) + \ker(T^{d+k})]$.

Proof. From Proposition 2.11, we know that $\widetilde{T_k} \in q\Phi(d)(\mathsf{H}/\ker(T^k))$. We apply now [9, Lemma 1.4], we deduce that there exists $\varepsilon > 0$ such that for all $\lambda \in \mathbb{C}$, $0 < |\lambda| < \varepsilon$, we have

$$\lambda I - \widetilde{T_k} \text{ is s-regular,} \tag{1}$$

$$\alpha(\lambda I - \widetilde{T_k}) = \dim(\ker(\widetilde{T_k}) \cap \operatorname{Im}(\widetilde{T_k}^d)), \tag{2}$$

$$\beta(\lambda I - \widetilde{T_k}) = \dim(\mathsf{H}/\ker(T^k))/[\operatorname{Im}(\widetilde{T_k}) + \ker(\widetilde{T_k}^d)]. \tag{3}$$

As $\ker(T^k) \subseteq \operatorname{Im}[(T_\lambda)^n]$, we have for all $n \in \mathbb{N}$,

$$\operatorname{Im}[(\lambda I - \widetilde{T_k})^n] = [\operatorname{Im}[(T_\lambda)^n] + \ker(T^k)]/\ker(T^k) = \operatorname{Im}[(T_\lambda)^n]/\ker(T^k)$$

and

$$\ker[(\lambda I - \widetilde{T_k})^n] = \big(\ker[(T_\lambda)^n] + \ker(T^k)\big)/\ker(T^k).$$

(i) By (1), we obtain

$$\ker(T_\lambda) \subseteq \ker(T_\lambda) + \ker(T^k) \subseteq \operatorname{Im}[(T_\lambda)^n], \quad \forall\, n \in \mathbb{N}$$

and it follows that $\mathsf{Im}(T_\lambda)$ is closed. So T_λ is s-regular for all $0 < |\lambda| < \varepsilon$.

(ii) Since $\mathsf{ker}(T^k) \cap \mathsf{ker}(T_\lambda) = \{0\}$, it follows from (2) that

$$\begin{aligned}
\alpha(T_\lambda) &= \dim[\mathsf{ker}(T_\lambda) + \mathsf{ker}(T^k)]/\mathsf{ker}(T^k) \\
&= \alpha(\lambda I - \widetilde{T_k}) \\
&= \dim \mathsf{ker}(\widetilde{T_k}) \cap \mathsf{Im}(\widetilde{T_k}^d) \\
&= \dim \big([\mathsf{Im}(T^d) + \mathsf{ker}(T^k)] \cap \mathsf{ker}(T^{k+1})\big)/\mathsf{ker}(T^k) \\
&= \dim \big(\mathsf{Im}(T^d) \cap \mathsf{ker}(T^{k+1}) + \mathsf{ker}(T^k)\big)/\mathsf{ker}(T^k) \\
&= \dim \big(\mathsf{Im}(T^d) \cap \mathsf{ker}(T^{k+1})\big)/\big(\mathsf{Im}(T^d) \cap \mathsf{ker}(T^k)\big) \\
&= \dim \mathsf{ker}(S^{k+1})/\mathsf{ker}(S^k), \quad \text{where } S = T_{|\mathsf{Im}(T^d)} \\
&= \dim \mathsf{ker}(S) \cap \mathsf{Im}(S^k) \\
&= \dim \mathsf{ker}(T) \cap \mathsf{Im}(T^{d+k}).
\end{aligned}$$

(iii) From (3), we get

$$\begin{aligned}
\beta(T_\lambda) &= \beta(\lambda I - \widetilde{T_k}) \\
&= \dim \big(\mathsf{H}/\mathsf{ker}(T^k)\big)/\big(\mathsf{Im}(\widetilde{T_k}) + \mathsf{ker}(\widetilde{T_k}^d)\big) \\
&= \dim \mathsf{H}/\big(\mathsf{Im}(T) + \mathsf{ker}(T^{d+k})\big).
\end{aligned}$$

The proof is complete. □

Corollary 3.2. *Let $T \in \varphi(\mathsf{H})$ and $k \in \mathbb{N}$. Then $\sigma^k_{q\Phi}(T)$ and $\sigma^\infty_{q\Phi}(T)$ are closed.*

For $T \in \varphi(\mathsf{H})$, we consider the following :

$$\mathsf{E}(T) = \{\lambda \in \sigma(T) \ : \lambda \text{ an isolated point, } \boldsymbol{a}(T_\lambda) < +\infty, \ \boldsymbol{d}(T_\lambda) = m < +\infty \text{ and } \mathsf{Im}[(T_\lambda)^m] \text{ is closed}\}.$$

Let's recall that if $\varrho(T) \neq \emptyset$, (see, [8, Theorem 2.1])

$$\mathsf{E}(T) = \{\lambda \in \sigma(T) \ : \boldsymbol{a}(T_\lambda) = \boldsymbol{d}(T_\lambda) < +\infty\}.$$

Theorem 3.3. *Let $T \in \varphi(\mathsf{H})$ and $k \in \mathbb{N}$. Then*

$$\partial\sigma(T) \cap \varrho^k_{q\Phi}(T) = \partial\sigma(T) \cap \varrho^\infty_{q\Phi}(T) = \mathsf{E}(T).$$

Proof. The case $\varrho(T) = \emptyset$ is trivial, so assume that $\varrho(T) \neq \emptyset$. Clearly, the following inclusions hold :

$$\mathsf{E}(T) \subseteq \partial\sigma(T) \cap \varrho^k_{q\Phi}(T) \subseteq \partial\sigma(T) \cap \varrho^\infty_{q\Phi}(T).$$

For the reverse inclusions, let $\mu \in \partial\sigma(T) \cap \varrho^\infty_{q\Phi}(T)$, we denote by $R = \mu I - T$. Let $k,\, d \in \mathbb{N}$ such that $R \in k\text{-}q\Phi(d)(\mathsf{H})$. We know from Lemma 3.1, that there exists $\varepsilon > 0$ such that

$$\alpha(\lambda I - R) = \dim \mathsf{ker}(R) \cap \mathsf{Im}(R^{d+k}) \quad \text{and} \quad \beta(\lambda I - R) = \dim \mathsf{H}/[\mathsf{Im}(R) + \mathsf{ker}(R^{d+k})],$$

for all $0 < |\lambda| < \varepsilon$. Since $\varrho(R) \cap \{\lambda \in \mathbb{C} \ : 0 < |\lambda| < \varepsilon\} \neq \emptyset$, we deduce that

$$\alpha(\lambda I - R) = \beta(\lambda I - R) = 0, \quad \forall\, 0 < |\lambda| < \varepsilon.$$

This leads to $\boldsymbol{a}(R) = \boldsymbol{d}(R) \leq d + k$ and $\mu \in \mathsf{E}(T)$. This completes the proof. □

We recall that $T \in \mathscr{B}(\mathsf{H})$ is called algebraic if $P(T) = 0$ for some nonzero polynomial P. Arguing as in the proof of [2, Theorem 1.5], we get the following result :

$$T \text{ is algebraic } \iff \sigma(T) = \{\lambda_1,\, \lambda_2, \ldots, \lambda_n\} = \mathsf{E}(T).$$

In the following theorem, we show that the operators whose k-quasi-Fredholm spectrum is empty are exactly the algebraic operators.

Theorem 3.4. *Let $T \in \mathscr{B}(\mathsf{H})$ and $k \in \mathbb{N}$, then the following conditions are equivalent :*

(*i*) $\sigma^k_{q\Phi}(T) = \emptyset$;
(*ii*) $\sigma^\infty_{q\Phi}(T) = \emptyset$;
(*iii*) T *is algebraic.*

Proof. "$(i) \Longrightarrow (iii)$" We have $\varrho^k_{q\Phi}(T) = \mathbb{C}$, this implies that $\mathsf{E}(T) = \varrho^k_{q\Phi}(T) \cap \partial\sigma(T) = \partial\sigma(T) \neq \emptyset$ and hence $\sigma(T) = \mathsf{E}(T)$. Consequently, T is algebraic.

"$(iii) \Longrightarrow (i)$" T is algebraic implies that $\sigma(T) = \mathsf{E}(T) = \varrho^k_{q\Phi}(T) \cap \partial\sigma(T) \subseteq \varrho^k_{q\Phi}(T)$. Therefore $\varrho^k_{q\Phi}(T) = \mathbb{C}$.

In the same way, we obtain the following equivalence :

$$\sigma^\infty_{q\Phi}(T) = \emptyset \Longleftrightarrow T \quad \text{is algebraic.}$$

This completes the proof of the proposition. □

4. A spectral mapping theorem for pseudo-quasi-Fredholm

For $T : \mathsf{D}(T) \subseteq \mathsf{H} \longrightarrow \mathsf{H}$, we denote by

$$\mathsf{do}(T) = \inf\{n \in \mathbb{N} : \mathsf{D}(T^n) = \mathsf{D}(T^{n+1})\},$$

where the infimum over the empty set is taken to be $+\infty$ (see, [4, page 31]). We remark that if $\mathsf{do}(T) < +\infty$, then

$$\mathsf{D}(T^{\mathsf{do}(T)}) = \mathsf{D}(T^{\mathsf{do}(T)+n}) \subseteq \mathsf{D}(T^n), \quad \forall\ n \in \mathbb{N}.$$

Consequently $T\big(\mathsf{D}(T^{\mathsf{do}(T)})\big) = T\big(\mathsf{D}(T^{\mathsf{do}(T)+1})\big) \subseteq \mathsf{D}(T^{\mathsf{do}(T)})$.

Of course, there exist operators such that $\mathsf{do}(T) = +\infty$ and operators such that $\mathsf{do}(T) < +\infty$. This can be illustrated in the following example.

Example 4.1.

(*i*) Let $\mathsf{H} = \mathsf{L}^2(\mathbb{R})$ and $n \in \mathbb{N}$, we define the subspace D_n of H by

$$\mathsf{D}_n = \Big\{ f \in \mathsf{H} : \int_{\mathbb{R}} t^{2n} |f(t)|^2 dt < +\infty \Big\},$$

and the operator T by

$$\begin{array}{rcccl} T : & \mathsf{D}(T) \subseteq \mathsf{H} & \longrightarrow & \mathsf{H} & \\ & f & \longmapsto & \psi f, & \text{with } \ \psi(t) = t. \end{array}$$

It is clear that $\mathsf{D}(T^n) = \mathsf{D}_n$ and hence $\mathsf{do}(T) = +\infty$. For $q \in \mathbb{N}$, we define

$$\begin{array}{rcccl} S : & \mathsf{D}(S) \subseteq \mathsf{H}/\mathsf{D}(T^q) & \longrightarrow & \mathsf{H}/\mathsf{D}(T^q) \\ & \overline{f} & \longmapsto & \overline{T(f)}. \end{array}$$

Since $\mathsf{D}(S^q) = \{0\}$ and $\mathsf{D}(S^{q-1}) \neq \{0\}$ (if $q > 0$), then $\mathsf{do}(S) = q$.

(*ii*) Let H be a separable Hilbert space and let $K : \mathsf{D}(K) \subseteq \mathsf{H} \longrightarrow \mathsf{H}$. Consider the linear operator $T : \mathsf{D}(T) \subseteq \bigotimes_{i=0}^\infty \mathsf{H} \longrightarrow \bigotimes_{i=0}^\infty \mathsf{H}$ defined by $T(h_0, h_1, h_2, \ldots) = (K(h_1), h_2, h_3, \ldots)$. Clearly, $\mathsf{D}(T^k) = \mathsf{H} \times \bigotimes_{i=1}^{i=k} \mathsf{D}(K) \times \bigotimes_{i=k+1}^\infty \mathsf{H}$. Hence $\mathsf{do}(T) = +\infty$ if $\mathsf{D}(K) \subsetneq \mathsf{H}$ and $\mathsf{do}(T) = 0$ when $\mathsf{D}(K) = \mathsf{H}$.

Let us consider the following class :

$$\begin{aligned} \Gamma(\mathsf{H}) = \{ & T : \mathsf{D}(T) \subseteq \mathsf{H} \longrightarrow \mathsf{H} \text{ paracomplete} : q = \mathsf{do}(T) < +\infty, \\ & \mathsf{D}(T^q) \text{ and } \mathsf{Im}(T_\lambda) + \mathsf{D}(T^q) \text{ are closed}, \forall\ \lambda \in \mathbb{C}\}. \end{aligned}$$

It is clear that $\mathscr{B}(\mathsf{H}) \subseteq \Gamma(\mathsf{H})$. Assume that T is a paracomplete operator such that $q = \mathsf{do}(T) < +\infty$. It is easy to see that if P is a complex polynomial, then $P(T)$ is paracomplete and $\mathsf{do}(P(T)) \leq q$. Furthermore, if P is a non-constant complex polynomial, then $\mathsf{D}([P(T)]^n) = \mathsf{D}(T^q)$, for all $n \geq \mathsf{do}(P(T))$. We will show that if $T \in \Gamma(\mathsf{H})$, then $P(T) \in \Gamma(\mathsf{H})$, for all complex polynomial P. Set $q = \mathsf{do}(T)$ and define

$$\begin{array}{rcccc} \overline{T} & : & \mathsf{D}(\overline{T}) \subseteq \mathsf{H}/\mathsf{D}(T^q) & \longrightarrow & \mathsf{H}/\mathsf{D}(T^q) \\ & & \overline{x} & \longmapsto & \overline{Tx}. \end{array}$$

Let $\lambda \in \mathbb{C}$ and $\overline{x} \in \ker(\lambda I - \overline{T})$, then $T_\lambda x \in \mathsf{D}(T^q)$. Clearly, $x \in \mathsf{D}(T^{q+1}) = \mathsf{D}(T^q)$ and $\overline{x} = 0$, so $\ker(\lambda I - \overline{T}) = \{0\}$. Let us remark that $\mathsf{Im}(\lambda I - \overline{T}) = [\mathsf{Im}(T_\lambda) + \mathsf{D}(T^q)]/\mathsf{D}(T^q)$ is closed. As in the proof of Proposition 2.11, we prove that $\lambda I - \overline{T}$ is paracomplete and so by [7, Proposition 2.2.3], $\lambda I - \overline{T} \in \varphi(\mathsf{H}/\mathsf{D}(T^q))$. Hence $\lambda I - \overline{T} \in \Phi_+(\mathsf{H}/\mathsf{D}(T^q))$. Now, let $P(Z) = (\lambda_1 - Z)^{\alpha_1}(\lambda_2 - Z)^{\alpha_2} \cdots (\lambda_m - Z)^{\alpha_m}$ be a complex polynomial. We know that if $S, L \in \varphi(\mathsf{H})$ such that $L \in \Phi_+(\mathsf{H})$ and $\mathsf{Im}(S)$ is closed, then $LS \in \varphi(\mathsf{H})$ and $\mathsf{Im}(LS)$ is closed. For $i, j \in \{1, 2, \ldots, m\}$, we have $\lambda_i I - \overline{T} \in \Phi_+(\mathsf{H}/\mathsf{D}(T^q))$ and $\mathsf{Im}(\lambda_j I - \overline{T})$ is closed, therefore $(\lambda_i I - \overline{T})(\lambda_j I - \overline{T}) \in \varphi(\mathsf{H}/\mathsf{D}(T^q))$ and $\mathsf{Im}[(\lambda_i I - \overline{T})(\lambda_j I - \overline{T})]$ is closed. Since $\ker[(\lambda_i I - \overline{T})(\lambda_j I - \overline{T})] = \{0\}$, then $(\lambda_i I - \overline{T})(\lambda_j I - \overline{T}) \in \Phi_+(\mathsf{H}/\mathsf{D}(T^q))$ and consequently $\mathsf{Im}(P(\overline{T})) = [\mathsf{Im}[P(T)] + \mathsf{D}(T^q)]/\mathsf{D}(T^q)$ is closed. Finally, we deduce that $\mathsf{Im}[P(T)] + \mathsf{D}(T^q) = \mathsf{Im}[P(T)] + \mathsf{D}[(P(T))^{\mathsf{do}(P(T))}]$ is closed and $P(T) \in \Gamma(\mathsf{H})$.

Example 4.2.

(*i*) Let H be a separable Hilbert space and let $K \in \varphi(\mathsf{H})$ such that $\mathsf{D}(K) \subsetneq \mathsf{H}$ is closed. Let $\mathcal{H} = \bigotimes_{i=0}^{3} \mathsf{H}$ and consider the linear operator $T : \mathcal{H} \longrightarrow \mathcal{H}$ defined by $T(h_0, h_1, h_2, h_3) = (K(h_1), h_2, h_3, h_3)$. Clearly,

$$\mathsf{D}(T^k) = \begin{cases} \mathsf{H} \times \mathsf{D}(K) \times \mathsf{H} \times \mathsf{H} & \text{if } k = 1, \\ \mathsf{H} \times \mathsf{D}(K) \times \mathsf{D}(K) \times \mathsf{H} & \text{if } k = 2, \\ \mathsf{H} \times \mathsf{D}(K) \times \mathsf{D}(K) \times \mathsf{D}(K) & \text{if } k \geq 3 \end{cases}$$

is closed. Hence $\mathsf{do}(T) = 3$. It is not difficult to see that

$$\mathsf{Im}(T_\lambda) + \mathsf{D}(T^3) = \begin{cases} \mathsf{H} \times \mathsf{H} \times \mathsf{H} \times \mathsf{D}(K) & \text{if } \lambda = 1, \\ \mathsf{H} \times \mathsf{H} \times \mathsf{H} \times \mathsf{H} & \text{if } \lambda \neq 1 \end{cases}$$

is closed. Since $T \in \varphi(\mathcal{H})$, it follows that $T \in \Gamma(\mathcal{H})$.

(*ii*) Let H be a separable Hilbert space and $\{e_n : n \in \mathbb{N}\}$ be an orthonormal basis of H. Define the following operators T and L on H by

$$\mathsf{D}(T) = \mathsf{D}(L) = \langle e_n : n \geq 2 \rangle, \quad T(e_n) = e_{n+1} \quad \text{and} \quad L(e_n) = e_{n-1}, \quad \forall\, n \geq 2.$$

It is clear that $\mathsf{D}(T^k) = \mathsf{D}(T)$ and $\mathsf{D}(L^k) = \langle e_n : n \geq 1 + k \rangle$, for all $k \geq 1$ and hence $\mathsf{do}(T) = 1$ and $\mathsf{do}(L) = +\infty$ ($L \notin \Gamma(\mathsf{H})$). Since $T \in \varphi(\mathsf{H})$, $\mathsf{Im}(T_\lambda) \subseteq \mathsf{D}(T)$ for all $\lambda \in \mathbb{C}$ and $\mathsf{D}(T)$ is closed, then $T \in \Gamma(\mathsf{H})$.

The following proposition generalizes [7, Proposition 3.3.2].

Proposition 4.3. *Let $T \in \varphi(\mathsf{H})$ and $k \in \mathbb{N}$ such that $\ker(T^k)$ is closed. If $T \in k\text{-}q\Phi(\mathsf{H})$, then*

$$\mathsf{Im}(T^i) + \ker(T^{k+j}) \quad \textit{is closed,} \quad \textit{for all} \quad i + j \geq q_k(T).$$

Proof. If $T \in k\text{-}q\Phi(\mathsf{H})$, then from Proposition 2.11, $\widetilde{T_k} \in q\Phi(q_k(T))(\mathsf{H}/\ker(T^k))$. But by [7, Proposition 3.3.2], we have

$$\mathsf{Im}[(\widetilde{T_k})^i] + \ker[(\widetilde{T_k})^j] = [\mathsf{Im}(T^i) + \ker(T^{k+j})]/\ker(T^k) \quad \text{is closed,} \quad \forall i + j \geq q_k(T).$$

Therefore
$$\mathsf{Im}(T^i) + \mathsf{ker}(T^{k+j}) \quad \text{is closed}, \quad \forall i+j \geq q_k(T).$$
and the proof of the proposition is complete. □

For $T \in \varphi(\mathsf{H})$ and M a subspace of H, we define $T_{|\mathsf{M}}$ as the restriction of T to M viewed as a map from M onto M.

The next lemma is used in order to show Lemmas 4.5 and 4.8.

Lemma 4.4. *Let T be a paracomplete operator on H and P be a non-constant complex polynomial. If $q = \mathsf{do}(T) < +\infty$ and $\mathsf{D}(T^q)$ is closed, then*
 (*i*) *$T_{|\mathsf{D}(T^q)}$ is a bounded operator,*
 (*ii*) *$\mathsf{ker}[P(T)] = \mathsf{ker}[P(T_{|\mathsf{D}(T^q)})]$ is closed,*
 (*iii*) *$\mathsf{Im}\big([P(T)]^n\big) \subseteq \mathsf{D}(T^q)$, for all $n \geq q$.*

Proof. (*i*) Let $\widehat{T}$ (resp. $T_{|\mathsf{D}(T^q)}$) be the restriction of T to $\mathsf{D}(T^q)$ viewed as map from $\mathsf{D}(T^q)$ onto H (resp. $\mathsf{D}(T^q)$ onto $\mathsf{D}(T^q)$). From [7, Proposition 2.1.4, Proposition 2.1.5], it follows that $\widehat{T}$ is a bounded operator. Since for all $x \in \mathsf{D}(T^q)$, we have $\|Tx\| = \|\widehat{T}x\| \leq \|\widehat{T}\|\,\|x\|$, then $T_{|\mathsf{D}(T^q)}$ is also a bounded operator.

(*ii*) Since $\mathsf{ker}[P(T)] \subseteq \mathsf{D}\big([P(T)]^q\big) = \mathsf{D}(T^q)$, then $\mathsf{ker}[P(T)] = \mathsf{ker}[P(T_{|\mathsf{D}(T^q)})]$ is closed.

(*iii*) Let $y \in \mathsf{Im}\big([P(T)]^n\big)$, then there exists $x \in \mathsf{D}\big([P(T)]^n\big) = \mathsf{D}(T^q) = \mathsf{D}\big([P(T)]^{n+q}\big)$ such that $y = [P(T)]^n x$ i.e., $y \in \mathsf{D}(T^q)$. This completes the proof. □

Lemma 4.5. *Let $T \in \varphi(\mathsf{H})$, $m \in \mathbb{N}\backslash\{0\}$ and $k \in \mathbb{N}$.*
 (*i*) *If $q = \mathsf{do}(T) < +\infty$ and $\mathsf{D}(T^q)$ is closed, then*
$$T \in k\text{-}q\Phi(\mathsf{H}) \Longrightarrow T^m \in k\text{-}q\Phi(\mathsf{H}).$$
 (*ii*) *If $T \in \Gamma(\mathsf{H})$, then*
$$T^m \in k\text{-}q\Phi(\mathsf{H}) \Longrightarrow T \in pq\Phi(\mathsf{H}).$$

Proof. (*i*) Let $n \in \mathbb{N}\backslash\{0\}$ and $d = q_k(T)$. Since $d+k \geq q_0(T)$ (see Lemma 2.1), it follows from [7, Proposition 3.1.1] that
$$\mathsf{ker}[(T^n)^j] \subseteq \mathsf{Im}(T^n) + \mathsf{ker}(T^{d+k}) \subseteq \mathsf{Im}(T^n) + \mathsf{ker}[(T^n)^{(d+k)}], \quad \forall\, j \in \mathbb{N},$$
and so $q_0(T^n) \leq d+k$. Hence, by Lemma 2.1, we obtain $q_k(T^n) \leq d$. In the other hand, from Lemma 4.4, we have $\mathsf{ker}(T^j)$ is closed for all $j \in \mathbb{N}$ and by Proposition 4.3, we know that $\mathsf{Im}(T^{nd}) + \mathsf{ker}(T^{nk})$ and $\mathsf{Im}(T^n) + \mathsf{ker}(T^{n(d+k)})$ are closed, this proves that $[\mathsf{Im}(T^{nd}) + \mathsf{ker}(T^{nk})] \cap \mathsf{ker}(T^{n(k+1)})$ is closed. Since $d_n = q_k(T^n) \leq d$, then
$$\big(\mathsf{Im}[(T^n)^{d_n}] + \mathsf{ker}[(T^n)^k]\big) \cap \mathsf{ker}[(T^n)^{k+1}] = \big(\mathsf{Im}[(T^n)^d] + \mathsf{ker}[(T^n)^k]\big) \cap \mathsf{ker}[(T^n)^{k+1}]$$
and
$$\mathsf{Im}(T^n) + \mathsf{ker}(T^{n(d_n+k)}) = \mathsf{Im}(T^n) + \mathsf{ker}(T^{n(d+k)})$$
are closed. It follows now from Corollary 2.12, that $T^n \in k\text{-}q\Phi(\mathsf{H})$.

(*ii*) Let $l \in \mathbb{N}$ such that $l\,m > \mathsf{do}(T)$, then by (*i*), $T^n \in k\text{-}q\Phi(\mathsf{H})$, with $n = l\,m$. Let $d = q_k(T^n) = \max\{q_0(T^n) - k,\, 0\}$, then $d+k \geq q_0(T^n)$. For all $j \in \mathbb{N}$, by [7, Proposition 3.1.1], we see that
$$\mathsf{ker}(T^j) \subseteq \mathsf{ker}[(T^n)^j] \subseteq \mathsf{Im}(T^n) + \mathsf{ker}(T^{n(d+k)}) \subseteq \mathsf{Im}(T) + \mathsf{ker}(T^{n(d+k)})$$
and hence $q_0(T) \leq n(d+k)$. Let $\alpha = k\,n + n\,d - d \geq k$ and $d_\alpha = q_\alpha(T)$. Now by Lemma 2.1, we get
$$d_\alpha \leq q_k(T) = \max\{q_0(T) - k,\, 0\} \leq n(d+k) - k.$$

Therefore

$$\mathsf{Im}(T)+\mathsf{ker}(T^{d_\alpha+\alpha})=\mathsf{Im}(T)+\mathsf{ker}(T^{q_k(T)+\alpha})=\mathsf{Im}(T)+\mathsf{ker}(T^{q_k(T)+k}).$$

Since $q_k(T)+k\leq n(d+k)\leq n(d+k)+n-1$, we deduce

$$\mathsf{Im}(T)+\mathsf{ker}(T^{d_\alpha+\alpha})=\mathsf{Im}(T)+\mathsf{ker}(T^{n(d+k)+n-1}).$$

But $n>\mathsf{do}(T)$, then $\mathsf{D}(T^{n-1})=\mathsf{D}(T^q)$ and $\mathsf{Im}(T^n)\subseteq\mathsf{D}(T^q)$. We have by Lemma 4.4 that $S=T_{|\mathsf{D}(T^q)}$ is a bounded operator, so that

$$\begin{aligned}[\mathsf{Im}(T)+\mathsf{ker}(T^{d_\alpha+\alpha})]\cap\mathsf{D}(T^q) &= [\mathsf{Im}(T)+\mathsf{ker}(T^{n(d+k)+n-1})]\cap\mathsf{D}(T^{n-1})\\ &= T^{-(n-1)}\big(\mathsf{Im}(T^n)+\mathsf{ker}(T^{n(d+k)})\big)\\ &= S^{-(n-1)}\big(\mathsf{Im}(T^n)+\mathsf{ker}(T^{n(d+k)})\big)\end{aligned}$$

is closed. As $[\mathsf{Im}(T)+\mathsf{ker}(T^{d_\alpha+\alpha})]+\mathsf{D}(T^q)=\mathsf{Im}(T)+\mathsf{D}(T^q)$ is closed, we infer by [7, Proposition 2.1.1] and Lemma 2.9 that $\mathsf{Im}(T)+\mathsf{ker}(T^{d_\alpha+\alpha})$ is closed. In the other hand, from Proposition 4.3, for all $i\geq d$ the subspace $\mathsf{Im}(T^{n\,i})+\mathsf{ker}(T^{k\,n})$ is closed. Suppose that $i\geq\max\{2\,d+k,\,1\}$, since $\mathsf{Im}(T^{n\,i-(n\,d-d)})+\mathsf{ker}(T^\alpha)\subseteq\mathsf{D}(T^q)=\mathsf{D}(T^{(n\,d-d)})$ and $\mathsf{Im}(T^{n\,i})+\mathsf{ker}(T^{k\,n})\subseteq\mathsf{D}(T^q)$ (see Lemma 4.4), then

$$\begin{aligned}\mathsf{Im}(T^{n\,i-(n\,d-d)})+\mathsf{ker}(T^\alpha) &= [\mathsf{Im}(T^{n\,i-(n\,d-d)})+\mathsf{ker}(T^\alpha)]\cap\mathsf{D}(T^{(n\,d-d)})\\ &= T^{-(n\,d-d)}\big(\mathsf{Im}(T^{n\,i})+\mathsf{ker}(T^{n\,k})\big)\\ &= S^{-(n\,d-d)}\big(\mathsf{Im}(T^{n\,i})+\mathsf{ker}(T^{n\,k})\big)\end{aligned}$$

is closed. This implies that $\mathsf{Z}=[\mathsf{Im}(T^{n\,i-(n\,d-d)})+\mathsf{ker}(T^\alpha)]\cap\mathsf{ker}(T^{\alpha+1})$ is closed. We have

$$n\,i-(n\,d-d)=n(i-d)+d\geq n(d+k)+d\geq n(d+k)\geq q_k(T)\geq d_\alpha,$$

thus $\mathsf{Z}=[\mathsf{Im}(T^{d_\alpha})+\mathsf{ker}(T^\alpha)]\cap\mathsf{ker}(T^{\alpha+1})$ is closed. Hence by Corollary 2.12, it follows that $T\in\alpha\text{-}q\Phi(d_\alpha)(\mathsf{H})$. This completes the proof. □

As an immediate consequence of Proposition 2.7 and Lemma 4.5, we obtain the following result.

Corollary 4.6. *Let $T\in\mathscr{B}(\mathsf{H})$. The following conditions are equivalent :*

(i) T has topological uniform descent;
(ii) T^n has topological uniform descent for all $n\in\mathbb{N}$;
(iii) T^n has topological uniform descent for some $n\in\mathbb{N}$.

The next lemma is used to prove Lemma 4.8.

Lemma 4.7. *Let $k\in\mathbb{N}$ and $T\in\varphi(\mathsf{H})$ such that $\mathsf{ker}(T^n)$ is closed for all $n\in\mathbb{N}$. If $T\in k\text{-}q\Phi(\mathsf{H})$, then $T\in(k+1)\text{-}q\Phi(\mathsf{H})$.*

Proof. Let $T\in k\text{-}q\Phi(\mathsf{H})$, from Lemma 2.1, $d=q_{k+1}(T)\leq q_k(T)<+\infty$ and hence

$$(1)\qquad [\mathsf{Im}(T^d)+\mathsf{ker}(T^{k+1})]\cap\mathsf{ker}(T^{k+2})=[\mathsf{Im}(T^{d+q_k(T)})+\mathsf{ker}(T^{k+1})]\cap\mathsf{ker}(T^{k+2})$$

and

$$(2)\qquad \mathsf{Im}(T)+\mathsf{ker}(T^{d+k+1})=\mathsf{Im}(T)+\mathsf{ker}(T^{q_k(T)+k+1})=\mathsf{Im}(T)+\mathsf{ker}(T^{q_k(T)+k}).$$

Since by Proposition 4.3, we know that $\mathsf{Im}(T^{d+q_k(T)})+\mathsf{ker}(T^{k+1})$ is closed, then it follows from (1) and (2) that $T\in(k+1)\text{-}q\Phi(\mathsf{H})$, and this completes the proof. □

The next lemma is used to prove Corollary 4.10.

Lemma 4.8. *Let $T:\mathsf{D}(T)\subseteq\mathsf{H}\longrightarrow\mathsf{H}$ be a paracomplete operator. Let $A=P(T)$, $B=Q(T)$, where P and Q are relatively prime polynomials, and $k\in\mathbb{N}$.*

(*i*) $q_k(A^nB^n) = \max\{q_k(A^n),\, q_k(B^n)\}$, *for all* $n \in \mathbb{N}$.

(*ii*) *If* $q = \mathsf{do}(T) < +\infty$ *and* $\mathsf{D}(T^q)$ *is closed, then*

$$A,\, B \in k\text{-}q\Phi(\mathsf{H}) \Longrightarrow AB \in k\text{-}q\Phi(\mathsf{H}).$$

(*iii*) *If* $T \in \Gamma(\mathsf{H})$, *then*

$$A,\, B \in pq\Phi(\mathsf{H}) \Longleftrightarrow AB \in pq\Phi(\mathsf{H}).$$

Proof. (*i*) For $n,\, k \in \mathbb{N}$, we denote by $Z_n^k(T) = [\mathsf{Im}(T^n) + \mathsf{ker}(T^k)] \cap \mathsf{ker}(T^{k+1})$. By [4, Lemma 4.4], we see

$$\begin{aligned}
Z_n^k(AB) &= [\mathsf{Im}(A^nB^n) + \mathsf{ker}(A^kB^k)] \cap \mathsf{ker}(A^{k+1}B^{k+1})\\
&= [\mathsf{Im}(A^n) \cap \mathsf{Im}(B^n) + \mathsf{ker}(A^k) + \mathsf{ker}(B^k)] \cap [\mathsf{ker}(A^{k+1}) + \mathsf{ker}(B^{k+1})]\\
&= [[\mathsf{Im}(A^n) + \mathsf{ker}(A^k)] \cap \mathsf{Im}(B^n) + \mathsf{ker}(B^k)] \cap [\mathsf{ker}(A^{k+1}) + \mathsf{ker}(B^{k+1})]\\
&= [\mathsf{Im}(A^n) + \mathsf{ker}(A^k)] \cap [\mathsf{Im}(B^n) + \mathsf{ker}(B^k)] \cap [\mathsf{ker}(A^{k+1}) + \mathsf{ker}(B^{k+1})]\\
&= [\mathsf{Im}(A^n) + \mathsf{ker}(A^k)] \cap [\mathsf{ker}(A^{k+1}) + (\mathsf{Im}(B^n) + \mathsf{ker}(B^k)) \cap \mathsf{ker}(B^{k+1})]\\
&= [\mathsf{Im}(A^n) + \mathsf{ker}(A^k)] \cap \mathsf{ker}(A^{k+1}) + [\mathsf{Im}(B^n) + \mathsf{ker}(B^k)] \cap \mathsf{ker}(B^{k+1})\\
&= Z_n^k(A) + Z_n^k(B)
\end{aligned}$$

and

$$Z_n^k(A) \cap Z_n^k(B) \subseteq \mathsf{ker}(A^{k+1}) \cap \mathsf{ker}(B^{k+1}) = \{0\}.$$

Therefore

$$q_k(A^nB^n) = \max\{q_k(A^n),\, q_k(B^n)\}, \quad \forall\, n \in \mathbb{N}.$$

(*ii*) First, recall that from Lemma 4.4, we get $\mathsf{ker}(A^k)$ and $\mathsf{ker}(B^k)$ are closed, for all $k \in \mathbb{N}$. For $j,\, n \in \mathbb{N}$, we have

$$\begin{aligned}
\mathsf{Im}(A^nB^n) + \mathsf{ker}(A^jB^j) &= \mathsf{Im}(A^n) \cap \mathsf{Im}(B^n) + \mathsf{ker}(A^j) + \mathsf{ker}(B^j)\\
&= [\mathsf{Im}(A^n) + \mathsf{ker}(A^j)] \cap [\mathsf{Im}(B^n) + \mathsf{ker}(B^j)].
\end{aligned} \tag{1}$$

Assume that $A,\, B \in k\text{-}q\Phi(\mathsf{H})$ and let $d = q_k(AB) = \max\{q_k(A),\ q_k(B)\}$. In particular, this allows us to see

$$\mathsf{Im}(A) + \mathsf{ker}(A^{k+d}) \quad \text{and} \quad \mathsf{Im}(B) + \mathsf{ker}(B^{k+d}) \quad \text{are closed.} \tag{2}$$

Furthermore, from Proposition 4.3, it follows that

$$\mathsf{Im}(A^d) + \mathsf{ker}(A^k) \quad \text{and} \quad \mathsf{Im}(B^d) + \mathsf{ker}(B^k) \quad \text{are closed.} \tag{3}$$

Thus, taking into account of the equalities (1), (2), (3) and Corollary 2.12, we deduce that $AB \in k\text{-}q\Phi(\mathsf{H})$.

(*iii*) Taking into account of [7, Proposition 2.1.3] and Lemma 2.9, we obtain that $Z_n^k(A)$ (resp. $Z_n^k(B)$) is paracomplete and applying [7, Proposition 2.1.1], we conclude that

$$Z_n^k(AB) \quad \text{is closed} \quad \Longrightarrow \quad Z_n^k(A) \quad \text{and} \quad Z_n^k(B) \quad \text{are closed.} \tag{4}$$

Since for $j \in \mathbb{N}$ and $n \geq \mathsf{do}(T)$, we have

$$[\mathsf{Im}(A^n) + \mathsf{ker}(A^j)] + [\mathsf{Im}(B^n) + \mathsf{ker}(B^j)] = \mathsf{Im}(A^n) + \mathsf{Im}(B^n) = \mathsf{D}(T^q),$$

it follows from [7, Proposition 2.1.1, Proposition 2.1.3], Lemma 2.9 and (1) that

$$\mathsf{Im}(A^n\, B^n) + \mathsf{ker}(A^j\, B^j) \quad \text{is closed} \quad \Longleftrightarrow \quad \begin{cases} \mathsf{Im}(A^n) + \mathsf{ker}(A^j),\\ \mathsf{Im}(B^n) + \mathsf{ker}(B^j) \end{cases} \text{are closed,} \quad \forall\, n \geq \mathsf{do}(T). \tag{5}$$

Assume that $AB \in k\text{-}q\Phi(\mathsf{H})$, then $A^nB^n \in k\text{-}q\Phi(\mathsf{H})$, for $n \geq \mathsf{do}(T)$ according to Lemma 4.5. In particular $Z_d^k(A^nB^n)$ and $\mathsf{Im}(B^nA^n) + \mathsf{ker}[(A^nB^n)^{k+d}]$ are closed, with $d = q_k(A^nB^n)$. Since $q_k(A^n) \leq d$, taking into account of (4) and (5), we deduce that $Z_{q_k(A^n)}^k(A^n) = Z_d^k(A^n)$ and $\mathsf{Im}(A^n) + \mathsf{ker}[(A^n)^{k+q_k(A^n)}] = \mathsf{Im}(A^n) + \mathsf{ker}[(A^n)^{k+d}]$ are closed. Therefore by Corollary 2.12, we obtain that $A^n \in k\text{-}q\Phi(\mathsf{H})$ and hence $A \in pq\Phi(\mathsf{H})$ according to Lemma 4.5. Consequently if $AB \in pq\Phi(\mathsf{H})$, then $A,\ B \in pq\Phi(\mathsf{H})$.

Suppose, conversely, that $A, B \in pq\Phi(\mathsf{H})$, then there exists $k_1, k_2 \in \mathbb{N}$ such that $A \in k_1\text{-}q\Phi(\mathsf{H})$ and $B \in k_2\text{-}q\Phi(\mathsf{H})$. Now from Lemma 4.7, it follows that $A, B \in k\text{-}q\Phi(\mathsf{H})$, with $k = \max\{k_1, k_2\}$. Finally, by (ii), we obtain $AB \in pq\Phi(\mathsf{H})$. This completes the proof. □

Using Proposition 2.7, [10, Lemma 12.8] and the proof of Lemma 4.8, one proves the following result.

Corollary 4.9. *Let $T, S, L, R \in \mathscr{B}(\mathsf{H})$ be mutually commuting operators, satisfying $TR + LS = I$. Then T has topological uniform descent if and only if the same holds for S.*

Corollary 4.10. *Let $T \in \varphi(\mathsf{H})$ and $P(Z) = (\lambda_1 - Z)^{m_1}(\lambda_2 - Z)^{m_2} \cdots (\lambda_s - Z)^{m_s}$ be a complex polynomial such that $m_i \neq 0$ for all $i = 1, 2, \ldots, s$.*

(i) Let $k \in \mathbb{N}$, if $q = \mathsf{do}(T) < +\infty$ and $\mathsf{D}(T^q)$ is closed, then

$$\forall\, 1 \le i \le s, \quad \lambda_i \in \varrho^k_{q\Phi}(T) \Longrightarrow 0 \in \varrho^k_{q\Phi}(P(T)).$$

(ii) If $T \in \Gamma(\mathsf{H})$, then

$$0 \in \varrho^\infty_{q\Phi}(P(T)) \Longleftrightarrow \lambda_i \in \varrho^\infty_{q\Phi}(T), \quad \forall\, 1 \le i \le s.$$

Proof. From Lemmas 4.5 and 4.8, it follows that

$$\begin{aligned} \forall\, 1 \le i \le s,\ \ \lambda_i \in \varrho^k_{q\Phi}(T) &\implies 0 \in \bigcap_{1\le i\le s} \varrho^k_{q\Phi}(\lambda_i I - T) \\ &\implies 0 \in \bigcap_{1\le i\le s} \varrho^k_{q\Phi}[(\lambda_i I - T)^{m_i}] \\ &\implies 0 \in \varrho^k_{q\Phi}(P(T)) \end{aligned}$$

and

$$\begin{aligned} 0 \in \varrho^\infty_{q\Phi}(P(T)) &\iff 0 \in \bigcap_{1\le i\le s} \varrho^\infty_{q\Phi}[(\lambda_i I - T)^{m_i}] \\ &\iff 0 \in \bigcap_{1\le i\le s} \varrho^\infty_{q\Phi}(\lambda_i I - T) \\ &\iff \lambda_i \in \varrho^\infty_{q\Phi}(T), \quad \forall\, 1 \le i \le s. \end{aligned}$$

This completes the proof. □

Corollary 4.11. *Let $T \in \mathscr{B}(\mathsf{H})$ and $P(Z) = (\lambda_1 - Z)^{m_1}(\lambda_2 - Z)^{m_2} \cdots (\lambda_s - Z)^{m_s}$ be a complex polynomial such that $m_i \neq 0$ for all $i = 1, 2, \ldots, s$. The following conditions are equivalent :*

(i) $P(T)$ has topological uniform descent;

(ii) $\lambda_i I - T$ has topological uniform descent for all $1 \le i \le s$.

Now we give a spectral mapping theorem which is our main result.

Theorem 4.12. *Let $T \in \varphi(\mathsf{H})$ and P be a non-constant complex polynomial.*

(i) If $k \in \mathbb{N}$, $q = \mathsf{do}(T) < +\infty$ and $\mathsf{D}(T^q)$ is closed, then

$$\sigma^k_{q\Phi}(P(T)) \subseteq P(\sigma^k_{q\Phi}(T)).$$

(ii) If $T \in \Gamma(\mathsf{H})$, then

$$P(\sigma^\infty_{q\Phi}(T)) = \sigma^\infty_{q\Phi}(P(T)).$$

In particular, the topological uniform descent spectrum of a bounded operator on a Hilbert space satisfies the non-constant polynomial version of the spectral mapping theorem.

Proof. (i) Let $\lambda \in \sigma^k_{q\Phi}(P(T))$ and suppose that $\lambda - P(Z) = (\mu_1 - Z)^{m_1} \cdots (\mu_s - Z)^{m_s}$. From Corollary 4.10, it follows that there exists $i \in \{1, 2, \ldots, s\}$ such that $\mu_i \in \sigma^k_{q\Phi}(T)$. Hence $\lambda = P(\mu_i) \in P(\sigma^k_{q\Phi}(T))$.

(ii) From Corollary 4.10, it follows that

$$\begin{aligned} \lambda \in P(\sigma^\infty_{q\Phi}(T)) &\iff \lambda = P(\mu), \text{ with } \mu \in \sigma^\infty_{q\Phi}(T), \\ &\iff \lambda - P(Z) = (\mu - Z)^k Q(Z), \text{ with } Q(\mu) \neq 0, \\ &\iff \lambda \in \sigma^\infty_{q\Phi}(P(T)), \end{aligned}$$

which completes the proof. □

Question 1. *Let $T \in \Gamma(\mathsf{H})$, $k \in \mathbb{N}$ and P be a non-constant complex polynomial. It is not clear at present whether $P(\sigma^k_{q\Phi}(T)) = \sigma^k_{q\Phi}(P(T))$?*

Corollary 4.13. *Let $T \in \varphi(\mathsf{H})$ such that $q = \mathsf{do}(T) < +\infty$ and $\mathsf{D}(T^q)$ is closed, and P be a complex polynomial having no roots in $\sigma^k_{q\Phi}(T)$, for $k \in \mathbb{N}$, then $P(T)$ is a k-quasi-Fredholm operator.*

Corollary 4.14. *Let $T \in \Gamma(\mathsf{H})$ and P be a complex polynomial having no roots in $\sigma^\infty_{q\Phi}(T)$, then $P(T)$ is pseudo-quasi-Fredholm. Furthermore, $P(T)$ has topological uniform descent, when $T \in \mathscr{B}(\mathsf{H})$.*

The next lemma is used to prove Theorem 4.16.

Lemma 4.15. *Let $T, L \in \mathscr{B}(\mathsf{H})$ such that $TL = LT$. If L is invertible, then for all $k \in \mathbb{N}$, we have $T \in k\text{-}q\Phi(\mathsf{H})$ if and only if $TL \in k\text{-}q\Phi(\mathsf{H})$.*

Proof. For $n \in \mathbb{N}$, we know that $\mathsf{ker}(T^n) = \mathsf{ker}(T^n L^n)$ and $\mathsf{Im}(T^n) = \mathsf{Im}(T^n L^n)$. For every $k, n, i \in \mathbb{N}$, we deduce that $q_k(T) = q_k(TL)$, $\mathsf{Im}(T^i) + \mathsf{ker}(T^n)$ is closed if and only if $\mathsf{Im}(L^i T^i) + \mathsf{ker}(L^n T^n)$ is closed and $[\mathsf{Im}(T^i) + \mathsf{ker}(T^k)] \cap \mathsf{ker}(T^{k+1})$ is closed if and only if $[\mathsf{Im}(L^i T^i) + \mathsf{ker}(L^k T^k)] \cap \mathsf{ker}(L^{k+1} T^{k+1})$ is closed. Therefore,

$$T \in k\text{-}q\Phi(\mathsf{H}) \iff TL \in k\text{-}q\Phi(\mathsf{H}).$$

This completes the proof. □

The spectral mapping theorem holds for the pseudo-quasi-Fredholm spectrum.

Theorem 4.16. *Let $T \in \mathscr{B}(\mathsf{H})$ and f be an analytic function in a neighborhood of the usual spectrum $\sigma(T)$ and not locally constant in $\sigma(T)$. For $k \in \mathbb{N}$, we have*

$$\sigma^k_{q\Phi}\big(f(T)\big) \subseteq f\big(\sigma^k_{q\Phi}(T)\big) \quad \text{and} \quad f\big(\sigma^\infty_{q\Phi}(T)\big) = \sigma^\infty_{q\Phi}\big(f(T)\big).$$

So, the topological uniform descent spectrum of a bounded operator on a Hilbert space satisfies the spectral mapping theorem.

Proof. Let $\mu \in \mathbb{C}$ and f be an analytic function in a neighborhood of $\sigma(T)$. Since $\sigma(T)$ is a compact subset of $\mathbb{C}$, the function $f(z) - f(\mu)$ possesses at most a finite number of zeros in $\sigma(T)$. So

$$f(z) - f(\mu) = (z - \mu)^{m_0} (z - \lambda_1)^{m_1} \cdots (z - \lambda_n)^{m_n} g(z),$$

where $g(z)$ is a non-vanishing analytic function on $\sigma(T)$. Using the functional calculus we deduce that :

$$f(T) - f(\mu)I = (T - \mu I)^{m_0} (T - \lambda_1 I)^{m_1} \cdots (T - \lambda_n I)^{m_n} g(T),$$

where $g(T)$ is an invertible operator. Therefore

$$[f(T) - f(\mu)I]\big(g(T)^{-1}\big) = (T - \mu I)^{m_0} (T - \lambda_1 I)^{m_1} \cdots (T - \lambda_n I)^{m_n}.$$

So from Corollary 4.10 and Lemma 4.15, it follows that

$$\begin{aligned}\mu \in \sigma^{\infty}_{q\Phi}(T) &\iff [f(T)-f(\mu)I]\big(g(T)^{-1}\big) \notin pq\Phi(\mathsf{H})\\ &\iff f(T)-f(\mu)I \notin pq\Phi(\mathsf{H})\\ &\iff f(\mu)\in \sigma^{\infty}_{q\Phi}\big(f(T)\big).\end{aligned}$$

In the same way, we obtain that

$$\sigma^{k}_{q\Phi}\big(f(T)\big) \subseteq f\big(\sigma^{k}_{q\Phi}(T)\big).$$

This proves the theorem. □

Corollary 4.17. *Let $T \in \mathscr{B}(\mathsf{H})$ and f be an analytic function in a neighborhood of the usual spectrum $\sigma(T)$ having no roots in $\sigma^{\infty}_{q\Phi}(T)$ (resp. $\sigma^{k}_{q\Phi}(T)$, for $k \in \mathbb{N}$) and not locally constant in $\sigma(T)$. Then $f(T)$ is a pseudo-quasi-Fredholm (resp. k-quasi-Fredholm) operator.*

Remark 4.18. Recall that if $T \in \varphi(\mathsf{H})$ such that $\varrho^{+}_{e}(T) \neq \emptyset$, then $\mathsf{ker}(P(T))$ is closed, for all complex polynomial P. Thus, the first assertion of Lemma 4.5 and the second assertion of Lemma 4.8 are true also for a closed operator T on a Hilbert space such that $\varrho^{+}_{e}(T) \neq \emptyset$ and not necessarily $q = \mathsf{do}(T) < +\infty$ and $\mathsf{D}(T^{q})$ is closed. Hence, we can prove that all results in Section 4 related to the k-quasi-Fredholm spectrum remain valid for an operator $T \in \varphi(\mathsf{H})$ such that $\varrho^{+}_{e}(T) \neq \emptyset$ without the assumption that $q = \mathsf{do}(T) < +\infty$ and $\mathsf{D}(T^{q})$ is closed.

5. The k-quasi-Fredholm and finite-dimensional perturbations

For two subspaces M and N of H, we write $\mathsf{M} \overset{e}{\subset} \mathsf{N}$ if there exists a finite-dimensional subspace V of H such that $\mathsf{M} \subset \mathsf{N} + \mathsf{V}$, i.e. $\dim \mathsf{M}/(\mathsf{M}\cap\mathsf{N}) = \dim(\mathsf{M}+\mathsf{N})/\mathsf{N} < +\infty$. Similarly, we write $\mathsf{M} \overset{e}{=} \mathsf{N}$ if both $\mathsf{M} \overset{e}{\subset} \mathsf{N}$ and $\mathsf{N} \overset{e}{\subset} \mathsf{M}$.

The elementary next lemma is used to show Lemma 5.2.

Lemma 5.1. *Let $T \in \varphi(\mathsf{H})$ and $F \in \mathscr{B}(\mathsf{H})$ such that $\dim \mathsf{Im}(F) < +\infty$, $\mathsf{Im}(F) \subset \mathsf{D}(T)$ and $TFx = FTx$, for all $x \in \mathsf{D}(T)$. Then for every $n \in \mathbb{N}$, we have*

$$\mathsf{ker}[(T+F)^n] \overset{e}{=} \mathsf{ker}(T^n) \quad \text{and} \quad \mathsf{Im}[(T+F)^n] \overset{e}{=} \mathsf{Im}(T^n).$$

In particular,

$$\mathsf{ker}[(T+F)^n] + \mathsf{Im}[(T+F)^i] \overset{e}{=} \mathsf{ker}(T^n) + \mathsf{Im}(T^i), \quad \forall\, n,\, i \in \mathbb{N}.$$

Proof. For $n \in \mathbb{N}$, we define

$$\begin{array}{rccc} \theta: & \mathsf{ker}[(T+F)^n] & \longrightarrow & \mathsf{Im}(F) \\ & x & \longmapsto & T^n x, \end{array} \quad \text{and} \quad \begin{array}{rccc} \psi: & \mathsf{ker}(T^n) & \longrightarrow & \mathsf{Im}(F) \\ & x & \longmapsto & (T+F)^n x. \end{array}$$

We have

$$\begin{aligned}\dim \mathsf{ker}[(T+F)^n]/\big(\mathsf{ker}[(T+F)^n]\cap \mathsf{ker}(T^n)\big) &= \dim \mathsf{ker}[(T+F)^n]/\mathsf{ker}(\theta)\\ &\leq \dim \mathsf{Im}(F) < +\infty\end{aligned}$$

and

$$\begin{aligned}\dim \mathsf{ker}(T^n)/\big(\mathsf{ker}[(T+F)^n]\cap \mathsf{ker}(T^n)\big) &= \dim \mathsf{ker}(T^n)/\mathsf{ker}(\psi)\\ &\leq \dim \mathsf{Im}(F) < +\infty.\end{aligned}$$

This implies that

$$\mathsf{ker}[(T+F)^n] \overset{e}{=} \mathsf{ker}(T^n), \quad \forall\, n \in \mathbb{N}.$$

Since $(T+F)^n - T^n$ is a finite dimensional operator, then $\mathsf{Im}[(T+F)^n] \overset{e}{=} \mathsf{Im}(T^n)$. This completes the proof. □

Lemma 5.2. *Let* $T \in \varphi(\mathsf{H})$ *and* $F \in \mathscr{B}(\mathsf{H})$ *such that* $\dim \mathsf{Im}(F) < +\infty$, $\mathsf{Im}(F) \subset \mathsf{D}(T)$ *and* $TFx = FTx$, *for all* $x \in \mathsf{D}(T)$. *Then*

$$q_0(T) < +\infty \Longleftrightarrow q_0(T+F) < +\infty.$$

Proof. "$\Longrightarrow$" Let $q_0(T) = d < +\infty$, $\mathsf{M} = \mathsf{Im}(T^d)$ and put $\widetilde{T} = T_{|\mathsf{M}}$. Then $\mathsf{ker}(\widetilde{T}) \subseteq \mathsf{Im}^\infty(\widetilde{T})$ and $\widetilde{T}(\mathsf{Im}^\infty(T)) = \mathsf{Im}^\infty(T)$. Indeed, we have

$$\mathsf{ker}(\widetilde{T}) = \mathsf{ker}(T) \cap \mathsf{Im}(T^d) = \mathsf{ker}(T) \cap \mathsf{Im}(T^{d+n}) \subseteq \mathsf{Im}(\widetilde{T}^n), \quad \forall\, n \in \mathbb{N}$$

and so $\mathsf{ker}(\widetilde{T}) \subseteq \mathsf{Im}^\infty(\widetilde{T})$. Now let $z \in \mathsf{Im}^\infty(T) = \mathsf{Im}^\infty(\widetilde{T})$, then there exists $x \in \mathsf{D}(\widetilde{T})$ such that $z = \widetilde{T}x$. Moreover, for every $n \in \mathbb{N}$, there exists $y \in \mathsf{D}(\widetilde{T}^{n+1}) \subseteq \mathsf{D}(\widetilde{T}^n)$ such that $\widetilde{T}^{n+1}y = \widetilde{T}x$, so $x - \widetilde{T}^n y \in \mathsf{ker}(\widetilde{T}) \subseteq \mathsf{Im}^\infty(\widetilde{T}) \subseteq \mathsf{Im}(\widetilde{T}^n)$. Therefore $x \in \mathsf{Im}^\infty(\widetilde{T}) = \mathsf{Im}^\infty(T)$.

It clearly suffices to consider only the case when $\dim \mathsf{Im}(F) = 1$. As in the proof of [6, Theorem, page 194], it is possible to show that $\mathsf{ker}(\widetilde{T}) \overset{e}{\subset} \mathsf{Im}^\infty(T+F)$. We know that if $\mathsf{M} \overset{e}{\subset} \mathsf{N}$ and $\mathsf{M} \overset{e}{\subset} \mathsf{L}$, then $\mathsf{M} \overset{e}{\subset} \mathsf{N} \cap \mathsf{L}$. Since by Lemma 5.1, we have

$$\mathsf{ker}(T+F) \cap \mathsf{Im}[(T+F)^d] \subseteq \mathsf{ker}(T+F) \overset{e}{\subset} \mathsf{ker}(T)$$

and

$$\mathsf{ker}(T+F) \cap \mathsf{Im}[(T+F)^d] \subseteq \mathsf{Im}[(T+F)^d] \overset{e}{\subset} \mathsf{Im}(T^d),$$

then we can deduce that

$$\mathsf{ker}(T+F) \cap \mathsf{Im}[(T+F)^d] \overset{e}{\subset} \mathsf{ker}(T) \cap \mathsf{Im}(T^d).$$

Hence,

$$\mathsf{ker}(T+F) \cap \mathsf{Im}[(T+F)^d] \overset{e}{\subset} \mathsf{ker}(T) \cap \mathsf{Im}(T^d) = \mathsf{ker}(\widetilde{T}) \overset{e}{\subset} \mathsf{Im}^\infty(T+F)$$

and since $\mathsf{ker}(T+F) \cap \mathsf{Im}[(T+F)^d] \subseteq \mathsf{ker}(T+F)$, so

$$\mathsf{ker}(T+F) \cap \mathsf{Im}[(T+F)^d] \overset{e}{\subset} \mathsf{ker}(T+F) \cap \mathsf{Im}^\infty(T+F).$$

This implies that

$$\alpha = \dim(\mathsf{ker}(T+F) \cap \mathsf{Im}[(T+F)^d])/(\mathsf{ker}(T+F) \cap \mathsf{Im}^\infty(T+F)) < +\infty.$$

Let $n \geq d$ and $\alpha_n = \dim(\mathsf{ker}(T+F) \cap \mathsf{Im}[(T+F)^d])/(\mathsf{ker}(T+F) \cap \mathsf{Im}[(T+F)^n])$. It is clear that the sequence $(\alpha_n)_{n \geq d}$ is increasing and $\alpha_n \leq \alpha$, for all $n \geq d$. Then there exist $n_0 \geq d$ and $\beta \leq \alpha$ such that $\alpha_n = \beta$, for all $n \geq n_0$. Let $n \geq n_0$, since

$$\mathsf{ker}(T+F) \cap \mathsf{Im}[(T+F)^{n+1}] \subseteq \mathsf{ker}(T+F) \cap \mathsf{Im}[(T+F)^n] \subseteq \mathsf{ker}(T+F) \cap \mathsf{Im}[(T+F)^d],$$

we deduce that

$$\alpha_{n+1} = \alpha_n + \dim(\mathsf{ker}(T+F) \cap \mathsf{Im}[(T+F)^n])/(\mathsf{ker}(T+F) \cap \mathsf{Im}[(T+F)^{n+1}]).$$

Thus, $\dim(\mathsf{ker}(T+F) \cap \mathsf{Im}[(T+F)^n])/(\mathsf{ker}(T+F) \cap \mathsf{Im}[(T+F)^{n+1}]) = \alpha_{n+1} - \alpha_n = 0$. It follows from this that

$$\mathsf{ker}(T+F) \cap \mathsf{Im}[(T+F)^n] = \mathsf{ker}(T+F) \cap \mathsf{Im}[(T+F)^{n_0}], \quad \forall\, n \geq n_0.$$

This means that $q_0(T+F) \leq n_0$.

"$\Longleftarrow$" If $q_0(T+F) < +\infty$, from the first sense $q_0(T) = q_0(T+F-F) < +\infty$.

This finishes the proof of the lemma. □

The following corollary is a straightforward consequence of Lemma 2.1 and Lemma 5.2.

Corollary 5.3. *Let $T \in \varphi(\mathsf{H})$ and $F \in \mathscr{B}(\mathsf{H})$ such that $\dim \mathsf{Im}(F) < +\infty$, $\mathsf{Im}(F) \subset \mathsf{D}(T)$ and $TFx = FTx$, for all $x \in \mathsf{D}(T)$. Then*

$$q_k(T) < +\infty \Longleftrightarrow q_k(T+F) < +\infty, \quad \forall\, k \in \mathbb{N}.$$

Recall that if T and F are bounded operators such $\dim \mathsf{Im}(F) < +\infty$, then T is quasi-Fredholm if and only if $T+F$ is quasi-Fredholm (see [6, Theorem]). We generalize this result to the class of k-quasi-Fredholm operators as follows :

Theorem 5.4. *Let $T \in \varphi(\mathsf{H})$ such that $\varrho_e^+(T) \neq \emptyset$. Let $F \in \mathscr{B}(\mathsf{H})$ such that $\dim \mathsf{Im}(F) < +\infty$, $\mathsf{Im}(F) \subset \mathsf{D}(T)$ and $TFx = FTx$, for all $x \in \mathsf{D}(T)$. Then for all $k \in \mathbb{N}$, we have $\sigma^k_{q\Phi}(T+F) = \sigma^k_{q\Phi}(T)$ and $\sigma^\infty_{q\Phi}(T+F) = \sigma^\infty_{q\Phi}(T)$.*

Proof. Let $k \in \mathbb{N}$ and $T \in k\text{-}q\Phi(\mathsf{H})$. By Corollary 5.3, we have $d = \max\{q_k(T), q_k(T+F)\} < +\infty$. It follows from Proposition 4.3 that $\mathsf{Im}(T^d) + \mathsf{ker}(T^k)$ and $\mathsf{Im}(T) + \mathsf{ker}(T^{d+k})$ are closed subspaces. From Lemma 5.1, we deduce that $\mathsf{Im}[(T+F)^d] + \mathsf{ker}[(T+F)^k]$ and $\mathsf{Im}(T+F) + \mathsf{ker}[(T+F)^{d+k}]$ are closed subspaces. Since $d_1 = q_k(T+F) \le d$, then $\mathsf{Im}(T+F) + \mathsf{ker}[(T+F)^{d_1+k}]$ and $\big(\mathsf{Im}[(T+F)^{d_1}] + \mathsf{ker}[(T+F)^k]\big) \cap \mathsf{ker}[(T+F)^{k+1}]$ are closed and hence $T+F \in k\text{-}q\Phi(\mathsf{H})$. Consequently, $\sigma^k_{q\Phi}(T+F) = \sigma^k_{q\Phi}(T)$ and

$$\sigma^\infty_{q\Phi}(T+F) = \bigcap_{k\ge 0} \sigma^k_{q\Phi}(T+F) = \bigcap_{k\ge 0} \sigma^k_{q\Phi}(T) = \sigma^\infty_{q\Phi}(T).$$

This completes the proof. □

As consequence of Proposition 2.7 and Theorem 5.4 we derive the following corollary :

Corollary 5.5. *Let T, $F \in \mathscr{B}(\mathsf{H})$ such that $TF = FT$ and $\dim \mathsf{Im}(F) < +\infty$. Then T has topological uniform descent if and only if the same holds for $T+F$.*

Remark 5.6.

(*i*) Let $k \in \mathbb{N}$. It is clear that if $T = 0$, then $T \in k\text{-}q\Phi(\mathsf{H})$ and if K is a one-to-one compact operator (so $\mathsf{Im}(K^n)$ is not closed for all $n \in \mathbb{N}\backslash\{0\}$), then $K \notin pq\Phi(\mathsf{H})$. Therefore if $T \in pq\Phi(\mathsf{H})$ and K is a compact operator such that $TK = KT$, then it is not necessary that $T + K \in pq\Phi(\mathsf{H})$.

(*ii*) Let H be the Hilbert space with an orthonormal basis $\{e_n : n \in \mathbb{N}\}$. Let $T = 0$ and $S \in \mathscr{B}(\mathsf{H})$ be defined by

$$S(e_n) = 2^{-n} e_{n+1}, \quad \forall\, n \in \mathbb{N}.$$

It is clear that S is quasi-nilpotent and $TS = ST$. Since $\mathsf{Im}(S)$ is not closed and $\mathsf{ker}(S) = \{0\}$, it follows that $T+S$ is not pseudo-quasi-Fredholm. Therefore if $T \in pq\Phi(\mathsf{H})$ and S is a quasi-nilpotent operator such that $TS = ST$, then it is not necessary that $T + S \in pq\Phi(\mathsf{H})$.

Several questions still remain unanswered. Some of these are :

Question 2. *Let $T \in \varphi(\mathsf{H})$ and $F \in \mathscr{B}(\mathsf{H})$ such that $\mathsf{Im}(F) \subset \mathsf{D}(T)$ and $TFx = FTx$, for all $x \in \mathsf{D}(T)$.*

(*i*) *If $\dim \mathsf{Im}(F^n) < +\infty$, for some $n \in \mathbb{N}$, can we prove that $\sigma^\infty_{q\Phi}(T+F) = \sigma^\infty_{q\Phi}(T)$?*

(*ii*) *Suppose that F is a nilpotent operator. We know from* [3, Theorem 4.3] *that*

$$\sigma^0_{q\Phi}(T) = \sigma^0_{q\Phi}(T+F).$$

Can we prove that $\sigma^k_{q\Phi}(T) = \sigma^k_{q\Phi}(T+F)$, for all $k \ge 1$ or $\sigma^\infty_{q\Phi}(T) = \sigma^\infty_{q\Phi}(T+F)$?

(*iii*) *If F is s-regular, can we prove that $\sigma^\infty_{q\Phi}(T+F) = \sigma^\infty_{q\Phi}(T)$?*

Remark 5.7. Let $k \in \mathbb{N}$. The set of all k-quasi-Fredholm (resp. pseudo-quasi-Fredholm) operators is not open. Indeed, consider the Hilbert space H with an orthonormal basis $\{e_{i,\,j},\ i,\,j \text{ integers},\ \ i \geq 1\}$. Let $T \in \mathscr{B}(\mathsf{H})$ be defined by

$$T(e_{i,\,j}) = \begin{cases} e_{i,\,j+1} & \text{if } j \neq 0, \\ 0 & \text{if } j = 0. \end{cases}$$

Clearly $\mathsf{ker}(T)$ is the subspace of H spanned by $\{e_{i,\,0} : i \geq 1\}$, $\mathsf{ker}(T) \subseteq \bigcap\limits_{n \geq 0} \mathsf{Im}(T^n)$ and $\mathsf{Im}(T)$ is closed, so that T is k-quasi-Fredholm, for all $k \geq 0$.

Let $\varepsilon > 0$. Define $S_\varepsilon \in \mathscr{B}(\mathsf{H})$ by

$$S_\varepsilon(e_{i,\,j}) = \begin{cases} \dfrac{\varepsilon}{i+1}\, e_{i,\,1} & \text{if } j = 0, \\ 0 & \text{if } j \neq 0. \end{cases}$$

Clearly $\|S_\varepsilon\| = \varepsilon$ and S_ε is an infinite dimensional compact operator so that $\mathsf{Im}(S_\varepsilon)$ is not closed. Let M denote the closed subspace of H spanned by $\{e_{i,\,1},\ i \geq 1\}$. We have $\mathsf{Im}(T) \perp \mathsf{M}$ and $\mathsf{Im}(S_\varepsilon) \subseteq \mathsf{M}$, so that $(T + S_\varepsilon)x \in \mathsf{M}$ implies $x \in \mathsf{ker}(T)$ and $(T + S_\varepsilon)x = S_\varepsilon x$. Thus $\mathsf{Im}(T + S_\varepsilon) \cap \mathsf{M} = S_\varepsilon(\mathsf{ker}(T)) = \mathsf{Im}(S_\varepsilon)$, so that $\mathsf{Im}(T + S_\varepsilon)$ is not closed. Therefore $T + S_\varepsilon$ is not pseudo-quasi-Fredholm because $\mathsf{ker}(T + S_\varepsilon) = \{0\}$.

6. *pq*-index of pseudo-quasi-Fredholm

In this section, we will associate to each pseudo-quasi-Fredholm operator an index "*pq*-index" which coincide with the usual index in the case of a semi-Fredholm operator.

For $T \in \varphi(\mathsf{H})$ and $n,\, k \in \mathbb{N}$, we denote by

$$\alpha_n^k(T) = \dim \mathsf{ker}(T^k) \cap \mathsf{Im}(T^n),$$
$$\beta_n^k(T) = \dim \mathsf{Im}(T^n)/\mathsf{Im}(T^{n+k}).$$

The essential ascent and the essential descent of $T \in \varphi(\mathsf{H})$ are defined by

$$\boldsymbol{d_e}(T) = \inf\{n \in \mathbb{N}\ : \beta_n^1(T) < +\infty\},$$
$$\boldsymbol{a_e}(T) = \inf\{n \in \mathbb{N}\ : \alpha_n^1(T) < +\infty\},$$

respectively, whenever these minima exist. If no such numbers exist the essential ascent and the essential descent of T are defined to be $+\infty$.

Define

$$\mathscr{A}(\mathsf{H}) = \{T \in \varphi(\mathsf{H}) : \mathsf{D}(T^i) + \mathsf{Im}(T^j) = \mathsf{H},\ \ \forall\, i,\, j \in \mathbb{N}\}.$$

Clearly, $\mathscr{A}(\mathsf{H}) \neq \emptyset$, because $T \in \mathscr{A}(\mathsf{H})$, when T is a closed surjective operator.

For $T \in \mathscr{A}(\mathsf{H})$, we can see the following

$$\begin{aligned} \beta_n^k(T) &= \dim \mathsf{Im}(T^n)/\mathsf{Im}(T^{n+k}), \\ &= \dim \mathsf{D}(T^n)/[\mathsf{Im}(T^k) + \mathsf{ker}(T^n)] \cap \mathsf{D}(T^n), \\ &= \dim[\mathsf{D}(T^n) + \mathsf{Im}(T^k)]/[\mathsf{Im}(T^k) + \mathsf{ker}(T^n)], \\ &= \dim \mathsf{H}/[\mathsf{Im}(T^k) + \mathsf{ker}(T^n)]. \end{aligned}$$

We note from [4, Lemma 2.2] that if $\boldsymbol{a_e}(T) < +\infty$, then

$$q_0(T) = \inf\{n \in \mathbb{N} : \alpha_n^1(T) = \alpha_{n+1}^1(T)\} < +\infty,$$

and we also note from [4, Lemma 2.5] that if $T \in \mathscr{A}(\mathsf{H})$ such that $\boldsymbol{d_e}(T) < +\infty$, then

$$q_0(T) = \inf\{n \in \mathbb{N} : \beta_n^1(T) = \beta_{n+1}^1(T)\} < +\infty.$$

We start our study with the following lemma.

Lemma 6.1. *Let $T \in \mathscr{A}(\mathsf{H})$ such that $\mathsf{ker}(T^n) \subseteq \mathsf{Im}(T)$, for all $n \in \mathbb{N}$. Then*

$$\alpha(T^n) = n\,\alpha(T), \quad \beta(T^n) = n\,\beta(T), \quad \forall\, n \in \mathbb{N}\backslash\{0\}.$$

Proof. Let $n \in \mathbb{N}\backslash\{0\}$, and we consider the following map :

$$\begin{array}{cccc} \theta & : & \mathsf{ker}(T^n) & \longrightarrow & \mathsf{ker}(T^{n-1}) \\ & & x & \longmapsto & Tx\,. \end{array}$$

Clearly θ is a surjective linear operator and hence $\alpha(T^n) = \alpha(T) + \alpha(T^{n-1}) = n\,\alpha(T)$. Now, we define the following linear operator :

$$\begin{array}{cccc} S & : & \mathsf{D}(T^{n-1}) & \longrightarrow & \mathsf{H}/\mathsf{Im}(T^n) \\ & & x & \longmapsto & \overline{T^{n-1}x}. \end{array}$$

Since $\mathsf{ker}(S) = [\mathsf{Im}(T) + \mathsf{ker}(T^{n-1})] \cap \mathsf{D}(T^{n-1}) = \mathsf{Im}(T) \cap \mathsf{D}(T^{n-1})$, we deduce that

$$\begin{array}{rcl} \mathsf{Im}(T^{n-1})/\mathsf{Im}(T^n) & \approx & \mathsf{D}(T^{n-1})/[\mathsf{Im}(T) \cap \mathsf{D}(T^{n-1})] \\ & \approx & [\mathsf{D}(T^{n-1}) + \mathsf{Im}(T)]/\mathsf{Im}(T) \\ & \approx & \mathsf{H}/\mathsf{Im}(T). \end{array}$$

But, $\mathsf{Im}(T^n) \subseteq \mathsf{Im}(T^{n-1}) \subseteq \mathsf{H}$, so

$$\dim \mathsf{H}/\mathsf{Im}(T^n) = \dim \mathsf{H}/\mathsf{Im}(T^{n-1}) + \dim \mathsf{Im}(T^{n-1})/\mathsf{Im}(T^n).$$

Therefore

$$\beta(T^n) = \beta(T^{n-1}) + \beta(T) = n\,\beta(T).$$

This completes the proof. □

Lemma 6.2. *Let $T \in \mathscr{A}(\mathsf{H})$ such that $\min\{\boldsymbol{d_e}(T),\, \boldsymbol{a_e}(T)\} < +\infty$ and let $p = q_0(T) < +\infty$. Then for all $n \geq p$, we have*

$$\alpha_n^k(T) = k\,\alpha_p^1(T), \quad \beta_n^k(T) = k\,\beta_p^1(T), \quad \forall\, k \in \mathbb{N}\backslash\{0\}.$$

Proof. Let $m \geq p$ and let $\widetilde{T_m}$ be the operator induced by T on $\mathsf{H}/\mathsf{ker}(T^m)$. Since $\mathsf{ker}[(\widetilde{T_m})^n] \subseteq \mathsf{Im}(\widetilde{T_m})$, for every $n \in \mathbb{N}$, by Lemma 6.1, we get

$$\beta_m^k(T) = \beta(\widetilde{T_m}^{\,k}) = k\,\beta(\widetilde{T_m}) = k\,\beta_m^1(T) = k\,\beta_p^1(T), \quad \forall\, k \geq 1$$

and

$$\alpha_m^k(T) = \alpha(\widetilde{T_m}^{\,k}) = k\,\alpha(\widetilde{T_m}) = k\,\alpha_m^1(T) = k\,\alpha_p^1(T), \quad \forall\, k \geq 1.$$

This completes the proof. □

Remark 6.3. Let $k,\, d \in \mathbb{N}$ and $T \in k\text{-}q\Phi(d)(\mathsf{H})$ such that $\boldsymbol{a_e}(T) < +\infty$ or $\boldsymbol{d_e}(T) < +\infty$. Let $m = \min\{\boldsymbol{a_e}(T),\, \boldsymbol{d_e}(T)\}$, we denote by

$$\delta_m^k(T) = \alpha_m^k(T) - \beta_m^k(T) \in \mathbb{Z} \cup \{-\infty,\, +\infty\}.$$

If $T \in \mathscr{A}(\mathsf{H})$ from [4, Lemma 2.2, Lemma 2.5], we deduce that $\delta_m^k(T) = \delta_n^k(T)$, for all $n \geq m$. Therefore for $k \in \mathbb{N}\backslash\{0\}$, by Lemma 6.2, we obtain

$$\begin{array}{rcl} \delta_m^k(T) = \delta_{q_0(T)}^k(T) & = & \alpha_{q_0(T)}^k(T) - \beta_{q_0(T)}^k(T) \\ & = & k\,\alpha_{q_0(T)}^1(T) - k\,\beta_{q_0(T)}^1(T) \\ & = & k\,\delta_{q_0(T)}^1(T) \\ & = & k\,\delta_m^1(T). \end{array}$$

Remark 6.3 enables us to define the pq-index of pseudo-quasi-Fredholm operator.

Definition 6.4. We say that an operator $T \in pq\Phi(\mathsf{H})$ possesses pq-index if $\ell = \min\{\boldsymbol{a_e}(T),\, \boldsymbol{d_e}(T)\} < +\infty$, in this case the pq-index of T is defined by

$$\mathrm{ind}_{pq}(T) = \alpha_\ell^1(T) - \beta_\ell^1(T) \in \mathbb{Z} \cup \{-\infty,\, +\infty\}.$$

Example 6.5.

(*i*) Let T be a pseudo-quasi-Fredholm operator such that $\boldsymbol{a}(T) < +\infty$ (resp. $\boldsymbol{d}(T) < +\infty$, $\max\{\boldsymbol{a}(T),\, \boldsymbol{d}(T)\} < +\infty$), then T possesses a pq-index and $\operatorname{ind}_{pq}(T) \leq 0$ (resp. $\operatorname{ind}_{pq}(T) \geq 0$, $\operatorname{ind}_{pq}(T) = 0$).

(*ii*) Let H be the Hilbert space with an orthonormal basis $\{e_{i,\,j} : i,\, j \in \mathbb{N}\backslash\{0\}\}$. Let $T \in \mathscr{B}(\mathsf{H})$ be defined by

$$T(e_{i,\,j}) = \begin{cases} 0 & \text{if } i = 1, \\ e_{i,\,j+1} & \text{if } i \geq 2. \end{cases}$$

Clearly $\mathsf{ker}(T^k)$ (resp. $\mathsf{Im}(T^k)$) is the subspace of H spanned by $\{e_{1,\,j} : j \geq 1\}$ (resp. $\{e_{i,\,j} : i \geq 2,\, j \geq k+1\}$), for all $k \geq 1$, so that $q_0(T) = \boldsymbol{a}(T) = \boldsymbol{a_e}(T) = 1$ and $\boldsymbol{d_e}(T) = +\infty$. Since $\mathsf{Im}(T)$ is closed and $\mathsf{Im}(T) \perp \mathsf{ker}(T)$, then $\mathsf{Im}(T) + \mathsf{ker}(T)$ is closed, this implies that T is k-quasi-Fredholm of degree $q_k(T) = \max\{1-k,\, 0\}$, for every $k \in \mathbb{N}$ and the pq-index of T is equal to

$$\operatorname{ind}_{pq}(T) = \alpha_1^1(T) - \beta_1^1(T) = -\infty.$$

Moreover, $T \notin \Phi_{\pm}(\mathsf{H})$, but there exists $\varepsilon > 0$ such that $\lambda I - T \in \Phi_+(\mathsf{H})$ and $\alpha(\lambda I - T) = 0$, for all $\lambda \in \mathbb{C}$ and $0 < |\lambda| < \varepsilon$ according to Lemma 3.1.

Remark 6.6. Let $k \in \mathbb{N}$ and $T \in \varphi(\mathsf{H})$ such that $\varrho(T) \neq \emptyset$ (in particular $T \in \mathscr{A}(\mathsf{H})$). If $T \in k\text{-}q\Phi(\mathsf{H})$ possesses pq-index, then $T^n \in k\text{-}q\Phi(\mathsf{H})$ and $\operatorname{ind}_{pq}(T^n) = n \operatorname{ind}_{pq}(T)$, for all $n \in \mathbb{N}\backslash\{0\}$. Indeed, by Lemma 4.5 and Remark 4.18, it follows that $T^n \in k\text{-}q\Phi(\mathsf{H})$ and by [4, Lemma 2.1], we infer that T^n possesses pq-index. Let $d = q_0(T^n)$, since

$$\mathsf{ker}(T^j) \subseteq \mathsf{ker}[(T^n)^j] \subseteq \mathsf{Im}(T^n) + \mathsf{ker}[(T^n)^d] \subseteq \mathsf{Im}(T) + \mathsf{ker}(T^{dn}), \quad \forall\, j \in \mathbb{N},$$

then $l = q_0(T) \leq n\, d$. From Remark 6.3, we obtain

$$\begin{aligned} \operatorname{ind}_{pq}(T^n) &= \alpha_d^1(T^n) - \beta_d^1(T^n) \\ &= \alpha_{n\,d}^n(T) - \beta_{n\,d}^n(T) \\ &= \delta_{n\,d}^n(T) = \delta_l^n(T) = n\, \delta_l^1(T) = n \operatorname{ind}_{pq}(T). \end{aligned}$$

Proposition 6.7. *Let $T \in \varphi(\mathsf{H})$ such that $\varrho_e^+(T) \neq \emptyset$ and $k \in \mathbb{N}$. If $\boldsymbol{a_e}(T) < +\infty$, then*

$$T \in k\text{-}q\Phi(\mathsf{H}) \Longleftrightarrow \mathsf{Im}(T) + \mathsf{ker}(T^{\boldsymbol{a_e}(T)}) \quad \textit{is closed.}$$

Proof. " $\Longrightarrow$ " Let $d = q_k(T)$, by Lemma 2.1, we have $d + k \geq q_0(T) \geq \boldsymbol{a_e}(T)$ and as $\mathsf{Im}(T) + \mathsf{ker}(T^{d+k})$ is closed, then from [4, Lemma 3.3], we get $\mathsf{Im}(T) + \mathsf{ker}(T^{\boldsymbol{a_e}(T)})$ is closed.

" $\Longleftarrow$ " Since $\boldsymbol{a_e}(T)$ is finite, then $q_0(T)$ is also finite and hence $d = q_k(T) = \max\{q_0(T) - k,\, 0\} < +\infty$ according to Lemma 2.1. As $d + k \geq q_0(T) \geq \boldsymbol{a_e}(T)$, then we can deduce from [4, Lemma 3.3], that $\mathsf{Im}(T) + \mathsf{ker}(T^{d+k})$ is closed. Let $m = \max\{d,\, \boldsymbol{a_e}(T)\}$, we have $\dim \mathsf{Im}(T^m) \cap \mathsf{ker}(T^{k+1}) < +\infty$, this gives that

$$\mathsf{Im}(T^d) \cap \mathsf{ker}(T^{k+1}) + \mathsf{ker}(T^k) = \mathsf{Im}(T^m) \cap \mathsf{ker}(T^{k+1}) + \mathsf{ker}(T^k) \quad \text{is closed.}$$

Hence, $T \in k\text{-}q\Phi(\mathsf{H})$ and the proof of the lemma is complete. □

Proposition 6.8. *Let $T \in \mathscr{A}(\mathsf{H})$ such that $\varrho_e^+(T) \neq \emptyset$ and $\boldsymbol{d_e}(T) < +\infty$. Then*

$$T \in k\text{-}q\Phi(\mathsf{H}), \quad \forall\, k \geq \boldsymbol{d_e}(T).$$

Proof. For $n \in \mathbb{N}$ and $i \in \mathbb{N}\backslash\{0\}$, we have

$$\beta_n^1(T) \leq \beta_n^i(T) = \beta(\widetilde{T_n}^{\,i}) \leq i\, \beta(\widetilde{T_n}) = i\, \beta_n^1(T),$$

where $\widetilde{T_n}$ is the operator induced by T on $\mathsf{H}/\mathsf{ker}(T^n)$. This implies that

$$\beta_n^1(T) < +\infty \Longleftrightarrow \beta_n^i(T) < +\infty.$$

Let $k \geq \boldsymbol{d_e}(T)$ and $d = q_k(T) = \max\{q_0(T) - k,\ 0\} < +\infty$. Since

$$\dim \mathsf{H}/[\mathsf{Im}(T) + \mathsf{ker}(T^{d+k})] = \beta^1_{d+k}(T) < +\infty$$

and

$$\dim \mathsf{H}/[\mathsf{Im}(T^d) + \mathsf{ker}(T^k)] = \beta^d_k(T) < +\infty,$$

then $\mathsf{Im}(T) + \mathsf{ker}(T^{d+k})$ and $[\mathsf{Im}(T^d) + \mathsf{ker}(T^k)] \cap \mathsf{ker}(T^{k+1})$ are closed (see Lemma 2.9 and [7, Proposition 2.1.1]). This completes the proof. □

Remark 6.9. By Propositions 6.7 and 6.8, we remark that, we can replace the hypothesis of Definition 6.4 by : let $T \in \mathscr{A}(\mathsf{H})$ such that $\varrho^+_e(T) \neq \emptyset$ and $\boldsymbol{d_e}(T) < +\infty$ or $\boldsymbol{a_e}(T) < +\infty$ and $\mathsf{Im}(T) + \mathsf{ker}(T^{\boldsymbol{a_e}(T)})$ is closed. If additionally $T \in \mathscr{B}(\mathsf{H})$, then T is semi-B-Fredholm and the *pq*-index coincide with the index of a semi-B-Fredholm operator [1].

Theorem 6.10. *Let $k \in \mathbb{N}$ and $T \in k\text{-}q\Phi(\mathsf{H})$ such that $\varrho(T) \neq \emptyset$. Let $F \in \mathscr{B}(\mathsf{H})$ such that* $\dim \mathsf{Im}(F) < +\infty$, $\mathsf{Im}(F) \subset \mathsf{D}(T)$ *and $TFx = FTx$, for all $x \in \mathsf{D}(T)$. If T possesses pq-index, then $T + F \in k\text{-}q\Phi(\mathsf{H})$, $T + F$ possesses pq-index and* $\mathrm{ind}_{pq}(T + F) = \mathrm{ind}_{pq}(T)$.

Proof. From Theorem 5.4, we have $T + F \in k\text{-}q\Phi(\mathsf{H})$. According to Lemma 2.1 and Corollary 5.3, $d = \max\{q_k(T),\ q_k(T + F)\}$ and $p = \max\{q_0(T),\ q_0(T + F)\}$ are finite. By Lemma 3.1, we know that there exists $\lambda \in \mathbb{C}\backslash\{0\}$ such that

$$\alpha(T_\lambda) = \alpha^1_{d+k}(T) = \alpha^1_p(T),\ \beta(T_\lambda) = \beta^1_{d+k}(T) = \beta^1_p(T),$$
$$\alpha(\lambda I - T - F) = \alpha^1_{d+k}(T + F) = \alpha^1_p(T + F),$$
$$\beta(\lambda I - T - F) = \beta^1_{d+k}(T + F) = \beta^1_p(T + F).$$

So, $T_\lambda \in \Phi_\pm(\mathsf{H})$, consequently $(T + F)_\lambda \in \Phi_\pm(\mathsf{H})$ and

$$l = \min\{\boldsymbol{a_e}(T + F),\ \boldsymbol{d_e}(T + F)\} \leq p.$$

Now since $j = \min\{\boldsymbol{a_e}(T),\ \boldsymbol{d_e}(T)\} \leq p$, then

$$\begin{aligned} \mathrm{ind}_{pq}(T) &= \alpha^1_j(T) - \beta^1_j(T) = \alpha^1_p(T) - \beta^1_p(T) = \mathrm{ind}(T_\lambda) \\ &= \mathrm{ind}[(T + F)_\lambda] = \alpha^1_p(T + F) - \beta^1_p(T + F) \\ &= \alpha^1_l(T + F) - \beta^1_l(T + F) = \mathrm{ind}_{pq}(T + F). \end{aligned}$$

This completes the proof. □

Remark 6.11. Let $k \in \mathbb{N}$ and $T \in k\text{-}q\Phi(\mathsf{H})$ such that $\varrho(T) \neq \emptyset$. From the proof of Theorem 6.10, we see that if T possesses *pq*-index, then there exists $\varepsilon > 0$ such that $T_\lambda \in \Phi_\pm(\mathsf{H})$ and $\mathrm{ind}(T_\lambda) = \mathrm{ind}_{pq}(T)$, for every $0 < |\lambda| < \varepsilon$.

Theorem 6.12. *Let $d, k \in \mathbb{N}$, $T \in k\text{-}q\Phi(d)(\mathsf{H})$ and $V \in \mathscr{B}(\mathsf{H})$. Suppose that T is a bounded operator that commutes with V and $V - T$ is sufficiently small and invertible, then :*

(*i*) *V is a s-regular operator,*
(*ii*) *$\alpha^1_n(V) = \alpha^1_{d+k}(T)$, for all $n \geq 0$,*
(*iii*) *$\beta^1_n(V) = \beta^1_{d+k}(T)$, for all $n \geq 0$.*

Proof. It follows from Lemma 2.1 and Proposition 2.7 that T has topological uniform descent for $n \geq d + k$. The result now follows from [5, Theorem 4.7]. □

Corollary 6.13. *Let $T, V \in \mathscr{B}(\mathsf{H})$ such that $TV = VT$ and V is sufficiently small and invertible. If $T \in pq\Phi(\mathsf{H})$, then $T + V \in pq\Phi(\mathsf{H})$.*

Corollary 6.14. *Let $d, k \in \mathbb{N}$, $T \in k\text{-}q\Phi(d)(\mathsf{H})$ and $V \in \mathscr{B}(\mathsf{H})$. Suppose that T is a bounded operator that commutes with V and $V - T$ is sufficiently small and invertible, then :*

(*i*) *V has infinite ascent or descent if and only if T does.*

(*ii*) *V is onto if and only if T has finite descent.*
(*iii*) *V is one-to-one (or bounded below) if and only if T has finite ascent.*
(*iv*) *V is invertible if and only if $0 \in \mathsf{E}(T)$.*
(*v*) *V is semi-Fredholm if and only if T possesses pq-index. If $V \in \Phi_{\pm}(\mathsf{H})$, then*

$$\mathrm{ind}_{pq}(T) = \mathrm{ind}(V) = \alpha_n^1(V) - \beta_n^1(V), \quad \forall\, n \geq 0.$$

Theorem 6.15. *Let $V, T \in pq\Phi(\mathsf{H})$. Suppose that $V, T \in \mathscr{B}(\mathsf{H})$ such that $TV = VT$ and $V - T$ is sufficiently small, then T possesses pq-index if and only if V possesses pq-index. If T or V possesses pq-index, then*

$$\mathrm{ind}_{pq}(T) = \mathrm{ind}_{pq}(V).$$

Proof. Let $k_1, k_2, d_1, d_2 \in \mathbb{N}$ such that $T \in k_1\text{-}q\Phi(d_1)(\mathsf{H})$ and $V \in k_2\text{-}q\Phi(d_2)(\mathsf{H})$, then T and V having topological uniform descent for $n \geq \max\{d_1 + k_1, d_2 + k_2\}$. Now the proof follows from [5, Theorem 4.6]. □

7. Examples

In this section we present some examples that are applications of the abstract theory of the pseudo-quasi-Fredholm.

Example 7.1. In $\mathsf{H} = L^2([0, 1])$ define the second-order differential operator T by

$$\mathsf{D}(T) = \{u \in \mathsf{H}^2([0, 1]) : u'(0) + u'(1) = 0,\, u(0) = 0\}, \quad Tu = -u'',$$

where $\mathsf{H}^2([0, 1])$ denotes the subspace of H consisting of all functions $u \in \mathsf{C}^1([0, 1])$ with u' absolutely continuous on $[0, 1]$ and $u'' \in \mathsf{H}$. Then T is a discrete operator in H. In [4, Example 3.12], it is proved that $\sigma(T) = \{\lambda_i\}_{i=1}^{\infty}$ where $\lambda_i = (2i-1)^2\pi^2$, and $\boldsymbol{a}(\lambda_i I - T) = \boldsymbol{d}(\lambda_i I - T) = 2$, for $i = 1, 2, \ldots$. This shows that $q_0(\lambda_i I - T) = 2$,

$$\mathsf{Im}(\lambda_i I - T) + \mathsf{ker}[(\lambda_i I - T)^n] = \mathsf{H},$$

$$\mathsf{Im}[(\lambda_i I - T)^n] \cap \mathsf{ker}[(\lambda_i I - T)^{j+1}] + \mathsf{ker}[(\lambda_i I - T)^j] = \mathsf{ker}[(\lambda_i I - T)^j],$$

for all $j \in \mathbb{N}$, $n \geq 2$ and $i \geq 1$. For $i \geq 1$ and $k \in \mathbb{N}$, by Lemma 2.1, we obtain $\lambda_i I - T$ is k-quasi-Fredholm of degree $d_k = \max\{2 - k, 0\}$. Hence $\mathbb{C} = \varrho(T) \cup \sigma(T) \subseteq \varrho^k_{q\Phi}(T)$ i.e., $\sigma^k_{q\Phi}(T) = \sigma^{\infty}_{q\Phi}(T) = \emptyset$, for all $k \in \mathbb{N}$.

Remark 7.2. If $T \in \mathscr{B}(\mathsf{H})$ by Theorem 3.4, we observe that

$$\sigma^{\infty}_{q\Phi}(T) = \emptyset \Longrightarrow \sigma(T) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} = \mathsf{E}(T), \tag{1}$$

for some $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$. From Example 7.1 the conclusion (1) fails when $\mathsf{D}(T) \subsetneq \mathsf{H}$.

Example 7.3. Consider the operator S defined on $\ell^2(\mathbb{N})$ by

$$S(x_1, x_2, x_3, \ldots) = \Big(\frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4}, \cdots\Big)$$

and the operator T defined on $\ell^2(\mathbb{N}) \times \ell^2(\mathbb{N})$ by

$$T\big((x_1, x_2, x_3, \ldots), (y_1, y_2, y_3, \ldots)\big) = \big((0, x_2, x_3, \ldots), S(y_1, y_2, y_3, \ldots)\big).$$

(*a*) It is clear that S is a quasi-nilpotent operator and $\dim \mathsf{ker}(S^n) = n$, for all $n \in \mathbb{N}$. Thus, $\sigma^{\infty}_{q\Phi}(S) \subseteq \sigma^k_{q\Phi}(S) \subseteq \sigma(S) = \{0\}$, for all $k \in \mathbb{N}$. Suppose that $\sigma^{\infty}_{q\Phi}(S) = \emptyset$, then by Theorem 3.4, T is algebraic. This implies that $\mathsf{E}(S) = \{0\}$, which is a contradiction because $\boldsymbol{a}(S) = +\infty$. It follows that $\sigma^{\infty}_{q\Phi}(S) = \{0\}$ and hence $\sigma^k_{q\Phi}(S) = \{0\}$, for all $k \in \mathbb{N}$. Let f be an analytic function in a neighborhood of the usual spectrum $\sigma(S)$ and not locally constant in a neighborhood of 0 and $f(0) \neq 0$, then by Corollary 4.17, $f(S)$ is a k-quasi-Fredholm operator, for all $k \in \mathbb{N}$.

(b) Let $F \in \mathscr{B}(\ell^2(\mathbb{N}) \times \ell^2(\mathbb{N}))$ be defined by

$$F\big((x_1, x_2, x_3, \ldots), (y_1, y_2, y_3, \ldots)\big) = \big((x_1, 0, 0, \ldots), (0, 0, 0, \ldots)\big).$$

Note that $(T+F)(x, y) = (x, Sy)$, for all $x, y \in \ell^2(\mathbb{N})$, which implies that $\sigma^k_{pq}(T + F) = \sigma^k_{pq}(I) \cup \sigma^k_{pq}(S) = \{0\}$, because $\sigma^k_{pq}(I) = \emptyset$, for all $k \in \mathbb{N}$. Furthermore, since $\dim \mathsf{Im}(F) = 1$ and $TF = FT = 0$, by Theorem 5.4, it follows that

$$\sigma^k_{pq}(T) = \sigma^k_{pq}(T+F) = \{0\}, \quad \forall\, k \in \mathbb{N}.$$

Example 7.4. For each $n \in \mathbb{N}\backslash\{0\}$, set

$$\nu(n) = \max\{k \in \mathbb{N} : 2^k \text{ divides } n\}.$$

Let $T \in \mathscr{B}(\ell^2(\mathbb{N}))$ be defined by

$$T\Big(\sum_{n=0}^{+\infty} x_n e_n\Big) = \sum_{n=1}^{+\infty} \frac{1}{2^{\nu(n)}}\, x_n e_n,$$

with $\{e_n : n \in \mathbb{N}\}$ is an orthonormal basis of $\ell^2(\mathbb{N})$.

(a) We remark that $\mathsf{ker}(T)$ is the subspace of $\ell^2(\mathbb{N})$ spanned by e_0, which gives $\boldsymbol{a_e}(T) = 0$. Since $\mathsf{Im}(T)$ is easily seen to be non-closed, it follows from Proposition 6.7 that

$$T \notin k\text{-}q\Phi(\ell^2(\mathbb{N})), \quad \forall\, k \in \mathbb{N}.$$

Now Proposition 6.8 gives $\boldsymbol{d_e}(T) = +\infty$.

(b) It is not difficult to see that

$$\sigma(T) = \{0\} \cup \Big\{\lambda_n = \frac{1}{2^n} : n \in \mathbb{N}\Big\}.$$

Besides, for each $n \in \mathbb{N}$, $\mathsf{ker}(\lambda_n I - T)$ is the closed subspace of $\ell^2(\mathbb{N})$ spanned by $\{e_{2^n(2j+1)} : j \in \mathbb{N}\}$, and $\mathsf{Im}(\lambda_n I - T) = \mathsf{ker}(\lambda_n I - T)^{\perp}$. It follows that $\boldsymbol{a}(\lambda_n I - T) = \boldsymbol{d}(\lambda_n I - T) = 1$. Since $\mathsf{Im}[(\lambda_n I - T)^i] + \mathsf{ker}[(\lambda_n I - T)^j] = \ell^2(\mathbb{N})$ and $\mathsf{Im}[(\lambda_n I - T)^i] \cap \mathsf{ker}[(\lambda_n I - T)^j] = \{0\}$, for all $i, j \geq 1$, it follows that $\lambda_n \in \varrho^k_{q\Phi}(T)$, for all $n, k \in \mathbb{N}$. This shows that $\mathbb{C}\backslash\{0\} \subseteq \varrho^k_{q\Phi}(T)$, and as $0 \in \sigma^k_{q\Phi}(T)$, we obtain

$$\sigma^\infty_{q\Phi}(T) = \sigma^k_{q\Phi}(T) = \{0\}, \quad \forall\, k \in \mathbb{N}.$$

(c) Since for all $\lambda \in \sigma(T)\backslash\{0\}$, we have $\boldsymbol{a}(\lambda I - T) = \boldsymbol{d}(\lambda I - T) = 1$, it follows that $\lambda I - T \in pq\Phi(\ell^2(\mathbb{N}))$ possesses pq-index, for all $\lambda \in \mathbb{C}\backslash\{0\}$. Furthermore, since $\max\{\boldsymbol{a}(\lambda I - T),\ \boldsymbol{d}(\lambda I - T)\} \leq 1$, for all $\lambda \in \mathbb{C}\backslash\{0\}$, by Remark 6.3, we deduce that

$$\mathrm{ind}_{pq}(\lambda I - T) = \alpha^1_1(\lambda I - T) - \beta^1_1(\lambda I - T) = 0.$$

(d) Fix $c \in \mathbb{C}$ and consider the polynomial P defined by $P(Z) = c$. Then $P(T) = cI$. Since $\sigma^\infty_{q\Phi}(T)$ is nonempty, it follows that

$$P(\sigma^\infty_{q\Phi}(T)) = \{c\}.$$

However, $\varrho^\infty_{q\Phi}(P(T)) = \mathbb{C}$: indeed, $\mathbb{C}\backslash\{c\} = \varrho(cI) \subseteq \varrho^\infty_{q\Phi}(cI)$, and $cI - cI$ (that is, the zero operator on $\ell^2(\mathbb{N})$) is pseudo-quasi-Fredholm. Consequently, $\varrho^\infty_{q\Phi}(P(T)) = \mathbb{C}$ and

$$\sigma^\infty_{q\Phi}(P(T)) = \emptyset \neq P(\sigma^\infty_{q\Phi}(T)).$$

Hence the conclusion of Theorem 4.12 fails in the presence of a constant complex polynomial.

Example 7.5. Consider the infinite-dimensional complex Hilbert space $\mathsf{H} = \mathbb{C}^3 \times \ell^2(\mathbb{N})$ and the operator $T \in \mathscr{B}(\mathsf{H})$ defined by

$$T\Big((z_1,\, z_2,\, z_3),\, \sum_{n=0}^{+\infty} x_n e_n\Big) = \Big((z_2,\, 0,\, 0),\, z_3 e_0 + \sum_{n=0}^{+\infty} x_{n+1} e_n\Big),$$

where $\{e_n : n \in \mathbb{N}\}$ is an orthonormal basis of $\ell^2(\mathbb{N})$.

(a) We remark that

$$\mathsf{ker}(T) = \big\{\big((z_1, z_2,\, z_3),\, (x_n)_{n\in\mathbb{N}}\big) \in \mathsf{H} : z_2 = 0,\, x_1 = -z_3,\, x_n = 0,\, \forall\, n \geq 2\big\}$$

and

$$\mathsf{Im}(T) = \big\{\big((z_1, z_2,\, z_3),\, (x_n)_{n\in\mathbb{N}}\big) \in \mathsf{H} : z_2 = z_3 = 0\big\}.$$

Hence $\alpha(T) = 3$ and $\beta(T) = 2$, and consequently

(1) $\mathsf{Im}(T^i) \cap \mathsf{ker}(T^{j+1}) + \mathsf{ker}(T^j)$ and $\mathsf{Im}(T) + \mathsf{ker}(T^j)$ are closed, $\forall\, i,\, j \in \mathbb{N}$.

We observe that, for all $k \geq 2$,

$$T^k\Big((z_1,\, z_2,\, z_3),\, \sum_{n=0}^{+\infty} x_n e_n\Big) = \Big((0,\, 0,\, 0),\, \sum_{n=0}^{+\infty} x_{n+k} e_n\Big). \tag{2}$$

Hence

$$\mathsf{Im}(T^k) = \{0\} \times \ell^2(\mathbb{N}), \quad \forall\, k \geq 2.$$

Therefore,

$$\mathsf{ker}(T) \cap \mathsf{Im}(T) = \big\{\big((z_1,\, z_2,\, z_3),\, (x_n)_{n\in\mathbb{N}}\big) \in \mathsf{H} : z_2 = z_3 = 0,\; x_n = 0,\, \forall\, n \geq 1\big\},$$

and, for all $k \geq 2$,

$$\mathsf{ker}(T) \cap \mathsf{Im}(T^k) = \big\{\big((z_1,\, z_2,\, z_3),\, (x_n)_{n\in\mathbb{N}}\big) \in \mathsf{H} : z_1 = z_2 = z_3 = 0,\; x_n = 0,\, \forall\, n \geq 1\big\}.$$

Thus

$$q_0(T) = \inf\{k \in \mathbb{N} : \mathsf{ker}(T) \cap \mathsf{Im}(T^k) = \mathsf{ker}(T) \cap \mathsf{Im}(T^m),\; \forall\, m \geq k\} = 2.$$

For $k \geq 2$, by using (1) and Lemma 2.1, we obtain that T is a quasi-Fredholm (resp. 1-quasi-Fredholm, k-quasi-Fredholm) operator of degree $d = 2$ (resp. $d = 1$, $d = 0$).

(b) Recall that the reduced minimum modulus of a non-zero operator $A \in \mathscr{B}(\mathsf{H})$ is defined by

$$\gamma(A) = \inf\{\|Ax\| : x \in \mathsf{ker}(A)^\perp \text{ and } \|x\| = 1\}.$$

If $A = 0$ then we take $\gamma(A) = +\infty$. Now let $S \in \ell^2(\mathbb{N})$ be defined by

$$S\Big(\sum_{n=0}^{+\infty} x_n e_n\Big) = \sum_{n=0}^{+\infty} x_{n+2} e_n.$$

We note from (2) that

$$(\lambda I - T^2)(z,\, x) = \big(\lambda\, z,\, (\lambda I - S)x\big), \quad \forall\, (z,\, x) \in \mathbb{C}^3 \times \ell^2(\mathbb{N}), \quad \forall\, \lambda \in \mathbb{C}. \tag{3}$$

It is clear that S is Fredholm ($\alpha(S) = 2$, $\beta(S) = 0$) and $\gamma(S) = \|S\| = 1$. Therefore, for all $\lambda_1, \lambda_2 \in \mathbb{C}$ such that $|\lambda_1| < 1 = \gamma(S)$ and $|\lambda_2| > 1 = \|S\|$, we have $\lambda_1 I - S$ is Fredholm and $\lambda_2 I - S$ is invertible. Since T is Fredholm it follows from (3) that $\lambda I - T^2$ is Fredholm for all $\lambda \in \mathbb{C}$ such that $|\lambda| \neq 1$. Consequently, $\sigma^\infty_{q\Phi}(T^2) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and $\sigma^k_{q\Phi}(T^2) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, for all $k \in \mathbb{N}$. Now by Theorem 4.12, we see that if $\lambda \in \sigma^\infty_{q\Phi}(T)$ then $|\lambda^2| = 1$, this implies that $|\lambda| = 1$. Hence

$$\sigma^\infty_{q\Phi}(T) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = 1\}.$$

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