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AN ANALOGUE OF THE LOGARITHMIC (u,v)-DERIVATIVE AND ITS APPLICATION

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Abstract. We study an analogue of the logarithmic (u,v)-derivative. The last one has many interesting properties and good ways to calculate it. To show how it can be used we apply it to a model class of nowhere monotone functions that are composition of Salem function and nowhere differentiable functions.

1. Introduction

We are interested in continuous functions that are both singular (different from constant, but have a derivative equal to zero, almost everywhere, in terms of Lebesgue measure) and nowhere monotone (don't have any interval of monotonicity). Their theory is poor enough and is exhausted by a few separate examples. It is possible to expand the range of such objects by the superposition of singular and nowhere monotone functions. In a model example of a pair of known simple representatives of the class of singular functions and the class of nowhere monotone functions, we discuss the problems of a detailed study of differential properties of complex functions and propose a new toolkit for their study.

A singularly continuous Salem function, which depends on the parameter of $q_0\in(0;1)$, is defined on $[0;1]$ by

$$(1)\quad S(x)=S\left(\Delta^2_{\alpha_1(x)\alpha_2(x)\ldots\alpha_n(x)\ldots}\right)=\alpha_1 q_{1-\alpha_1}+\sum_{k=2}^{\infty}\left(\alpha_k q_{1-\alpha_k}\prod_{j=1}^{k-1}q_{\alpha_j}\right)\equiv\Delta^{Q_2}_{\alpha_1\alpha_2\ldots\alpha_n\ldots},$$

where $q_1\equiv 1-q_0$, $\alpha_n q_{1-\alpha_n}\equiv\beta_{\alpha_n}$, $\Delta^2_{\alpha_1\alpha_2\ldots\alpha_n\ldots}=\sum_{n=1}^{\infty}2^{-n}\alpha_n$ is the classical binary representation of a number, $\alpha_n\in A_2=\{0,1\}$.

For $q=1/2$, the function $S(x)$ is linear and, for $q\neq 1/2$, it is singularly continuous. Its properties have been studied in [6].

For a given set of three parameters (g_0,g_1,g_2), where $g_0+g_1+g_2=1$, $g_0=g_2\in\left(\frac{1}{2};1\right)$, the function g on $[0;1]$ is defined as

$$(2)\qquad g(x)=g\left(\Delta^3_{a_1(x)a_2(x)\ldots a_n(x)\ldots}\right)=\delta_{a_1}+\sum_{k=2}^{\infty}\left(\delta_{a_k}\prod_{j=1}^{\infty}g_{a_j}\right)\equiv\Delta^{G_3}_{a_1a_2\ldots a_n\ldots},$$

where $\delta_0=0$, $\delta_1=g_0$, $\delta_2=g_0+g_1$, $\Delta^3_{a_1a_2\ldots a_n\ldots}=\sum_{n=1}^{\infty}3^{-n}a_n$ is a classical ternary representation of a number, $a_n\in A_3=\{0,1,2\}$.

The function g is continuous on $[0,1]$, nowhere monotone, non-differentiable.

The object of our consideration is the continuous functions $\psi(x)=S\big(g(x)\big)$ and $\varphi(x)=g\big(S(x)\big)$. Moreover, each of them on any segment of the domain of definition is a function of unbounded variation, which ensures the existence of infinite levels of the function. It

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is clear that these functions have non-trivial local properties, a study of which requires a use of nontraditional approaches. Note that even calculating the value of the function at a given point is not easy, let alone calculating the actual derivative.

In the following four paragraphs we provide brief facts about the (u,v)-derivative and its analogue, the logarithmic (u,v)-derivative and, accordingly, an analogue of the logarithmic (u,v)-derivative, which provides the sought toolkit for these functions.

2. Key concepts and statements

Let $\mathcal{P}$ be a set of pairs (u,v) of all infinitesimal functions at zero, such, that for each pair there exists a number $\delta > 0$ such that for $\forall h \in O^*_\delta$ we have $\mathsf{u}(h) \neq -\mathsf{v}(h)$. Let $\Delta^{\mathsf{u}(h)}_{\mathsf{v}(h)} f(x_0) := f\big(x_0 + \mathsf{u}(h)\big) - f\big(x_0 - \mathsf{v}(h)\big)$, $\Delta^{\mathsf{u}(h)}_{\mathsf{v}(h)} x := \mathsf{u}(h) + \mathsf{v}(h)$.

Definition 1. Let $(\mathsf{u},\mathsf{v}) \in \mathcal{P}$. A finite or infinite limit (if it exists)

$$\mathfrak{D}^{\mathsf{u}}_{\mathsf{v}} f(x_0) = \lim_{h\to 0} \frac{\Delta^{\mathsf{u}(h)}_{\mathsf{v}(h)} f(x_0)}{\Delta^{\mathsf{u}(h)}_{\mathsf{v}(h)} x} = \lim_{h\to 0} \frac{f\big(x_0 + \mathsf{u}(h)\big) - f\big(x_0 - \mathsf{v}(h)\big)}{\mathsf{u}(h) + \mathsf{v}(h)} \tag{3}$$

is called the (u,v)-derivative of the function f at the point x_0.

The first time the (u,v)-derivative was introduced in [7]. This concept is useful for the tasks of uncovering uncertainties and establishing the fact of singularity and non-differentiability.

The following structures are related to the (u,v)-derivative.

- Wen Chen has defined a *fractal derivative* as the limit $\lim_{t_1\to t} \frac{u(t_1)-u(t)}{t_1^\alpha - t^\alpha}$ in [4, 3]. It is easy to show that $\lim_{t_1\to t} \frac{u(t_1)-u(t)}{t_1^\alpha - t^\alpha} = \frac{t^{1-\alpha}}{\alpha} \mathfrak{D}^{t_1 - t}_0 u(t)$ if $t \neq 0$, and is equal to the fractal velocity [5] if $t = 0$.
- In [2], the *conformable fractional derivate* was defined to be the limit $T_\alpha(f)(t) = \lim_{\varepsilon\to 0} \frac{f\left(t+\varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon}$. It is easy to get that $T_\alpha(f)(t) = t^{1-\alpha} \mathfrak{D}^{ht^{1\alpha}}_0 f(t)$.

Given the design of the analogue of the (u,v)–derivative as the limit $\boxtimes^{\mathsf{u}(h)}_{\mathsf{v}(h)} f(x) = \lim_{h\to 0} \frac{\square^{\mathsf{u}(h)}_{\mathsf{v}(h)} f(x)}{\square^{\mathsf{u}(h)}_{\mathsf{v}(h)} x}$, where $\square^{\mathsf{u}(h)}_{\mathsf{v}(h)} f(x)$ is oscillation of the function f on the segment with the endpoints $x + \mathsf{u}(h)$ and $x - \mathsf{v}(h)$, and a pair of functions $(\mathsf{u},\mathsf{v}) \in \mathcal{P}^+$ ($\mathcal{P}^+$ contains all pairs of $\mathcal{P}$ satisfying the inequality $\mathsf{u}(h)\cdot\mathsf{v}(h) \geq 0$ in certain punctured neighborhood of zero). The usage of the analogue of the (u,v)-derivative allowed to show that there is a model class of functions containing singular functions that have unbounded variation on each segment from the domain of definition.

To simplify the study of compositions of functions, there was introduced a logarithmic (u,v)-derivative.

Let a function f and a pair of functions $(\mathsf{u},\mathsf{v}) \in \mathcal{P}$ be given. Set the number (if it exists) for fixed x_0 from the domain of definition of f,

$$\mathfrak{L}^{\mathsf{u}}_{\mathsf{v}} f(x_0) \equiv \lim_{h\to 0} \frac{\ln|\Delta^{\mathsf{u}}_{\mathsf{v}} f(x_0)|}{\ln|\Delta^{\mathsf{u}}_{\mathsf{v}} x|} = \lim_{h\to 0} \frac{\ln\left|f\big(x_0 + \mathsf{u}(h)\big) - f\big(x_0 - \mathsf{v}(h)\big)\right|}{\ln|\mathsf{u}(h) + \mathsf{v}(h)|}, \tag{4}$$

which will be called the logarithmic (u, v)-derivative of the function f at the point x_0.

In [5] the *fractal velocity* of fractional order $0 \leq \beta \leq 1$ was defined as

$$v^\beta_\pm[f](x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{\pm h^\beta}. \tag{5}$$

Obviously $\mathfrak{L}^{\pm h}_0 f(x)$ is a number such that for all $\beta < \mathfrak{L}^{\pm h}_0 f(x)$ we have $v^\beta_\pm[f](x) = 0$ and that, for all $\beta > \mathfrak{L}^{\pm h}_0 f(x)$, $v^\beta_\pm[f](x) = \infty$.

Extending logarithmic (u,v)-derivative to vector functions we can see that the same relation is obtained with the *fractal gradient* defined in [1] as

$$\bigtriangledown T=\Gamma(1+\alpha)\lim_{x_B-x_a\to L_0}\frac{T_B-T_A}{(x_B-x_a)^{\alpha}}. \tag{6}$$

3. $\Lambda^{\mathsf{u}}_{\mathsf{v}}$ and it properties

Denote by $\mathcal{P}^{\oplus}=\left\{(\mathsf{u},\mathsf{v})\in\mathcal{P}^{+}:\mathsf{u}(h)\geq 0,\forall h\in O\left(\mathsf{u},\mathsf{v}\right)\right\}$.

Let $(\mathsf{u},\mathsf{v})\in\mathcal{P}^{+}$. We will use the following notation:

$$\Lambda^{\mathsf{u}}_{\mathsf{v}}f(x_0)=\lim_{h\to 0}\frac{\ln\Box^{\mathsf{u}}_{\mathsf{v}}f(x_0)}{\ln\Box^{\mathsf{u}}_{\mathsf{v}}x}. \tag{7}$$

A proof of the following three propositions are based on this notation.

Proposition 1. *We have* $\Lambda^{\mathsf{u}}_{\mathsf{v}}f(x_0)\in([0;+\infty)\cup\{-\infty\})$.

Proposition 2. *The following conditions hold true:*

- *if* $f'(x_0)\in\mathbb{R}\backslash\{0\}$ *then* $\Lambda^{\mathsf{u}}_{\mathsf{v}}f(x_0)=1$*;*
- *if* $\Lambda^{\mathsf{u}}_{\mathsf{v}}f(x_0)<1$ *then the function is non-differentiable at* x_0*;*
- *if* $\Lambda^{\mathsf{u}}_{\mathsf{v}}f(x_0)>1$ *then* $\mathfrak{D}^{\mathsf{u}}_{\mathsf{v}}(x_0)=0$.

Proposition 3. *Let for functions* f*,* g *and a pair of functions* $(\mathsf{u},\mathsf{v})\in\mathcal{P}^{+}$ *there exist an infinitesimal sequence* (h_n) *such that* $\mathsf{u}(h_n)\mathsf{v}(h_n)>0\ \forall\in\mathbb{N}$.

The following conditions hold true:

- *If* $\Lambda^{\mathsf{u}}_{\mathsf{v}}f(x_0)>0$ *then the function is continuous at* x_0*;*
- $\Lambda^{\mathsf{u}}_{\mathsf{v}}A=+\infty$ *where* $A\in\mathbb{R}$*;*
- *If the value of* $J(x_0)=\overline{\lim\limits_{h\to 0}}f(x_0+h)-\underline{\lim\limits_{h\to 0}}f(x_0-h)$ *is a non-zero real number then* $\Lambda^{\mathsf{u}}_{\mathsf{v}}f(x_0)=0$*;*
- *If* $J(x_0)=\infty$ *then* $\Lambda^{\mathsf{u}}_{\mathsf{v}}f(x_0)=-\infty$.

Proposition 4. *If a function* f *is continuous at* x_0 *then* $\Lambda^{\mathsf{u}}_{\mathsf{v}}f(x_0)=\Lambda^{\mathsf{u}}_{\mathsf{v}}\left|f\right|(x_0)$.

Proof. If there exists an (u,v)-neighborhood of x_0 (meaning an interval with endpoints at the points $x_0+\mathsf{u}(h)$, $x_0-\mathsf{v}$) such that $f\geq 0$, then from the equality $|f|=f$, we have $\Lambda^{\mathsf{u}}_{\mathsf{v}}f(x_0)=\Lambda^{\mathsf{u}}_{\mathsf{v}}\left|f\right|(x_0)$. On the other side, if $f\leq 0$, then from $\Box^{\mathsf{u}}_{\mathsf{v}}f(x_0)=\Box^{\mathsf{u}}_{\mathsf{v}}(-1\cdot f)(x_0)$ we came to the equality.

Let in some (u,v)-neighborhood of the point x_0 the function f have different signs. Then the following inequality holds: $\frac{1}{2}\Box^{\mathsf{u}}_{\mathsf{v}}f(x_0)\leq\Box^{\mathsf{u}}_{\mathsf{v}}\left|f\right|(x_0)\leq\Box^{\mathsf{u}}_{\mathsf{v}}f(x_0)$. Taking into account the equality $\frac{1}{2}\Box^{\mathsf{u}}_{\mathsf{v}}f(x_0)=\Box^{\mathsf{u}}_{\mathsf{v}}\left(\frac{1}{2}f\right)(x_0)$,

$$\ln\Box^{\mathsf{u}}_{\mathsf{v}}\left(\frac{1}{2}f\right)(x_0)\leq\ln\Box^{\mathsf{u}}_{\mathsf{v}}\left|f\right|(x_0)\leq\ln\Box^{\mathsf{u}}_{\mathsf{v}}f(x_0),$$

$$\frac{\ln\Box^{\mathsf{u}}_{\mathsf{v}}\left(\frac{1}{2}f\right)(x_0)}{\ln\Box^{\mathsf{u}}_{\mathsf{v}}x}\geq\frac{\ln\Box^{\mathsf{u}}_{\mathsf{v}}\left|f\right|(x_0)}{\ln\Box^{\mathsf{u}}_{\mathsf{v}}x}\geq\frac{\ln\Box^{\mathsf{u}}_{\mathsf{v}}f(x_0)}{\ln\Box^{\mathsf{u}}_{\mathsf{v}}x}.$$

To get $\Lambda^{\mathsf{u}}_{\mathsf{v}}f(x_0)=\Lambda^{\mathsf{u}}_{\mathsf{v}}\left|f\right|(x_0)$ we pass to limits in the last inequalities. □

Theorem 1. *Let for* f*,* g *be continuous functions and a pair of functions* $(\mathsf{u},\mathsf{v})\in\mathcal{P}^{+}$ *there exist finite values* $\Lambda^{\mathsf{u}}_{\mathsf{v}}f(x_0)\geq 0$ *and* $\Lambda^{\mathsf{u}}_{\mathsf{v}}g(x_0)\geq 0$*. Then*

(1) $\forall a\in\mathbb{R}\backslash\{0\}$ *and* $b\in\mathbb{R}$ *satisfies the equality* $\Lambda^{\mathsf{u}}_{\mathsf{v}}\left(a\cdot f+b\right)(x_0)=\Lambda^{\mathsf{u}}_{\mathsf{v}}f(x_0)$*;*
(2) *if* $\Lambda^{\mathsf{u}}_{\mathsf{v}}f(x_0)\neq\Lambda^{\mathsf{u}}_{\mathsf{v}}g(x_0)$ *then* $\Lambda^{\mathsf{u}}_{\mathsf{v}}\left(f+g\right)(x_0)=\min\left\{\Lambda^{\mathsf{u}}_{\mathsf{v}}f(x_0),\Lambda^{\mathsf{u}}_{\mathsf{v}}g(x_0)\right\}$*;*
(3) *if* $f(x_0)=0=g(x_0)$ *then* $\Lambda^{\mathsf{u}}_{\mathsf{v}}\big(f\cdot g\big)(x_0)\geq\Lambda^{\mathsf{u}}_{\mathsf{v}}f(x_0)+\Lambda^{\mathsf{u}}_{\mathsf{v}}g(x_0)$*;*
(4) *if* $f(x_0)=0=g(x_0)$ *and* $\lim_{h\to 0}\frac{\ln\sup_{t\in\Theta^{\mathsf{u}}_{\mathsf{v}}(x_0)}|f(t)g(t)|}{\ln\left(\sup_{t\in\Theta^{\mathsf{u}}_{\mathsf{v}}(x_0)}|f(t)|\cdot\sup_{t\in\Theta^{\mathsf{u}}_{\mathsf{v}}(x_0)}|g(t)|\right)}=m$ *exists then the following conditions hold:* $\Lambda^{\mathsf{u}}_{\mathsf{v}}\big(f\cdot g\big)(x_0)=m\left(\Lambda^{\mathsf{u}}_{\mathsf{v}}f(x_0)+\Lambda^{\mathsf{u}}_{\mathsf{v}}g(x_0)\right)$*;*

(5) *if* $\xi > 0$, *then we have* $\Lambda_{\mathsf{v}}^{\mathsf{u}} \left| f \right|^{\xi} (x_0) = \begin{cases} \xi \cdot \Lambda_{\mathsf{v}}^{\mathsf{u}} f(x_0), & \text{if } f(x_0) = 0 \\ \Lambda_{\mathsf{v}}^{\mathsf{u}} f(x_0), & \text{if } f(x_0) \neq 0 \end{cases}.$

Proof. 1. Assume that $\square_{\mathsf{v}}^{\mathsf{u}} (a \cdot f + b)(x_0) = |a| \square_{\mathsf{v}}^{\mathsf{u}} f(x_0)$. Then according to the definition we get $\Lambda_{\mathsf{v}}^{\mathsf{u}} (a \cdot f + b)(x_0) = \Lambda_{\mathsf{v}}^{\mathsf{u}} f(x_0)$.

2. Let $\Lambda_{\mathsf{v}}^{\mathsf{u}} f(x_0) = c$, $\Lambda_{\mathsf{v}}^{\mathsf{u}} g(x_0) = d$, $c < d$. In accordance to $\square_{\mathsf{v}}^{\mathsf{u}} f(x_0) = (\square_{\mathsf{v}}^{\mathsf{u}} x)^{c+\alpha(h)}$, $\square_{\mathsf{v}}^{\mathsf{u}} g(x_0) = (\square_{\mathsf{v}}^{\mathsf{u}} x)^{d+\beta(h)}$, where $\lim_{h\to 0} \alpha(h) = 0 = \lim_{h \to 0} \beta(h)$.

From $\square_{\mathsf{v}}^{\mathsf{u}} f(x_0) = \sup_{t_1, t_2 \in \Theta_{\mathsf{v}(h)}^{\mathsf{u}(h)}(x_0)} (f(t_1) - f(t_2))$ we get that $\square_{\mathsf{v}}^{\mathsf{u}} (f+g)(x_0) \leq \square_{\mathsf{v}}^{\mathsf{u}} f(x_0) + \square_{\mathsf{v}}^{\mathsf{u}} g(x_0)$. So,

$$\ln \square_{\mathsf{v}}^{\mathsf{u}} \Big(f + g \Big)(x_0) \leq \ln \Big(\square_{\mathsf{v}}^{\mathsf{u}} f(x_0) + \square_{\mathsf{v}}^{\mathsf{u}} g(x_0) \Big) \Rightarrow$$

$$\frac{\square_{\mathsf{v}}^{\mathsf{u}} \Big(f + g \Big)(x_0)}{\ln \square_{\mathsf{v}}^{\mathsf{u}} x} \geq \frac{\ln \Big(\square_{\mathsf{v}}^{\mathsf{u}} f(x_0) + \square_{\mathsf{v}}^{\mathsf{u}} g(x_0) \Big)}{\ln \square_{\mathsf{v}}^{\mathsf{u}} x} \Rightarrow$$

$$\Lambda_{\mathsf{v}}^{\mathsf{u}} (f+g)(x_0) \geq \lim_{h \to 0} \frac{\ln (\square_{\mathsf{v}}^{\mathsf{u}} f(x_0) + \square_{\mathsf{v}}^{\mathsf{u}} g(x_0))}{\ln \square_{\mathsf{v}}^{\mathsf{u}} x}$$

$$= \lim_{h \to 0} \left(c + \alpha(h) + \frac{\ln \Big(1 + (\square_{\mathsf{v}}^{\mathsf{u}} x)^{d - c + \beta(h) - \alpha(h)} \Big)}{\ln \square_{\mathsf{v}}^{\mathsf{u}} x} \right) = c = \Lambda_{\mathsf{v}}^{\mathsf{u}} f(x_0). \tag{8}$$

Since the values of the functions f and g at the point x_0 are equal to the zero, we have that

$$\square_{\mathsf{v}}^{\mathsf{u}} |f|(x_0) = \sup_{t \in \Theta_{\mathsf{v}}^{\mathsf{u}}(x_0)} |f(t)| - \inf_{t \in \Theta_{\mathsf{v}}^{\mathsf{u}}(x_0)} |f(t)| = \sup_{t \in \Theta_{\mathsf{v}}^{\mathsf{u}}(x_0)} |f(t)|,$$

$$\square_{\mathsf{v}}^{\mathsf{u}} |g|(x_0) = \sup_{t \in \Theta_{\mathsf{v}}^{\mathsf{u}}(x_0)} |g(t)| - \inf_{t \in \Theta_{\mathsf{v}}^{\mathsf{u}}(x_0)} |g(t)| = \sup_{t \in \Theta_{\mathsf{v}}^{\mathsf{u}}(x_0)} |g(t)|.$$

Whereas $\sup_{t \in \Theta_{\mathsf{v}}^{\mathsf{u}}(x_0)} |f(t) g(t)| \leq \sup_{t \in \Theta_{\mathsf{v}}^{\mathsf{u}}(x_0)} |f(t)| \cdot \sup_{t \in \Theta_{\mathsf{v}}^{\mathsf{u}}(x_0)} |g(t)|$ then

$$\Lambda_{\mathsf{v}}^{\mathsf{u}} |f \cdot g|(x_0) \geq \Lambda_{\mathsf{v}}^{\mathsf{u}} |f(x_0)| + \Lambda_{\mathsf{v}}^{\mathsf{u}} |g(x_0)|. \tag{9}$$

4. If $\lim_{h \to 0} \frac{\ln \sup_{t \in \Theta_{\mathsf{v}}^{\mathsf{u}}(x_0)} |f(t)g(t)|}{\ln \left(\sup_{t \in \Theta_{\mathsf{v}}^{\mathsf{u}}(x_0)} |f(t)| \cdot \sup_{t \in \Theta_{\mathsf{v}}^{\mathsf{u}}(x_0)} |g(t)| \right)} = m$ then

$$\begin{aligned}
&\Lambda_{\mathsf{v}}^{\mathsf{u}} (fg)(x_0) \\
&= \lim_{h \to 0} \left(\frac{\ln \sup_{t \in \Theta_{\mathsf{v}}^{\mathsf{u}}(x_0)} |f(t)g(t)|}{\ln \square_{\mathsf{v}}^{\mathsf{u}} x} \cdot \frac{\ln \left(\sup_{t \in \Theta_{\mathsf{v}}^{\mathsf{u}}(x_0)} |f(t)| \cdot \sup_{t \in \Theta_{\mathsf{v}}^{\mathsf{u}}(x_0)} |g(t)| \right)}{\ln \left(\sup_{t \in \Theta_{\mathsf{v}}^{\mathsf{u}}(x_0)} |f(t)| \cdot \sup_{t \in \Theta_{\mathsf{v}}^{\mathsf{u}}(x_0)} |g(t)| \right)} \right) \\
&= \lim_{h \to 0} \left(\frac{\ln \sup_{t \in \Theta_{\mathsf{v}}^{\mathsf{u}}(x_0)} |f(t)g(t)|}{\ln \left(\sup_{t \in \Theta_{\mathsf{v}}^{\mathsf{u}}(x_0)} |f(t)| \cdot \sup_{t \in \Theta_{\mathsf{v}}^{\mathsf{u}}(x_0)} |g(t)| \right)} \right. \\
&\qquad \left. \times \frac{\ln \left(\sup_{t \in \Theta_{\mathsf{v}}^{\mathsf{u}}(x_0)} |f(t)| \cdot \sup_{t \in \Theta_{\mathsf{v}}^{\mathsf{u}}(x_0)} |g(t)| \right)}{\ln \square_{\mathsf{v}}^{\mathsf{u}} x} \right) \\
&= m \Big(\Lambda_{\mathsf{v}}^{\mathsf{u}} f(x_0) + \Lambda_{\mathsf{v}}^{\mathsf{u}} g(x_0) \Big).
\end{aligned}$$

5.1. Let us consider the case where $f(x_0) = 0$.

From $\sup_{t\in\Theta_{\mathsf{v}}^{\mathsf{u}}(x)} |f(t)|^{\xi} = \Big(\sup_{t\in\Theta_{\mathsf{v}}^{\mathsf{u}}(x_0)} |f(t)|\Big)^{\xi}$ it follows that

$$\Lambda_{\mathsf{v}}^{\mathsf{u}}|f|^{\xi}(x_0) = \xi\Lambda_{\mathsf{v}}^{\mathsf{u}}|f|(x_0) = \xi\Lambda_{\mathsf{v}}^{\mathsf{u}} f(x_0).$$

5.2. Let $f(x_0) = d \neq 0$, and consider the function $\phi(x) = |f(x)| - |f(x_0)|$. We have $\Lambda_{\mathsf{v}}^{\mathsf{u}} f(x_0) = \Lambda_{\mathsf{v}}^{\mathsf{u}}|f|(x_0) = \Lambda_{\mathsf{v}}^{\mathsf{u}}\phi(x_0)$. It is obvious that

$$|f|^{\xi}(x) - 1 = d^{\xi}\left(1 + \frac{\phi(x)}{d}\right)^{\xi} - 1 = \phi(x) d^{\xi-1}\sum_{n=1}^{\infty}\left(a_n\left(\frac{\phi(x)}{d}\right)^{n-1}\right).$$

It is easy to show that $\Lambda_{\mathsf{v}}^{\mathsf{u}}(f\cdot g)(x_0) = \Lambda_{\mathsf{v}}^{\mathsf{u}} f(x_0)$ if exists such (u,v)-neighborhood of x_0 such that $0 < m \le |g(x)| \le M < \infty$. So, $\Lambda_{\mathsf{v}}^{\mathsf{u}}|f|^{\xi}(x_0) = \Lambda_{\mathsf{v}}^{\mathsf{u}}\psi(x_0) = \Lambda_{\mathsf{v}}^{\mathsf{u}} f(x_0)$. □

In the previous theorem, item 2, the condition $\Lambda_{\mathsf{v}}^{\mathsf{u}} f(x_0) \neq \Lambda_{\mathsf{v}}^{\mathsf{u}} g(x_0)$ is sufficient to ensure the equality $\Lambda_{\mathsf{v}}^{\mathsf{u}}(f+g)(x_0) = \min\{\Lambda_{\mathsf{v}}^{\mathsf{u}} f(x_0), \Lambda_{\mathsf{v}}^{\mathsf{u}} g(x_0)\}$.

Let $f(x) = x - s(x)$ and $g(x) = s(x)$ where $\Lambda_{\mathsf{v}}^{\mathsf{u}} s(x_0) = s_0 < 1$. Using the previous propositions we see that $\Lambda_{\mathsf{v}}^{\mathsf{u}} f(x_0) = s_0$, $\Lambda_{\mathsf{v}}^{\mathsf{u}} g(x_0) = s_0$, $\Lambda_{\mathsf{v}}^{\mathsf{u}}(f+g)(x_0) = \Lambda_{\mathsf{v}}^{\mathsf{u}}\Big(x - s(x) + s(x)\Big)(x_0) = \Lambda_{\mathsf{v}}^{\mathsf{u}} x = 1$. So, in this case the equality $\Lambda_{\mathsf{v}}^{\mathsf{u}}(f+g)(x_0) = \min\{\Lambda_{\mathsf{v}}^{\mathsf{u}} f(x_0), \Lambda_{\mathsf{v}}^{\mathsf{u}} g(x_0)\}$ doe not hold.

Let $g(x) = x\Big(1 - D(x)\Big)$, $f(x) = xD(x)$ where $D(x) = \begin{cases} 1, & x \in \mathbb{R}\backslash\mathbb{Q} \\ 0, & x \in \mathbb{Q}\end{cases}$. Then $f(0) = 0$, $g(0) = 0$ and $\Lambda_{\mathsf{v}}^{\mathsf{u}} f(0) = 1$, $\Lambda_{\mathsf{v}}^{\mathsf{u}} g(0) = 1$. The product is $f(x)g(x) = 0$. So, $\Lambda_{\mathsf{v}}^{\mathsf{u}}(f\cdot g)(0) = +\infty \ge \Lambda_{\mathsf{v}}^{\mathsf{u}} f(0) + \Lambda_{\mathsf{v}}^{\mathsf{u}} g(0)$.

4. Properties of $\Lambda f(x_0)$

Denote $\Lambda f(x_0)$ by $\Lambda_0^h f(x_0)$.

Proposition 5. *If $\Lambda f(x_0) > 0$ then the function f is continuous at x_0.*

Proof. From the definition of $\Lambda f(x_0)$ we get $\Box_0^h f(x_0) = |h|^{\Lambda f(x_0)+\alpha(h)}$, $\lim_{h\to 0}\alpha(h) = 0$.

Then $\lim_{h\to 0}\Box_0^h f(x_0) = \lim_{h\to 0}|h|^{\Lambda f(x_0)+\alpha(h)} = 0$. So, f is continuous at x_0. □

Lemma 1. *If $\Lambda f(x_0)$ exists then for all pairs $(\mathsf{u},\mathsf{v}) \in \mathcal{P}^{+}$ we have $\Lambda_{\mathsf{v}}^{\mathsf{u}} f(x_0) = \Lambda f(x_0)$.*

Proof. If $\Lambda f(x_0) = -\infty$ then the function has an infinite "jump" at the point x. So, $\Lambda_{\mathsf{v}}^{\mathsf{u}} f(x_0) = -\infty$.

Next, without losing the generality, we will assume that $(\mathsf{u},\mathsf{v}) \in \mathcal{P}^{\oplus}$.

Let us show that in the case of existence of $\Lambda f(x_0)$, the equality $\Lambda_h^h f(x_0) = \Lambda f(x_0)$ holds. Let $\sigma(h) = \max\left\{\Box_0^h f(x_0), \Box_h^0 f(x_0)\right\}$. Then $\sigma(h) \le \Box_h^h f(x_0) \le 2\sigma(h)$. So,

$$\frac{\ln\sigma(h)}{\ln|2h|} \ge \frac{\Box_h^h f(x_0)}{\ln|2h|} \ge \frac{\ln(2\sigma(h))}{\ln|2h|}.$$

To get $\Lambda f(x_0) = \Lambda_h^h f(x_0)$ we pass to the limit in the last inequalities.

Let $\mu = \mu(h) = \max\{\mathsf{u},\mathsf{v}\}$. From $\Box_0^{\mu} f(x_0) \le \Box_{\mathsf{v}}^{\mathsf{u}} f(x_0) \le \Box_{\mu}^{\mu} f(x_0)$ and $\frac{1}{2} \le \left|\frac{\mu}{\mathsf{u}+\mathsf{v}}\right| \le 1$ we have that

$$\frac{\ln\Box_0^{\mu} f(x_0)\cdot\ln\Box_0^{\mu} x}{\ln\Box_0^{\mu} x\cdot\ln\Box_{\mathsf{v}}^{\mathsf{u}} x} \ge \frac{\ln\Box_{\mathsf{v}}^{\mathsf{u}} f(x_0)}{\ln\Box_{\mathsf{v}}^{\mathsf{u}} x} \ge \frac{\ln\Box_{\mu}^{\mu} f(x_0)\cdot\ln\Box_{\mu}^{\mu} x}{\ln\Box_{\mu}^{\mu} x\cdot\ln\Box_{\mathsf{v}}^{\mathsf{u}} x}.$$

To get $\Lambda_{\mathsf{v}}^{\mathsf{u}} f(x_0) = \Lambda f(x_0)$ we pass to the limit at previous inequalities.

Let $\Lambda f(x) = +\infty$. Accordingly $\Box_0^h f(x) = |h|^{\alpha(h)}$, $\lim_{h\to 0}\alpha(h) = +\infty$ then $\Box_{\mathsf{v}}^{\mathsf{u}} f(x) \le 2\mu^{\min\{\alpha(h),\alpha(-h)\}}$. So,

$$\frac{\ln\Box_{\mathsf{v}}^{\mathsf{u}} f(x)}{\ln\Box_{\mathsf{v}}^{\mathsf{u}} x} \ge \frac{\ln 2}{\ln(\mu+\eta)} + \min\{\alpha(h),\alpha(-h)\}\frac{\ln\mu}{\ln(\mu+\eta)}. \tag{10}$$

To get $\Lambda^{\mathsf{u}}_{\mathsf{v}} f(x) = +\infty$ we pass to the limit in (10). $\square$

Theorem 2. *If $\Lambda f(\tau) \in \mathbb{R}$, $\Lambda g(x_0) \in \mathbb{R}$ exist, where $\tau = g(x_0)$, then*

$$\Lambda\big(f(g)\big)(x_0) = \Lambda f(\tau) \cdot \Lambda g(x_0).$$

Proof. From the definition of $\Lambda f(x_0)$ we get

$$\Lambda\big(f(g)\big)(x_0) = \lim_{h\to 0} \frac{\ln \square_0^h\big(f\,(g)\big)(x_0)}{\ln \square_0^h x} = \lim_{h\to 0} \left(\frac{\ln \square_0^h\big(f\,(g)\big)(x_0)}{\ln \square_0^h g(x_0)} \cdot \frac{\ln \square_0^h g(x_0)}{\ln \square_0^h x} \right).$$

According to the definition let us consider $\frac{\ln \square_0^h\left(f(g)\right)(x_0)}{\ln \square_0^h g(x_0)}$.

If $x \in \Theta_0^h(x_0)$ then $g(x) \in \Big[\inf_{t\in\Theta_0^h(x_0)} g(t); \sup_{t\in\Theta_0^h(x_0)} g(t)\Big]$. Let $u = u(h) = \sup_{t\in\Theta_0^h(x_0)} g(t) - g(x_0)$, $v = v(h) = g(x_0) - \inf_{t\in\Theta_0^h(x_0)} g(t)$. It is easy to see that $(u, v) \in \mathcal{P}^{\oplus}$. If $\tau = g(x_0)$, then $\square_0^h g(x_0) = \square_v^u \tau$, $\square_0^h\big(f\,(g)\big)(x_0) = \square_v^u f(\tau)$. So,

$$\Lambda\big(f(g)\big)(x_0) = \lim_{h\to 0} \left(\frac{\ln \square_v^u f\big(\tau\big)}{\ln \square_v^u \tau} \right) \cdot \lim_{h\to 0} \left(\frac{\ln \square_0^h g(x_0)}{\ln \square_0^h x} \right) = \Lambda f(\tau) \cdot \Lambda g(x_0). \tag{11}$$

$\square$

Proposition 6. *If $\mathfrak{L} f(x_0) \geq 0$ exist, then $\Lambda f(x_0) = \mathfrak{L} f(x_0)$.*

Proof. Let $\mathfrak{L} f(x_0) = c$. Then $\left|\Delta_0^h f(x_0)\right| = |h|^{c+\alpha(h)}$, $\lim_{h\to 0} \alpha(h) = 0$, $f(x_0 + h) = f(x_0) + s(h)\,|h|^{c+\alpha(h)}$ where $s(h) \in \{\pm 1\}$.

The exist $a = a(h)$, $b = b(h)$ such that $f(x_0 + a) = \sup_{t\in\Theta_0^h(x_0)} f(t)$, $f(x_0 + b) = \inf_{t\in\Theta_0^h(x_0)} f(t)$. So,

$$\square_0^h f(x_0) = f(x_0 + a) - f(x_0 + b) = s(a)\,|a|^{c+\alpha(a)} - s(b)\,|b|^{c+\alpha(b)} \leq 2\,|h|^{c+\mu(h)},$$

where $\mu(h) = \min\{\alpha(a), \alpha(b)\}$. Then

$$\ln\left|\Delta_0^h f(x_0)\right| \leq \square_0^h f(x_0) \leq \ln 2 + (c + \mu(h)) \ln |h|.$$

Using the equalities $|\Delta_0^h x| = |h| = \square_0^h x$ we get

$$\frac{\ln\left|\Delta_0^h f(x_0)\right|}{\ln|\Delta_0^h x|} \geq \frac{\ln \square_0^h f(x_0)}{\ln \square_0^h x} \geq \frac{\ln 2}{\ln|h|} + (c + \mu(h)) \frac{\ln|h|}{\ln|h|}.$$

Let us pass to the limit in last inequalities (for $h \to 0$),

$$\mathfrak{L} f(x_0) \geq \Lambda f(x_0) \geq \mathfrak{L} f(x_0) \Rightarrow \Lambda f(x_0) = \mathfrak{L} f(x_0). \tag{12}$$

$\square$

Theorem 3. *Let $(l_n; r_n)$ be a pair of infinitesimal sequences such that $l_n < l_{n+1} < x_0 < r_{n+1} < r_n$ for all $n \in N$ and $\lim_{n\to\infty} \frac{\ln(r_{n+1}-x_0)}{\ln(r_n - x_0)} = 1 = \lim_{n\to\infty} \frac{\ln(x_0 - l_{n+1})}{\ln(x_0 - l_n)}$. For $\Lambda f(x_0)$ to exist it is necessary and sufficient that the limits $\lim_{n\to\infty} \frac{\ln \square_0^{r_n - x_0} f(x_0)}{\ln(r_n - x_0)}$, $\lim_{n\to\infty} \frac{\ln \square^0_{x_0 - l_n} f(x_0)}{\ln(x_0 - l_n)}$ exist and be equal. If they exist, they are equal.*

Proof. If $\Lambda f(x_0)$ exists (finite or infinite) then $\lim_{n\to\infty} \frac{\ln \square_0^{r_n - x_0} f(x_0)}{\ln(r_n - x_0)}$ exists and equals $\Lambda f(x_0)$.

It is easy to observe that for positive $h \leq u_0$ there exists $n = n(h)$ such that $u_{n+1} < h \leq u_n$, where $u_n \equiv r_n - x_0$. Since $\square_0^a x = a$ and $\lim_{h\to 0} \frac{\ln u_{n+1}}{\ln u_n} = 1$, it is easy to show that $\lim_{h\to 0} \frac{\ln u_{n+1}}{\ln h} = 1 = \lim_{h\to 0} \frac{\ln u_n}{\ln h}$.

On the other side, $\square_0^{u_{n+1}} f(x_0) \le \square_0^h f(x_0) \le \square_0^{u_n} f(x_0)$. Then

$$\frac{\ln \square_0^{u_{n+1}} f(x_0)}{\ln \square_0^h x} \ge \frac{\ln \square_0^h f(x_0)}{\ln \square_0^h x} \ge \frac{\ln \square_0^{u_n} f(x_0)}{\ln \square_0^h x}. \tag{13}$$

To get $\lim_{n\to+\infty} \frac{\ln \square_0^{r_n - x_0} f(x_0)}{\ln \square_0^{r_n - x_0} x}$ and $\lim_{h\to 0+} \frac{\ln \square_0^h f(x_0)}{\ln \square_0^h x}$ we pass to the limit in (13). The other case is proved by similarly. □

Next, for simplicity, we will write $\square_{x_0 - l_n}^{r_n - x_0} f(x_0) = \underset{[l_n, r_n]}{\square} f(x_0)$.

Lemma 2. *Let (l_n, r_n) and $\left(\tilde{l}_n, \tilde{r}_n\right)$ be given pairs of infinitesimal sequences such that the following conditions are satisfied:*

(1) $l_n \le x_0 < r_n$, $x_0 < \tilde{l}_n < \tilde{r}_n$;
(2) $\lim_{n\to\infty} l_n = x_0 = \lim_{n\to\infty} r_n$, $\lim_{n\to\infty} \tilde{l}_n = x_0 = \lim_{n\to\infty} \tilde{r}_n$;
(3) $\left[\tilde{l}_n; \tilde{r}_n\right] \subset [x_0; r_n] \subset [l_n; r_n]$, *for all* $n \in \mathbb{N}$;
(4) $\lim_{n\to\infty} \frac{\ln \underset{[l_n;r_n]}{\square} f(x_0)}{\ln \underset{[\tilde{l}_n;\tilde{r}_n]}{\square} f(x_0)} = 1$, $\lim_{n\to\infty} \frac{\ln(r_n - l_n)}{\ln\left(\tilde{r}_n - \tilde{l}_n\right)} = 1$, $\lim_{n\to\infty} \frac{\ln(r_n - l_n)}{\ln(r_{n+1} - l_{n+1})} = 1$.

The value of the right-hand side limit $\Lambda f(x_0)$ and $\lim_{n\to\infty} \frac{\ln \underset{[l_n,r_n]}{\square} f(x_0)}{\ln(r_n - l_n)}$ exist simultaneously.

Proof. It is obvious that

$$\underset{[l_n,r_n]}{\square} f(x_0) \ge \underset{[x_0,r_n]}{\square} f(x_0) \ge \underset{[\tilde{l}_n,\tilde{r}_n]}{\square} f(x_0);\ r_n - l_n \ge r_n - x_0 \ge \tilde{r}_n - \tilde{l}_n. \tag{14}$$

Using the conditions of the theorem we have

$$\lim_{n\to\infty} \frac{\ln \underset{[l_n,r_n]}{\square} f(x_0)}{\ln \underset{[\tilde{l}_n,\tilde{r}_n]}{\square} f(x_0)} = \lim_{n\to\infty} \frac{\ln \underset{[x_0,r_n]}{\square} f(x_0)}{\ln \underset{[l_n,r_n]}{\square} f(x_0)} = \lim_{n\to\infty} \frac{\ln \underset{[x_0,r_n]}{\square} f(x_0)}{\ln \underset{[\tilde{l}_n,\tilde{r}_n]}{\square} f(x_0)} = 1, \tag{15}$$

$$\lim_{n\to\infty} \frac{\ln(r_n - l_n)}{\ln\left(\tilde{r}_n - \tilde{l}_n\right)} = \lim_{n\to\infty} \frac{\ln(r_n - x_0)}{\ln(r_n - l_n)} = \lim_{n\to\infty} \frac{\ln(r_n - x_0)}{\ln\left(\tilde{r}_n - \tilde{l}_n\right)} 1. \tag{16}$$

According to the Theorem 3 we obtain

$$\lim_{h\to 0+} \frac{\ln \square_0^h f(x_0)}{\ln \square_0^h x} = \lim_{n\to\infty} \frac{\ln \underset{[x_0,r_n]}{\square} f(x_0)}{\ln(r_n - x_0)} = \lim_{n\to\infty} \frac{\ln \underset{[l_n,r_n]}{\square} f(x_0)}{\ln(r_n - l_n)}. \tag{17}$$

□

Lemma 3. *Let (l_n, r_n) and $\left(\tilde{l}_n, \tilde{r}_n\right)$ be given pairs of infinitesimal sequences such that the following conditions are satisfied:*

(1) $l_n \le x_0 < r_n$, $\tilde{l}_n < \tilde{r}_n < x_0$;
(2) $\lim_{n\to\infty} l_n = x_0 = \lim_{n\to\infty} r_n$, $\lim_{n\to\infty} \tilde{l}_n = x_0 = \lim_{n\to\infty} \tilde{r}_n$;
(3) $\left[\tilde{l}_n, \tilde{r}_n\right] \subset [l_n; x_0] \subset [l_n, r_n]$ *for all* $n \in \mathbb{N}$;
(4) $\lim_{n\to\infty} \frac{\ln \underset{[l_n,r_n]}{\square} f(x_0)}{\ln \underset{[\tilde{l}_n,\tilde{r}_n]}{\square} f(x_0)} = 1$, $\lim_{n\to\infty} \frac{\ln(r_n - l_n)}{\ln\left(\tilde{r}_n - \tilde{l}_n\right)} = 1$, $\lim_{n\to\infty} \frac{\ln(r_n - l_n)}{\ln(r_{n+1} - l_{n+1})} = 1$.

The value of the left-hand side limit $\Lambda f(x_0)$ and $\lim_{n\to\infty} \frac{\ln \underset{[l_n,r_n]}{\square} f(x_0)}{\ln(r_n - l_n)}$ exist simultaneously.

5. Properties of $\Lambda_h^h f(x_0)$

Lemma 4. *Let there be given a strictly descending infinitesimal sequence of pairs of positive real numbers (τ_0) such that $\lim_{n\to\infty} \frac{\ln \tau_{n+1}}{\ln \tau_n} = 1$.*

In order for the limit $\Lambda_h^h f(x_0)$ to exist, it is necessary and sufficient that the limit $\lim_{n\to\infty} \frac{\ln \square_{\tau_n}^{\tau_n} f(x_0)}{\ln \square_{\tau_n}^{\tau_n} x}$ existed. If they exist, then they are equal.

Proof. From existence of $\Lambda_h^h f(x_0)$ we have that the limit $\lim_{n\to\infty} \frac{\ln \square_{\tau_n}^{\tau_n} f(x_0)}{\ln \square_{\tau_n}^{\tau_n} x}$ also exists.

Let $n = n(h)$ be such that $\tau_{n+1} < h \le \tau_n$. Then we have the following:

$$\square_{\tau_{n+1}}^{\tau_{n+1}} f(x_0) \le \square_h^h f(x_0) \le \square_{\tau_n}^{\tau_n} f(x_0),$$

$$\frac{\ln \square_{\tau_{n+1}}^{\tau_{n+1}} f(x_0) \cdot \ln \square_{\tau_{n+1}}^{\tau_{n+1}} x}{\ln \square_{\tau_{n+1}}^{\tau_{n+1}} x \cdot \ln \square_h^h x} \ge \frac{\ln \square_h^h f(x_0)}{\ln \square_h^h x} \ge \frac{\ln \square_{\tau_n}^{\tau_n} f(x_0) \cdot \ln \square_{\tau_n}^{\tau_n} x}{\ln \square_{\tau_n}^{\tau_n} x \cdot \ln \square_h^h x}, \tag{18}$$

$$\lim_{h\to 0} \frac{\ln \square_{\tau_{n+1}}^{\tau_{n+1}} f(x_0)}{\ln \square_{\tau_{n+1}}^{\tau_{n+1}} x} \ge \Lambda_h^h f(x_0) \ge \lim_{h\to 0} \frac{\ln \square_{\tau_n}^{\tau_n} f(x_0)}{\ln \square_{\tau_n}^{\tau_n} x}.$$

So, if the limit $\lim_{n\to\infty} \frac{\ln \square_{\tau_n}^{\tau_n} f(x_0)}{\ln \square_{\tau_n}^{\tau_n} x}$ exists, then there exists $\Lambda_h^h f(x_0)$, and they are equal. □

Note that from the inequality (18) we have the following:

$$\overline{\lim_{n\to\infty}} \frac{\ln \square_{\tau_n}^{\tau_n} f(x_0)}{\ln \square_{\tau_n}^{\tau_n} x} = \overline{\lim_{h\to 0}} \frac{\ln \square_h^h f(x_0)}{\ln \square_h^h x}, \quad \underline{\lim_{n\to\infty}} \frac{\ln \square_{\tau_n}^{\tau_n} f(x_0)}{\ln \square_{\tau_n}^{\tau_n} x} = \underline{\lim_{h\to 0}} \frac{\ln \square_h^h f(x_0)}{\ln \square_h^h x}. \tag{19}$$

Theorem 4. *Let there be given a sequence of pairs of real numbers (l_n, r_n) such that $\lim_{n\to\infty} l_n = x_0 = \lim_{n\to\infty} r_n$. In addition, we assume that $l_n < l_{n+1} < x_0 < r_{n+1} < r_n$ for all $n \in \mathbb{N}$ and*

$$\lim_{n\to\infty} \frac{\ln \max\{r_n - x_0, x_0 - l_n\}}{\ln \min\{r_n - x_0, x_0 - l_n\}} = 1 = \lim_{n\to\infty} \frac{\ln(r_{n+1} - l_{n+1})}{\ln(r_n - l_n)}. \tag{20}$$

Then $\lim_{n\to\infty} \frac{\ln \underset{[l_n;r_n]}{\square} f(x_0)}{\ln(r_n - l_n)}$ and $\Lambda_h^h f(x_0)$ exist or not simultaneously. If they exist, they are equal.

Proof. Let $u_n = r_n - x_0$, $v_n = x_0 - l_n$, $\mu_n = \min\{u_n, v_n\}$, $\eta_n = \max\{u_n, v_n\}$.

Let us show that under the given conditions, $\lim_{n\to\infty} \frac{\ln \mu_{n+1}}{\ln \mu_n} = 1 = \lim_{n\to\infty} \frac{\ln \eta_{n+1}}{\ln \eta_n}$. Indeed,

$$1 = \lim_{n\to\infty} \frac{\ln(r_{n+1} - l_{n+1})}{\ln(r_n - l_n)} = \lim_{n\to\infty} \frac{\ln \eta_{n+1} + \ln\left(1 + \frac{\mu_{n+1}}{\eta_{n+1}}\right)}{\ln \eta_n + \ln\left(1 + \frac{\mu_n}{\eta_n}\right)} = \lim_{n\to\infty} \frac{\ln \eta_{n+1}}{\ln \eta_n}.$$

From $\lim_{n\to\infty} \frac{\ln \max\{r_n - x_0, x_0 - l_n\}}{\ln \min\{r_n - x_0, x_0 - l_n\}} = \lim_{n\to\infty} \frac{\ln \eta_n}{\ln \mu_n} = 1$ we get $\lim_{n\to\infty} \frac{\ln \mu_{n+1}}{\ln \mu_n} = 1$.

Consider the inequality

$$\square_{\mu_n}^{\mu_n} f(x_0) \le \square_{v_n}^{u_n} f(x_0) \le \square_{\eta_n}^{\eta_n} f(x_0),$$

$$\frac{\ln \square_{\mu_n}^{\mu_n} f(x_0)}{\ln \square_{v_n}^{u_n} x} \ge \frac{\ln \square_{v_n}^{u_n} f(x_0)}{\ln \square_{v_n}^{u_n} x} \ge \frac{\ln \square_{\eta_n}^{\eta_n} f(x_0)}{\ln \square_{v_n}^{u_n} x}. \tag{21}$$

Passing to the limit we get

$$\overline{\lim_{n\to\infty}}\frac{\ln\square^{\mu_n}_{\mu_n}f(x_0)}{\ln\square^{u_n}_{v_n}x}\geq\overline{\lim_{n\to\infty}}\frac{\ln\square^{u_n}_{v_n}f(x_0)}{\ln\square^{u_n}_{v_n}x}\geq\overline{\lim_{n\to\infty}}\frac{\ln\square^{\eta_n}_{\eta_n}f(x_0)}{\ln\square^{u_n}_{v_n}x},$$

$$\underline{\lim_{n\to\infty}}\frac{\ln\square^{\mu_n}_{\mu_n}f(x_0)}{\ln\square^{u_n}_{v_n}x}\geq\underline{\lim_{n\to\infty}}\frac{\ln\square^{u_n}_{v_n}f(x_0)}{\ln\square^{u_n}_{v_n}x}\geq\underline{\lim_{n\to\infty}}\frac{\ln\square^{\eta_n}_{\eta_n}f(x_0)}{\ln\square^{u_n}_{v_n}x}.$$

By Lemma 4 and by the equality (19) we have

$$\overline{\lim_{n\to\infty}}\frac{\ln\square^{\mu_n}_{\mu_n}f(x_0)}{\ln\square^{u_n}_{v_n}x}=\overline{\lim_{h\to 0}}\frac{\ln\square^{h}_{h}f(x_0)}{\ln\square^{h}_{h}x}=\overline{\lim_{n\to\infty}}\frac{\ln\square^{\eta_n}_{\eta_n}f(x_0)}{\ln\square^{\eta_n}_{\eta_n}x},\tag{22}$$

$$\underline{\lim_{n\to\infty}}\frac{\ln\square^{\mu_n}_{\mu_n}f(x_0)}{\ln\square^{u_n}_{v_n}x}=\underline{\lim_{h\to 0}}\frac{\ln\square^{h}_{h}f(x_0)}{\ln\square^{h}_{h}x}=\underline{\lim_{n\to\infty}}\frac{\ln\square^{\eta_n}_{\eta_n}f(x_0)}{\ln\square^{\eta_n}_{\eta_n}x}.\tag{23}$$

In other words, $\overline{\lim_{n\to\infty}}\frac{\ln\square^{h}_{h}f(x_0)}{\ln\square^{h}_{h}x}=\overline{\lim_{n\to\infty}}\frac{\ln\square^{u_n}_{v_n}f(x_0)}{\ln\square^{u_n}_{v_n}x}$ and $\underline{\lim_{n\to\infty}}\frac{\ln\square^{h}_{h}f(x_0)}{\ln\square^{h}_{h}x}=\underline{\lim_{n\to\infty}}\frac{\ln\square^{u_n}_{v_n}f(x_0)}{\ln\square^{u_n}_{v_n}x}$. Therefore, from existence of $\lim_{n\to\infty}\frac{\ln\square^{u_n}_{v_n}f(x_0)}{\ln\square^{u_n}_{v_n}x}$ we obtain the fact of existence of $\Lambda^h_h f(x_0)$. □

Lemma 5. *The identity* $\Lambda^h_h f(x_0)=\lim_{h\to 0}\frac{\ln\max\{\square^h_0 f(x_0);\square^0_h f(x_0)\}}{\ln|h|}$ *holds true.*

Proof. Since

$$\max\{\square^h_0 f(x_0);\square^0_h f(x_0)\}\leq\square^h_h f(x_0)\leq 2\max\{\square^h_0 f(x_0);\square^0_h f(x_0)\},$$

we have

$$\frac{\ln\max\{\square^h_0 f(x_0);\square^0_h f(x_0)\}}{\ln 2+\ln|h|}\geq\frac{\ln\square^h_h f(x_0)}{\ln 2+\ln|h|}\geq\frac{\ln\left(2\max\{\square^h_0 f(x_0);\square^0_h f(x_0)\}\right)}{\ln 2+\ln|h|}.$$

Passing to the limit in the latter inequality, we get the necessary statement. □

5.1. **A study of differential properties of the functions ψ and φ.** This section presents results of a study of the functions ψ and φ, which were specified in the introduction.

The following three statements hold true.

Proposition 7. *For almost all numbers in the segment* $[0;1]$ *the equality* $\Lambda S(x)=-\frac{\ln(q_0(1-q_0))}{2\ln 2}$ *holds.*

To calculate the right-hand side value of $\Lambda S(x_0)$ with $x_0\in E_2$ Lemma 2 can be applied, where $l_n=\Delta^2_{\alpha_1\alpha_2\ldots\alpha_{P_n}(0)}$, $r_n=\Delta^2_{\alpha_1\alpha_2\ldots\alpha_{P_n}(1)}$, $\tilde{l}_n=\Delta^2_{\alpha_1\alpha_2\ldots\alpha_{P_n}1(0)}$, $\tilde{r}_n=r_n$, where P_n is the position number of the first digit of the n-th pair of digits (00) in the binary representation of the number x_0.

To calculate the left-hand side value of $\Lambda f(x_0)$ with $x_0\in E_2$ we can use lemma 3, where $l_n=\Delta^2_{\alpha_1\alpha_2\ldots\alpha_{P_n}(0)}$, $r_n=\Delta^2_{\alpha_1\alpha_2\ldots\alpha_{P_n}(1)}$, $\tilde{l}_n=l_n$, $\tilde{r}_n=\Delta^2_{\alpha_1\alpha_2\ldots\alpha_{P_n}0(1)}$, where P_n is the position number of the first digit of the n-th pair of digits (11) in the binary representation of the number x_0.

Proposition 8. *For all* $x\in[0;1]$ *estimate* $\frac{2g_0-1}{-\ln 3}\geq\Lambda g(x)\geq\frac{\ln g_0}{-\ln 3}$ *holds true.*

To calculate an estimate for the value of $\Lambda g(x_0)$, where $x_0\in E_3$ we use theorem 4, where $l_n=\Delta^3_{\alpha_1\alpha_2\ldots\alpha_{P_n}(0)}$, $r_n=\Delta^3_{\alpha_1\alpha_2\ldots\alpha_{P_n}(2)}$, where P_n is the position number of the n-th digit 1 in the ternary representation of the number x_0.

Taking into account the inequality $\Lambda g(x)\geq\Lambda^h_h g(x)$, the following statement is obvious.

Corollary 1. *If* $\frac{\ln g_0\cdot\ln(q_0(1-q_0))}{2\ln 3\cdot\ln 2}>1$, *then the function* φ *– is singular.*

Corollary 2. *If* $\frac{\ln(2g_0-1)\cdot\ln(q_0(1-q_0))}{2\ln 3\cdot\ln 2} < 1$, *then the function* φ *is non-differentiable almost everywhere.*

Corollary 3. *If the function* φ *is a singular function of unbounded variation, then* $q_0 \in \in \left(0; \frac{3-\sqrt{5}}{6}\right) \bigcup \left(\frac{3+\sqrt{5}}{6}; 1\right)$.

Using geometric probabilities it can be shown that with an arbitrary choice of the parameters p and q from the unit interval with probability $\approx 43.98\%$ we obtain a singular function (at the same time $\approx 2.49\%$ are singular functions of unbounded variation), $\approx 28\%$ are non-differentiable almost everywhere and the same percentage requires for an additional study.

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