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REPRESENTATIONS OF THE ORLICZ FIGÀ-TALAMANCA HERZ ALGEBRAS AND SPECTRAL SUBSPACES

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Abstract. Let G be a locally compact group. In this note, we characterise non-degenerate $*$-representations of $A_\Phi(G)$ and $B_\Phi(G)$. We also study spectral subspaces associated to a non-degenerate Banach space representation of $A_\Phi(G)$.

1. Introduction

Let G be a locally compact group. It is well known that there is a one to one correspondence between the unitary representations of G and the non-degenerate $*$-representations of $L^1(G)$ [5, p. 73]. Similarly, if X is any locally compact Hausdorff space, then there is a one to one correspondence between the cyclic $*$-representations of $C_0(X)$ and positive bounded Borel measures on X [8, p. 486]. The corresponding result for the Fourier algebra $A(G)$ of a locally compact group is due to Lau and Losert [10]. For more on the Fourier algebra see [4, 9]. Recently, Guex [11] extended the result of Lau and Losert to Figà-Talamanca Herz algebras. We refer the readers to [2] for more on Figà-Talamanca Herz algebras.

In [14], the authors have introduced and studied the L^Φ-versions of the Figà-Talamanca Herz algebras. Here L^Φ denotes the Orlicz space corresponding to the Young function Φ. The space $A_\Phi(G)$ is defined as the space of all continuous functions u, where u is of the form

$$u = \sum_{n=1}^{\infty} f_n * \check{g_n},$$

where $f_n \in L^\Phi(G)$, $g_n \in L^\Psi(G)$, (Φ, Ψ) is a pair of complementary Young functions satisfying the Δ_2-condition and

$$\sum_{n=1}^{\infty} N_\Phi(f_n)\|g_n\|_\psi < \infty.$$

It is shown in [14] that $A_\Phi(G)$ is a regular, tauberian, semisimple commutative Banach algebra with the Gelfand spectrum homeomorphic to G.

This paper has the modest aim of characterising the non-degenerate $*$-representations of $A_\Phi(G)$ in the spirit of [10]. This characterisation is given in Corollary 3.4. In Section 4, we show that any non-degenerate $*$-representation of $A_\Phi(G)$ can be extended uniquely to a non-degenerate $*$-representation of $B_\Phi(G)$. In Section 5, we provide an application to ergodic sequences.

Godement in his fundamental paper [6] on Wiener Tauberian theorems studied spectral subspaces associated to a certain Banach space representations. This result was extended to the Fourier algebra $A(G)$ by Parthasarathy and Prakash [12]. In Section 6, we also study spectral subspaces of $A_\Phi(G)$.

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2. Preliminaries

Let $\Phi : \mathbb{R} \to [0, \infty]$ be a convex function. Then Φ is called a Young function if it is symmetric and satisfies $\Phi(0) = 0$ and $\lim_{x\to\infty} \Phi(x) = +\infty$. If Φ is any Young function, then define Ψ as

$$\Psi(y) := \sup\{x|y| - \Phi(x) : x \geq 0\}, \quad y \in \mathbb{R}.$$

Then Ψ is also a Young function and is termed as the complementary function to Φ. Further, the pair (Φ, Ψ) is called a complementary pair of Young functions.

Let G be a locally compact group with a left Haar measure dx. We say that a Young function Φ satisfies the Δ_2-condition, denoted $\Phi \in \Delta_2$, if there exists a constant $K > 0$ and $x_0 > 0$ such that $\Phi(2x) \leq K\Phi(x)$ whenever $x \geq x_0$ if G is compact and the same inequality holds with $x_0 = 0$ if G is non compact.

The Orlicz space, denoted $L^\Phi(G)$, is a vector space consisting of measurable functions, defined as

$$L^\Phi(G) = \left\{ f : G \to \mathbb{C} : \text{f is measurable and } \int_G \Phi(\beta|f|)\, dx < \infty \text{ for some } \beta > 0 \right\}$$

The Orlicz space $L^\Phi(G)$ is a Banach space when equipped with the norm

$$N_\Phi(f) = \inf\left\{ k > 0 : \int_G \Phi\left(\frac{|f|}{k}\right) dx \leq 1 \right\}.$$

The above norm is called as the Luxemburg norm or Gauge norm. If (Φ, Ψ) is a complementary Young pair, then there is a norm on $L^\Phi(G)$, equivalent to the Luxemberg norm, given by,

$$\|f\|_\Phi = \sup\left\{ \int_G |fg| dx : \int_G \Psi(|g|) dx \leq 1 \right\}.$$

This norm is called as the Orlicz norm.

Let $C_c(G)$ denote the space of all continuous functions on G with compact support. If a Young function Φ satisfies the Δ_2 -condition, then $C_c(G)$ is dense in $L^\Phi(G)$. Further, if the complementary function Ψ is such that Ψ is continuous and $\Psi(x) = 0$ iff $x = 0$, then the dual of $(L^\Phi(G), N_\Phi(\cdot))$ is isometrically isomorphic to $(L^\Psi(G), \|\cdot\|_\Psi)$. In particular, if both Φ and Ψ satisfies the Δ_2-condition, then $L^\Phi(G)$ is reflexive.

For more details on Orlicz spaces, we refer the readers to [13].

Let Φ and Ψ be a pair of complementary Young functions satisfying the Δ_2 condition. Let

$$A_\Phi(G) = \left\{ u = \sum_{n=1}^{\infty} f_n * \check{g_n} : \{f_n\} \subset L^\Phi(G), \{g_n\} \in L^\Psi(G) \text{ and } \sum_{n=1}^{\infty} N_\Phi(f_n)\|g_n\|_\Psi < \infty \right\}.$$

Note that if $u \in A_\Phi(G)$ then $u \in C_0(G)$. If $u \in A_\Phi(G)$, define $\|u\|_{A_\Phi}$ as

$$\|u\|_{A_\Phi} := \inf\left\{ \sum_{n=1}^{\infty} N_\Phi(f_n)\|g_n\|_\Psi : u = \sum_{n=1}^{\infty} f_n * \check{g_n} \right\}.$$

The space $A_\Phi(G)$ equipped with the above norm and with the pointwise addition and multiplication becomes a commutative Banach algebra [14, Theorem 3.4]. In fact, $A_\Phi(G)$ is a commutative, regular and semisimple banach algebra with spectrum homeomorphic to G [14, Corollary 3.8]. This Banach algebra $A_\Phi(G)$ is called as the Orlicz Figà-Talamanca Herz algebra. Let

$$B_\Phi(G) := \{u \in C(G) : uv \in A_\Phi(G)\ \forall\ v \in A_\Phi(G)\}$$

equipped with the norm $\|u\|_{B_\Phi} = \sup\{\|uv\|_{A_\Phi} : v \in A_\Phi(G),\ \|v\|_{A_\Phi} = 1\}$. Then, with the above norm, $B_\Phi(G)$ becomes a commutative Banach algebra with pointwise addition and multiplication.

Let $\mathcal{B}(L^\Phi(G))$ be the linear space of all bounded linear operators on $L^\Phi(G)$ equipped with the operator norm. For a bounded complex Radon measure μ on G and $f \in L^\Phi(G)$, define $T_\mu : L^\Phi(G) \to L^\Phi(G)$ by $T_\mu(f) = \mu * f$. It is clear that $T_\mu \in \mathcal{B}(L^\Phi(G))$. Let $PM_\Phi(G)$ denote the closure of

$$\{T_\mu : \mu \text{ is a bounded complex Radon measure}\}$$

in $\mathcal{B}(L^\Phi(G))$ with respect to the ultraweak topology. It is proved in [14, Theorem 3.5], that for a locally compact group G, the dual of $A_\Phi(G)$ is isometrically isomorphic to $PM_\Psi(G)$. By [14, Theorem 3.6] singletons are sets of spectral synthesis for $A_\Phi(G)$. Further, every closed subgroup is a set of local synthesis for $A_\Phi(G)$.

Throughout this paper, G will denote a locally compact group with a fixed left Haar measure dx. Further Φ will always denote a Young function whose complementary Young function is Ψ and the pair (Φ, Ψ) satisfies the Δ_2-condition.

3. Non-degenerate $*$-representations of $A_\Phi(G)$

In this section, motivated by the results of [10, 11], we describe all the non-degenerate $*$-representations of $A_\Phi(G)$. Throughout this section and the next, $\mathcal{H}$ will denote a Hilbert space.

Proposition 3.1. *Let μ be a bounded positive Radon measure on G.*

(i) For each $u \in A_\Phi(G)$, the mapping $\pi_\mu(u) : f \mapsto uf$ is a bounded linear operator on $L^2(G, d\mu)$.
(ii) The mapping $u \mapsto \pi_\mu(u)$ defines a $*$-representation of $A_\Phi(G)$ on $\mathcal{B}(L^2(G, d\mu))$.
(iii) If μ is bounded, then π_μ is a cyclic representation of $A_\Phi(G)$ with the constant 1 function as cyclic vector.

Proof. (i) and (ii) are just a routine check.

(iii) We show that the constant 1 function is a cyclic vector. Since the measure μ is finite, the conclusion follows from the density of $A_\Phi(G) \cap C_c(G)$ in $C_c(G)$ with respect to the $L^2(G, d\mu)$-norm. □

Corollary 3.2. *If μ is a positive Radon measure on G (not necessarily bounded) then π_μ (defined as in Proposition 3.1) is non-degenerate.*

Proof. Let μ be a positive Radon measure on G. By [3, Pg. 33, 2.2.7], it is enough to show that the representation π_μ is a direct sum of cyclic representations. By [1, INT IV.77] and [1, INT V.14, Proposition 4], it follows that

$$L^2(G, d\mu) \cong \bigoplus_{\alpha \in \wedge} L^2(G, d\mu_\alpha),$$

where $\{\mu_\alpha\}_{\alpha \in \wedge}$ is a summable family of measures with pairwise disjoint support. Now the conclusion follows from (iii) of Proposition 3.1. □

In the next result, we characterise all cyclic $*$-representations.

Theorem 3.3. *Let $\{\pi, \mathcal{H}\}$ be a cyclic $*$-representation of $A_\Phi(G)$. Then there exists a bounded positive Radon measure μ such that π is unitarily equivalent to the representation $\{\pi_\mu, L^2(G, d\mu)\}$ given in Proposition 3.1.*

Proof. Let $u \in A_\Phi(G)$. Then, by [15, Pg. 22], it follows that $\|\pi(u)\|_{sp} \leq \|u\|_{sp}$. By [14, Theorem 3.4], $A_\Phi(G)$ is a commutative Banach algebra and hence the spectral norm and the operator norm for $\pi(u)$ coincides. Further, as $A_\Phi(G)$ is semi-simple and the fact that the spectrum of $A_\Phi(G)$ is G [14, Corollary 3.81], $\|u\|_{sp} = \|u\|_\infty$. Thus,

$$\|\pi(u)\|_{\mathcal{B}(\mathcal{H})} \leq \|u\|_\infty.$$

As a consequence of this inequality and the fact that $A_\Phi(G)$ is dense in $C_0(G)$, it follows that π extends to a $*$-representation of $C_0(G)$ on $\mathcal{H}$, still denoted as π. Note that π is a cyclic $*$-representation of the C^*-algebra $C_0(G)$. Let φ be the cyclic vector of the representation $\{\pi, C_0(G)\}$. Define $T_\varphi : C_0(G) \to \mathbb{C}$ as

$$T_\varphi(u) = \langle \pi(u)\varphi, \varphi\rangle, \ u \in C_0(G).$$

It is clear that T_φ is a positive linear functional on $C_0(G)$ and hence, by Riesz representation theorem, there exists a bounded positive Radon measure μ such that

$$T_\varphi(u) = \int_G u \ d\mu. \tag{3.1}$$

Let π_μ denote the cyclic $*$-representation of $A_\Phi(G)$ on $L^2(G, d\mu)$, given by Proposition 3.1.

We now claim that the representations π and π_μ of $A_\Phi(G)$ are unitarily equivalent. Since φ is a cyclic vector, in order to prove the above claim, it is enough to show that the correspondence $\pi(u)\varphi \mapsto u.1$ is an isometry and commutes with π and π_μ. Note that the above correspondence is well-defined by (3.1). Let T denote the above well-defined correspondence.

We now show that T is an isometry. Let $u \in A_\Phi(G)$. Then

$$\begin{aligned}\langle \pi(u)\varphi, \pi(u)\varphi\rangle =& \langle \pi^*(u)\pi(u)\varphi, \varphi\rangle \\ =& \langle \pi(\bar{u}u)\varphi, \varphi\rangle \ (\pi \text{ is a } *\text{-homomorphism}) \\ =& \int_G |u|^2 \ d\mu = \langle \varphi, \varphi\rangle.\end{aligned}$$

Finally, we show that T intertwines with π and π_μ. Let $u \in A_\Phi(G)$. Then, for $v \in A_\phi(G)$, we have,

$$\begin{aligned}T(\pi(u)(\pi(v)\varphi)) =& T((\pi(u)\pi(v))\varphi) \\ =& T(\pi(uv)\varphi) = uv.1 \\ =& \pi_\mu(u)(v.1) = \pi_\mu(u)(T(\pi(v)\varphi)).\end{aligned}$$

□

Here is the main result of this section, describing all the non-degenerate Hilbert space representations of $A_\Phi(G)$.

Corollary 3.4. *If $\{\pi, \mathcal{H}\}$ is any non-degenerate $*$-representation of $A_\Phi(G)$ then π is unitarily equivalent to $\{\pi_\mu, L^2(G, d\mu)\}$ for some positive measure μ.*

Proof. Let $\{\pi, \mathcal{H}\}$ be a non-degenerate $*$-representation of $A_\Phi(G)$. By [3, Proposition 2.2.7], π is a direct sum of cyclic $*$-representations $\{\pi_\alpha, \mathcal{H}_\alpha\}_{\alpha\in\wedge}$. For each $\alpha \in \wedge$, by Theorem 3.3, there exists a bounded positive measure μ_α such that the representations $\{\pi_\alpha, \mathcal{H}_\alpha\}$ and $\{\pi_{\mu_\alpha}, L^2(G, d\mu_\alpha)\}$ are unitarily equivalent.

Suppose that the family $\{\mu_\alpha\}_{\alpha\in\wedge}$ is summable. Let $\mu = \sum\limits_{\alpha\in\wedge} \mu_\alpha$. Then μ will be a positive measure and

$$\{\pi_\mu, L^2(G, d\mu)\} \cong \bigoplus_{\alpha\in\wedge} \{\pi_{\mu_\alpha}, L^2(G, d\mu_\alpha)\} \cong \bigoplus_{\alpha\in\wedge} \{\pi_\alpha, \mathcal{H}_\alpha\} \cong \{\pi, \mathcal{H}\}.$$

Thus, we are done if we can show that $\{\mu_\alpha\}_{\alpha\in\wedge}$ is a summable family. Let $f : G \to \mathbb{C}$ be a continuous function with compact support. Then $\bigoplus_{\alpha\in\wedge} f \in \bigoplus_{\alpha\in\wedge} L^2(G,\mu_\alpha)$ and hence,

$$\sum_{\alpha\in\wedge}\left(\int_G |f|^2 \ d\mu_\alpha\right)^{1/2} < \infty. \tag{3.2}$$

Now,

$$\begin{aligned}
\sum_{\alpha\in\wedge}|\mu_\alpha(f)| &= \sum_{\alpha\in\wedge}\left|\int_G f \ d\mu_\alpha\right| \le \sum_{\alpha\in\wedge}\int_G |f| \ d\mu_\alpha \\
&\le \sum_{\alpha\in\wedge}\left(\int_G |f|^2 \ d\mu_\alpha\right)^{1/2}\left(\int_G |1|^2 \ d\mu_\alpha\right)^{1/2} \\
&= \sum_{\alpha\in\wedge}\left(\int_G |f|^2 \ d\mu_\alpha\right)^{1/2}(\mu_\alpha(G))^{1/2} \\
&\le \sup_{\alpha\in\wedge}(\mu_\alpha(G))^{1/2}\sum_{\alpha\in\wedge}\left(\int_G |f|^2 \ d\mu_\alpha\right)^{1/2} \\
&\le \left(\sup_{\alpha\in\wedge}\mu_\alpha(G)\right)^{1/2}\sum_{\alpha\in\wedge}\left(\int_G |f|^2 \ d\mu_\alpha\right)^{1/2} < \infty.
\end{aligned}$$

The boundedness of $\sup_{\alpha\in\wedge}\mu_\alpha(G)$ follows from the uniform boundedness principle and from (3.2). □

4. Non-degenerate $*$-representations of $B_\Phi(G)$

In this section, we show that the non-degenerate representations described in the previous section can be extended uniquely to $B_\Phi(G)$.

Theorem 4.1. *Let $\{\pi,\mathcal{H}\}$ be a non-degenerate $*$-representation of $A_\Phi(G)$.*

(i) For each $u \in B_\Phi(G)$, there exists a unique operator $\widetilde{\pi}(u) \in \mathcal{B}(\mathcal{H})$ such that, $\forall\ v \in A_\Phi(G)$,

$$\widetilde{\pi}(u)\pi(v) = \pi(uv) \tag{4.1}$$

and

$$\widetilde{\pi}(v) = \pi(v). \tag{4.2}$$

(ii) The mapping $u \mapsto \widetilde{\pi}(u)$ defines a non-degenerate $*$-representation of $B_\Phi(G)$ on $\mathcal{H}$.

Proof. **(i)** Let π be a non-degenerate $*$-representation of $A_\Phi(G)$. By [3, Proposition 2.2.7], π is a direct sum of cyclic $*$-representations, say $\{\pi_\alpha,\mathcal{H}_\alpha\}_{\alpha\in\wedge}$. If we can prove (i) for each of these π_α's, then the argument for π is similar to the one given in Corollary 3.4. Thus, in order to prove this, we assume that the representation π is cyclic. Since π is a cyclic $*$-representation, by Theorem 3.3, π is unitarily equivalent to π_μ, for some bounded positive Radon measure μ. So, without loss of generality, let us assume that the non-degenerate $*$-representation of $A_\Phi(G)$ is π_μ for some bounded positive Radon measure μ.

Let $u \in B_\Phi(G)$. By Proposition 3.1, the space $\mathcal{K} := span\{\pi_\mu(v).1 : v \in A_\Phi(G)\}$ is dense in $L^2(G,d\mu)$. Define $\widetilde{\pi_\mu}(u) : \mathcal{K} \to L^2(G,d\mu)$ as

$$\widetilde{\pi_\mu}(u)(\pi_\mu(v).1) = \pi_\mu(uv).1.$$

It is clear that $\widetilde{\pi_\mu}(u)$ is linear. We now claim that $\widetilde{\pi_\mu}(u)$ is bounded. Let $v \in A_\Phi(G)$. Then

$$\begin{aligned}\|\widetilde{\pi_\mu}(u)\,(\pi_\mu(v).1)\,\|_2^2 =&\|\pi_\mu(uv).1\|_2^2\\ =&\int_G |\pi_\mu(uv).1|^2\ d\mu\\ =&\int_G |uv|^2\ d\mu\\ \le&\|u\|_\infty^2 \int_G |v|^2\ d\mu \le \|u\|_{B_\Phi}^2 \|\pi_\mu(v).1\|_2^2.\end{aligned}$$

Thus, $\widetilde{\pi_\mu}(u)$ extends to a bounded linear operator on $L^2(G, d\mu)$, still denoted $\widetilde{\pi_\mu}(u)$. Further, it is clear that, for $u \in B_\Phi(G)$ and $v \in A_\Phi(G)$, $\widetilde{\pi_\mu}(u)\pi_\mu(v) = \pi_\mu(uv)$. Now, let $v \in A_\Phi(G)$. Then, for $u \in A_\Phi(G)$,

$$\widetilde{\pi_\mu}(v)(\pi_\mu(u).1) = \pi_\mu(vu).1 = \pi_\mu(v)\,(\pi_\mu(u).1)\,.$$

Again, as $\mathcal{K}$ is dense in $L^2(G, d\mu)$, it follows that $\widetilde{\pi_\mu}(v) = \pi_\mu(v)$ for all $v \in A_\Phi(G)$.

Finally, uniqueness follows from condition (4.1).

(ii) Non-degeneracy of $\widetilde{\pi}$ follows from the fact that π is non-degenerate. Further, homomorphism property of $\widetilde{\pi}$ follows from (4.1). Now, we show that $\widetilde{\pi}$ preserves involution. Let $u \in B_\Phi(G)$. Then, for $v \in A_\Phi(G)$ and $\xi, \eta \in \mathcal{H}$, we have

$$\begin{aligned}\langle \widetilde{\pi}(u)^*\pi(v)\xi, \eta\rangle &= \langle \xi, \pi(\overline{v})\widetilde{\pi}(u)\eta\rangle\\ &= \langle \xi, \widetilde{\pi}(\overline{v})\widetilde{\pi}(u)\eta\rangle \text{ (by (4.2))}\\ &= \langle \xi, \widetilde{\pi}(u\overline{v})\eta\rangle \text{ ($\widetilde{\pi}$ is a homomorphism)}\\ &= \langle \xi, \pi(u\overline{v})\eta\rangle \text{ (by (4.2))}\\ &= \langle \xi, \pi(\overline{\overline{u}v})^*\eta\rangle \text{ (π preserves involution)}\\ &= \langle \pi(\overline{u}v)\xi, \eta\rangle\\ &= \langle \widetilde{\pi}(\overline{u})\pi(v)\xi, \eta\rangle. \text{ (by (4.1))}\end{aligned}$$

Since the representation π is non-degenerate, the space $\{\pi(u)\xi : u \in A_\Phi(G), \xi \in \mathcal{H}\}$ is dense in $\mathcal{H}$. Thus, it follows that $\widetilde{\pi}(u)^* = \widetilde{\pi}(\overline{u})$ for all $u \in B_\Phi(G)$. □

The following corollary is the converse of the above theorem.

Corollary 4.2. *Let $\{\pi, \mathcal{H}\}$ be a $*$-representation of $B_\Phi(G)$ such that $\pi|_{A_\Phi}$ is non-degenerate. Then, $\widetilde{\pi|_{A_\Phi}} = \pi$ and π is non-degenerate.*

Proof. Let $u \in B_\Phi(G)$ and $v \in A_\Phi(G)$. Then

$$\pi(u)\pi|_{A_\Phi}(v) = \pi(u)\pi(v) = \pi(uv) = \pi|_{A_\Phi}(uv).$$

Thus, by Theorem 4.1, it follows that $\widetilde{\pi|_{A_\Phi}} = \pi$. Again by Theorem 4.1, $\widetilde{\pi|_{A_\Phi}}$ is non-degenerate and hence it follows that the representation π is non-degenerate. □

5. Application to ergodic sequences in $A_\Phi(G)$

In this section, we discuss an application of ergodic sequences. This section is also motivated from [10] and [11].

Let

$$\begin{aligned}S_B^\Phi &= \big\{u \in B_\Phi(G) : \|u\|_{B_\Phi} = u(e) = 1\big\}\\ S_A^\Phi &= \big\{u \in A_\Phi(G) : \|u\|_{A_\Phi} = u(e) = 1\big\}.\end{aligned}$$

Before we proceed to the main result of this section, here we give an appropriate definition.

Definition 5.1. A sequence $\{u_n\} \subset S_B^\Phi$ is said to be strongly (resp. weakly) ergodic if for any non-degenerate $*$-representation $\{\pi, \mathcal{H}_\pi\}$ of $A_\Phi(G)$ the sequence $\{\widetilde{\pi}(u_n)\eta\}$ converges strongly (resp. weakly) to an element of $\mathcal{H}_f$, for every $\eta \in \mathcal{H}$, where

$$\mathcal{H}_f = \{\xi \in \mathcal{H} : \pi(u)\xi = \xi \ \forall \ u \in S_A^\Phi\}.$$

Our next theorem is the main result of this section.

Theorem 5.2. *For a sequence $\{u_n\}$ in S_B^Φ, the following statements are equivalent:*

(i) the sequence $\{u_n\}$ is strongly ergodic.

(ii) the sequence $\{u_n\}$ is weakly ergodic.

(iii) the sequence $\{u_n(x)\}$ converges to 0 for every $x \in G$ with $x \neq e$.

Proof. (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii). Fix $x \in G$ with $x \neq e$. Define $\pi : A_\Phi(G) \to \mathbb{C}$ as $\pi(u) = u(x)$. Then π defines a non-degenerate $*$-representation of $A_\Phi(G)$ on $\mathbb{C}$. By Theorem 4.1, the representation $\{\pi, \mathbb{C}\}$ can be extended uniquely to a non-degenerate $*$-representation $\widetilde{\pi}$ of $B_\Phi(G)$ on $\mathbb{C}$ such that $\widetilde{\pi}(u)z = u(x)z$ for all $u \in B_\Phi(G)$. Since $\{u_n\}$ is weakly ergodic the set $\{\mathbb{C}_f\}$ is non-empty. In order to prove (iii) it is enough to show that the set $\mathbb{C}_f$ consists only of the zero vector. Suppose to the contrary that there exists $0 \neq z \in \mathbb{C}_f$. Since G is Hausdorff, there exists an open set U containing e but not x. Let v denote the function given by [14, Proposition 5.5], corresponding the open set U. Then $v \in S_A^\Phi$ and $v(x)z = 0$, which is a contradiction. Thus the set $\mathbb{C}_f$ consists only of the zero vector. Hence (iii).

(iii) $\Rightarrow$ (i). Let π be a non-degenerate $*$-representation of $A_\Phi(G)$. By Corollary 3.4, π is unitarily equivalent to the representation $\{\pi_\mu, L^2(G, d\mu)\}$ for some positive measure μ defined on G. So, without loss of generality, let us assume that π is of the form π_μ for some positive measure μ on G. Let $\widetilde{\pi_\mu}$ denote the extension of π_μ from $A_\Phi(G)$ to $B_\Phi(G)$ as in Theorem 4.1. Let $f \in L^2(G, d\mu)$. We now claim that the sequence $\{\widetilde{\pi_\mu}(u_n)(f)\}$ converges strongly. As $L^2(G, d\mu)$ is complete, in order to prove the claim, it is enough to show that the sequence $\{\widetilde{\pi_\mu}(u_n)(f)\}$ is Cauchy. Note that, for any $n, m \in \mathbb{N}$,

$$|u_n(x) - u_m(x)|^2 |f(x)|^2 \leq 4|f(x)|^2 \ a.e.$$

Thus, by dominated convergence theorem and by (iv), we have,

$$\begin{aligned}
\|\widetilde{\pi_\mu}(u_n)(f) - \widetilde{\pi_\mu}(u_m)(f)\|_2^2 &= \int_G |\widetilde{\pi_\mu}(u_n)(f)(x) - \widetilde{\pi_\mu}(u_m)(f)(x)|^2 \ d\mu(x) \\
&= \int_G |u_n(x) - u_m(x)|^2 |f(x)|^2 \ d\mu(x) \\
&\to 0.
\end{aligned}$$

Let $g \in L^2(G, d\mu)$ denote the limit of the sequence $\{\widetilde{\pi_\mu}(u_n)(f)\}$. Our next claim is that g is a fixed point of $\pi_\mu(u)$ for each $u \in S_A^\Phi$. Again, this is a consequence of the dominated convergence theorem. $\square$

6. Spectral subspaces

In this section, we study the spectral subspaces associated to a non-degenerate Banach space representation of $A_\Phi(G)$. Our main aim in this section is to prove Corollary 6.9. Most of the ideas of this section are taken from [12].

Definition 6.1. Let $T \in PM_\Psi(G)$. Then the support of T is defined as

$$supp(T) = \{x \in G : u \in A_\Phi(G), u(x) \neq 0 \Rightarrow u.T \neq 0\}.$$

Here we recall some of the properties of the support of T in the form of a Lemma [7, Pg. 101].

Lemma 6.2.

(i) *If $T_1, T_2 \in PM_\Psi(G)$ then $supp(T_1 + T_2) \subseteq supp(T_1) \cup supp(T_2)$.*

(ii) *If $u \in A_\Phi(G)$ and $T \in PM_\Psi(G)$ then $supp(u.T) \subseteq supp(u) \cap supp(T)$.*

(iii) *If $c \in \mathbb{C}$ and $T \in PM_\Psi(G)$ then $supp(cT) \subseteq supp(T)$.*

(iv) *Let $T \in PM_\Psi(G)$ and let E be a closed subset of G. If a net $\{T_\alpha\} \subset PM_\Psi(G)$ converges weakly to T with $supp(T_\alpha) \subset E$ for all α, then $supp(T) \subset E$.*

Let X be a Banach space and let π be an algebra representation of $A_\Phi(G)$ on X. For $\varphi \in X$ and $x^* \in X^*$, define $T_{x^*,\varphi} : A_\Phi(G) \to \mathbb{C}$ as

$$\langle u, T_{x^*,\varphi} \rangle := \langle \pi(u)\varphi, x^* \rangle \ \forall\ u \in A_\Phi(G).$$

We say that the representation π is continuous if $T_{x^*,\varphi}$ is a continuous linear functional on $A_\Phi(G)$ for each $\varphi \in X$ and $x^* \in X^*$. It follows from uniform boundedness principle that the linear map $\pi : A_\Phi(G) \to \mathcal{B}(X)$ is norm continuous.

From now onwards, X will denote a Banach space and π an algebra representation of $A_\Phi(G)$ on X.

Let E be a closed subset of G. Define

$$X_E := \{\varphi \in X : supp(T_{x^*,\varphi}) \subseteq E \ \forall\ x^* \in X^*\}.$$

Remark 6.3. An immediate consequence of the above definition is that, if $E = G$ then $X_E = X$.

Lemma 6.4. *The set X_E is a closed π-invariant subspace of X.*

Proof. Note that for any $x^* \in X^*$, $\varphi_1, \varphi_2 \in X_E$ and $\alpha \in \mathbb{C}$, we have

$$T_{x^*,\varphi_1+\alpha\varphi_2} = T_{x^*,\varphi_1} + \alpha T_{x^*,\varphi_2}.$$

Thus, it follows from (i) and (iii) of Lemma 6.2 that X_E is a linear space. Further, closedness of X_E is an immediate consequence of (iv) from Lemma 6.2. Again, note that, for any $u \in A_\Phi(G), \varphi \in X$ and $x^* \in X^*$, we have $T_{x^*,\pi(u)\varphi} = u.T_{x^*,\varphi}$ and hence the invariance of X_E under π follows from (ii) of Lemma 6.2. □

The subspace X_E is called as the spectral subspace associated with the representation π and the closed set E.

Lemma 6.5. *Let π be a non-degenerate representation of $A_\Phi(G)$.*

(i) *The space $X_\emptyset = \{0\}$.*

(ii) *If $\{E_i\}$ is an arbitrary collection of closed subsets of G, then $X_{\bigcap_i E_i} = \bigcap_i X_{E_i}$.*

Proof. (i) is an easy consequence of the non-degeneracy of π, while (ii) is trivial. □

The following is an immediate corollary of Remark 6.3 and Lemma 6.5.

Corollary 6.6. *There exists a smallest closed non-empty set E of G such that $X_E = X$.*

Proposition 6.7. *Let K_1 and K_2 be disjoint compact subsets of G. Then $X_{K_1 \cup K_2} = X_{K_1} \oplus X_{K_2}$.*

Proof. The proof of this follows exactly as given in [12, Proposition 2 (iii)]. □

Theorem 6.8. *Let π be a non-degenerate representation of $A_\Phi(G)$ such that the only spectral subspaces are the trivial subspaces. Then there exists $x \in G$ such that $X_{\{x\}} = X$.*

Proof. Choose a smallest non-empty closed set E such that $X_E = X$, which is possible by Corollary 6.6. Suppose there exists $x, y \in E$ such that $x \neq y$. As G is locally compact and Hausdorff, there exists an open set U and a compact set K such that $x \in U \subset K$ and $y \notin K$. Since $A_\Phi(G)$ is regular, there exists $u \in A_\Phi(G)$ such that $u = 1$ on U and $supp(u) \subset K$.

Let $v \in A_\Phi(G)$ be arbitrary. Let $v_1 = v - uv$ and $v_2 = uv$ so that $v = v_1 + v_2$. Let $V = \{z \in G : v_1(z) \neq 0\}$. The choice of u tells us that $x \notin \overline{V}$. Again, using the regularity of $A_\Phi(G)$, choose a function $w \in A_\Phi(G)$ such that $w = 1$ on some open set W containing x and $supp(w) \cap V = \emptyset$. Further, it is clear that $v_1 w = 0$.

We now claim that $\pi(v) = 0$. Let $\varphi \in X$ and $x^* \in X^*$. If $z \in W$, then $w(z) = 1$ and hence $T_{x^*,\pi(v_1 w)\varphi} = 0$ as $T_{x^*,\pi(v_1 w)\varphi} = w.T_{x^*,\pi(v_1)\varphi}$. Thus $supp(T_{x^*,\pi(v_1)\varphi}) \subset W^c$. Therefore, using the non-degeneracy of π, it follows that, if $\pi(v_1)\varphi \neq 0$ then $X_{W^c} = X$ and hence, by the choice of the set E, it follows that E is a subset of W^c. On the other hand, $x \notin W^c$ and $x \in E$ and hence E is not a subset of W^c. Therefore, $\pi(v_1) = 0$. Similarly, one can show that $\pi(v_2) = 0$. Thus $\pi(v) = 0$. Since v is arbitrary, it follows that $\pi(v) = 0$ for all $v \in A_\Phi(G)$, which is a contradiction. Thus the set E is a singleton. □

Corollary 6.9. *Let π be a non-degenerate representation of $A_\Phi(G)$ such that the only spectral subspaces are the trivial subspaces. Then π is a character.*

Proof. By Theorem 6.8, there exists $x \in G$ such that $X_{\{x\}} = X$, i.e.,

$$supp(T_{x^*,\varphi}) \subset \{x\}$$

for all $\varphi \in X$ and $x^* \in X^*$. As singletons are sets of spectral synthesis for $A_\Phi(G)$ [14, Theorem 3.6 (i)], it follows that

$$T_{x^*,\varphi} = c\delta_x \tag{6.1}$$

for some $c \in \mathbb{C}$. Let $u \in A_\Phi(G)$ such that $u(x) = 1$. Then

$$c = c\langle u, \delta_x\rangle = \langle u, c\delta_x\rangle = \langle u, T_{x^*,\varphi}\rangle = \langle \pi(u)\varphi, x^*\rangle. \tag{6.2}$$

We now claim that π is a character. Let $v \in A_\Phi(G)$. Then, for $\varphi \in X$ and $x^* \in X^*$, we have

$$\begin{aligned}\langle \pi(v)\varphi, x^*\rangle =& \langle v, T_{x^*,\varphi}\rangle = \langle v, c\delta_x\rangle \text{ (by (6.1))}\\ =& c\langle v, \delta_x\rangle = \langle \pi(u)\varphi, x^*\rangle\langle v, \delta_x\rangle \text{ (by (6.2))}\\ =& v(x)\langle \pi(u)\varphi, x^*\rangle = \langle v(x)\pi(u)\varphi, x^*\rangle.\end{aligned}$$

Since φ and x^* are arbitrary, it follows that $\pi(v) = u(x)\pi(u)$. Now

$$\pi(u) = u(x)\pi(u) = u^2(x)\pi(u) = \pi(u^2) = \pi(u)^2,$$

i.e., $\pi(u)$ is a projection. As π is non-degenerate, it follows that $\pi(u)$ is the identity operator I on X. Thus

$$\pi(v) = v(x)I \ \forall\ v \in A_\Phi(G),$$

i.e., π is a character. □

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