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NORM INEQUALITIES FOR ACCRETIVE-DISSIPATIVE BLOCK MATRICES

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Abstract. Let $T = [T_{ij}] \in \mathbb{M}_{mn}(\mathbb{C})$ be accretive-dissipative, where $T_{ij} \in \mathbb{M}_n(\mathbb{C})$ for $i, j = 1, 2, ..., m$. Let f be a function that is convex and increasing on $[0, \infty)$ where $f(0) = 0$. Then

$$\left|\left|\left| f\left(\sum_{i<j} |T_{ij}|^2\right) + f\left(\sum_{i<j} |T_{ji}^*|^2\right) \right|\right|\right| \le \left|\left|\left| f\left(\frac{m^2-m}{2}|T|^2\right) \right|\right|\right|.$$

Also, if f is concave and increasing on $[0, \infty)$ where $f(0) = 0$, then

$$\left|\left|\left| f\left(\sum_{i<j} |T_{ij}|^2\right) + f\left(\sum_{i<j} |T_{ji}^*|^2\right) \right|\right|\right| \le (2m^2-2m) \left|\left|\left| f\left(\frac{|T|^2}{4}\right) \right|\right|\right|.$$

Нехай $T = T_{ij} \in \mathbb{M}_{mn}(\mathbb{C})$, де $T_{ij} \in \mathbb{M}_n(\mathbb{C})$ при $i, j = 1, 2, ..., m.$, – акретивно-дисипативна матриця. Нехай f - опукла функція, яка зростає на $[0, \infty)$, де $f(0) = 0$. Тоді

$$\left|\left|\left| f\left(\sum_{i<j} |T_{ij}|^2\right) + f\left(\sum_{i<j} |T_{ji}^*|^2\right) \right|\right|\right| \le \left|\left|\left| f\left(\frac{m^2-m}{2}|T|^2\right) \right|\right|\right|.$$

Також, якщо f є угнутою, зростає на $[0, \infty)$ і $f(0) = 0$, то

$$\left|\left|\left| f\left(\sum_{i<j} |T_{ij}|^2\right) + f\left(\sum_{i<j} |T_{ji}^*|^2\right) \right|\right|\right| \le (2m^2-2m) \left|\left|\left| f\left(\frac{|T|^2}{4}\right) \right|\right|\right|.$$

1. Introduction

Let $\mathbb{M}_n(\mathbb{C})$ be the algebra of all $n \times n$ complex matrices. A matrix $T \in \mathbb{M}_{mn}(\mathbb{C})$ can be partitioned as an $m \times m$ block matrix ($m \in \{2, 3, 4, ...\}$)

$$T = \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1m} \\ T_{21} & T_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ T_{m1} & T_{m2} & \cdots & T_{mm} \end{bmatrix}, \tag{1.1}$$

where $T_{ij} \in \mathbb{M}_n(\mathbb{C})$ for $i, j = 1, 2, ..., m$.

A matrix $T \in \mathbb{M}_{mn}(\mathbb{C})$ with Cartesian decomposition $T = \mathrm{Re}\,T + i\mathrm{Im}\,T$ is said to be accretive-dissipative if both $\mathrm{Re}\,T$ and $\mathrm{Im}\,T$ are positive semidefinite. We will represent $\mathrm{Re}\,T$ and $\mathrm{Im}\,T$ in our work as

$$\mathrm{Re}\,T = \tilde{A} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{12}^* & A_{22} & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ A_{1m}^* & \cdots & \cdots & A_{mm} \end{bmatrix} \text{ and } \mathrm{Im}\,T = \tilde{B} = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1m} \\ B_{12}^* & B_{22} & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ B_{1m}^* & \cdots & \cdots & B_{mm} \end{bmatrix},$$

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where $A_{ij}, B_{ij} \in \mathbb{M}_n(\mathbb{C})$ for $i, j = 1, 2, ..., m$.

A principal submatrix of a square matrix A is the matrix obtained by deleting any j rows and the corresponding j columns.

On $\mathbb{M}_n(\mathbb{C})$, a norm $|||\cdot|||$ satisfying the invariance property that $|||UAV||| = |||A|||$ for every $A, U, V \in \mathbb{M}_n(\mathbb{C})$ where U, V are unitary is said to be unitarily invariant.

For $A \in \mathbb{M}_n(\mathbb{C})$ and $B \in \mathbb{M}_{mn}(\mathbb{C})$, the inequality $|||A||| \leq |||B|||$ means that

$$|||A \oplus 0 \oplus ... \oplus 0||| \leq |||B|||\,,$$

where the direct sum $A \oplus 0 \oplus ... \oplus 0$ is the matrix in $\mathbb{M}_{mn}(\mathbb{C})$ defined by

$$A \oplus 0 \oplus ... \oplus 0 = \begin{bmatrix} A & 0 & \cdots & 0 \\ 0 & 0 & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix}.$$

The Ky Fan $k-$norms $||\cdot||_{(k)}$ $(k = 1, ..., n)$ are the norms defined on $\mathbb{M}_n(\mathbb{C})$ by $||T||_{(k)} = \sum_{j=1}^{k} s_j(T)$, $k = 1, ..., n$, where $s_1(T) \geq ... \geq s_n(T)$ are the eigen values of the matrix $|T| = (T^*T)^{1/2}$ arranged in decreasing order. The Ky Fan dominance principle asserts that, for every unitarily invariant norm, we have

$$|||A||| \leq |||B||| \;\; \Leftrightarrow \;\; ||A||_{(k)} \leq ||B||_{(k)} \text{ for } k = 1, ..., n. \tag{1.2}$$

Let ζ be the class of all functions f that are increasing and nonnegative on $[0, \infty)$ and satisfies the condition: If $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ are two decreasing sequences of nonnegative real numbers such that $\prod_{j=1}^{k} x_j \leq \prod_{j=1}^{k} y_j$ for $k = 1, 2, ..., n$, then $\prod_{j=1}^{k} f(x_j) \leq \prod_{j=1}^{k} f(y_j)$ for $k = 1, 2, ..., n$.

A nonnegative function f defined on $[0, \infty)$ is called submultiplicative if $f(mn) \leq f(m)f(n)$ whenever $m, n \in [0, \infty)$.

In [6],[12],[15], and [16], a norm inequalities that compare T with its diagonal blocks have been given.

In [8], it has been shown that for an accretive-dissipative 2×2 block matrix $T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \in \mathbb{M}_{2n}(\mathbb{C})$, we have

$$\left|\left|\left|f\left(|T_{12}|^2\right) + f\left(|T_{21}^*|^2\right)\right|\right|\right| \leq \left|\left|\left|f\left(|T|^2\right)\right|\right|\right| \tag{1.3}$$

$$\left|\left|\left|f\left(|T_{12}|^2\right) + f\left(|T_{21}^*|^2\right)\right|\right|\right| \leq 4\left|\left|\left|f\left(\frac{|T|^2}{4}\right)\right|\right|\right| \tag{1.4}$$

$$\left|\left|\left|f\left(|T_{12}|^2\right) + f\left(|T_{21}^*|^2\right)\right|\right|\right| \leq |||f^p\left(2\,|T_{11}|\right)|||^{1/p}\, |||f^q\left(2\,|T_{22}|\right)|||^{1/q} \tag{1.5}$$

and

$$\left|\left|\left|f\left(|T_{12}|^2\right) + f\left(|T_{21}^*|^2\right)\right|\right|\right| \leq 4\, |||f^p\left(|T_{11}|\right)|||^{1/p}\, |||f^q\left(|T_{22}|\right)|||^{1/q}\,, \tag{1.6}$$

where in the inequality (1.3) f is a function convex and increasing on $[0, \infty)$ with $f(0) = 0$, in the inequality (1.4) f is a function concave and increasing on $[0, \infty)$ with $f(0) = 0$, in the inequality (1.5) $f \in \zeta$ is submultiplicative convex function with $f(0) = 0$ and $p, q \in (0, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, and in the inequality (1.6) $f \in \zeta$ is submultiplicative concave function with $f(0) = 0$ and $p, q \in (0, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$.

In this paper, some norm inequalities concerning with accretive-dissipative block matrices in $\mathbb{M}_{mn}(\mathbb{C})$ $(m \in \{2,3,4,...\})$ are given. In Section 2, some unitarily invariant norm inequalities that compare the accretive-dissipative matrix T to its off-diagonal blocks, where T is partitioned as in (1.1) are derived. In Section 3, a unitarily invariant norm inequalities for functions $f \in \zeta$ are presented. In Section 4, some results for a 2×2 accretive-dissipative block matrices are given.

2. Some unitarily invariant norm inequalities

In this section, we give some unitarily invariant norm inequalities that compare the accretive-dissipative matrix T to its off-diagonal blocks, where T is partitioned as in (1.1). To start our work, we will use the following lemma (see [13]).

Lemma 2.1. *Let* $A = \begin{bmatrix} X & B \\ B^* & Y \end{bmatrix} \in \mathbb{M}_{2n}(\mathbb{C})$ *be positive semidefinite. Then*

$$2s_j(B) \leq s_j(A)$$

for $j = 1, 2, ..., n$.

The following lemma can be shown easily depending on the inequality (1.2).

Lemma 2.2. *Let* $X, Y \in \mathbb{M}_n(\mathbb{C})$ *be positive semidefinite, and let* f *be a function that is increasing and nonnegative on* $[0, \infty)$. *If* $s_j(X) \leq s_j(Y)$ *for* $j = 1, 2, ..., n$, *then*

$$|||f(X)||| \leq |||f(Y)||| \, .$$

The following lemma, which is essentially due to Fan and Hoffman [5], can be concluded from Lemma 3.2 in [12] or Proposition III.5.1 in [2, p. 73].

Lemma 2.3. *Let* $T \in \mathbb{M}_n(\mathbb{C})$ *be accretive-dissipative. Then*

$$s_j(\operatorname{Re} T) \leq s_j(T) \text{ and } s_j(\operatorname{Im} T) \leq s_j(T)$$

for $j = 1, 2, ..., n$.

Also, we need the following lemma (see [10, p. 149]) which is essential in our work.

Lemma 2.4. *Let* $X \in \mathbb{M}_n(\mathbb{C})$ *and let* Y *be a principal submatrix. Then*

$$s_j(Y) \leq s_j(X)$$

for $j = 1, 2, ..., n$.

In the following lemma, part (a) is an extension of Theorem 2.3 in [1] for n-tuples of operators (see also [9, Theorem 1]), a stronger version of part (b) of the lemma can be obtained by invoking an argument similar to that used in the proof of Proposition 4.1 in [14]. For various Jensen type matrix inequalities, we refer to [3] and references therein. Part (c) can be found in [11] and we can find part (d) in [4]. Henceforth, we assume that every function is continuous.

Lemma 2.5. *Let* $A_1, ..., A_n \in \mathbb{M}_n(\mathbb{C})$ *be positive and let* $\alpha_1, ..., \alpha_n$ *be positive real numbers such that* $\sum_{j=1}^{n} \alpha_j = 1$. *Then*

(a) $\left|\left|\left| f\left(\sum_{j=1}^{n} \alpha_j A_j\right) \right|\right|\right| \leq \left|\left|\left| \sum_{j=1}^{n} \alpha_j f(A_j) \right|\right|\right|$ *for every function* f *that is convex and nonnegative on* $[0, \infty)$.

(b) $\left|\left|\left| \sum_{j=1}^{n} \alpha_j f(A_j) \right|\right|\right| \leq \left|\left|\left| f\left(\sum_{j=1}^{n} \alpha_j A_j\right) \right|\right|\right|$ *for every function* f *that is concave and nonnegative on* $[0, \infty)$.

(c) $\left|\left|\left|\sum\limits_{j=1}^{n} f(A_j)\right|\right|\right| \leq \left|\left|\left|f\left(\sum\limits_{j=1}^{n} A_j\right)\right|\right|\right|$ *for every function f that is convex and non-negative on $[0,\infty)$ with $f(0)=0$.*

(d) $\left|\left|\left|f\left(\sum\limits_{j=1}^{n} A_j\right)\right|\right|\right| \leq \left|\left|\left|\sum\limits_{j=1}^{n} f(A_j)\right|\right|\right|$ *for every function f that is concave and non-negative on $[0,\infty)$.*

Note that the the Fan (dominance and maximum) principles (see, e.g., [2, pp. 24, 93] or [7, pp. 47, 82]) are essential in the proof of Lemma 2.5.

Our first main result in this section is the following theorem.

Theorem 2.6. *Let $T \in \mathbb{M}_{mn}(\mathbb{C})$ be a partitioned accretive-dissipative matrix as given in (1.1), and let f be a function that is convex and increasing on $[0,\infty)$ where $f(0)=0$. Then*

$$\left|\left|\left|f\left(\sum_{i<j}|T_{ij}|^2\right)+f\left(\sum_{i<j}\left|T_{ji}^*\right|^2\right)\right|\right|\right| \leq \left|\left|\left|f\left(\frac{m^2-m}{2}|T|^2\right)\right|\right|\right|. \tag{2.1}$$

Proof. Let $C_{ij} = \begin{bmatrix} T_{ii} & T_{ij} \\ T_{ji} & T_{jj} \end{bmatrix} \in \mathbb{M}_{2n}(\mathbb{C})$, then C_{ij} is a principal submatrix of T, it follows that C_{ij} is accretive-dissipative with Cartesian decomposition $C_{ij} = \begin{bmatrix} A_{ii} & A_{ij} \\ A_{ij}^* & A_{jj} \end{bmatrix} + i\begin{bmatrix} B_{ii} & B_{ij} \\ B_{ij}^* & B_{jj} \end{bmatrix}$. Using Lemma 2.1 and Lemma 2.2, we get that

$$\left|\left|\left|f\left((2m^2-2m)|A_{ij}|^2\right)\right|\right|\right| \leq \left|\left|\left|f\left(\frac{m^2-m}{2}(\operatorname{Re}C_{ij})^2\right)\right|\right|\right| \tag{2.2}$$

and

$$\left|\left|\left|f\left((2m^2-2m)|B_{ij}|^2\right)\right|\right|\right| \leq \left|\left|\left|f\left(\frac{m^2-m}{2}(\operatorname{Im}C_{ij})^2\right)\right|\right|\right|. \tag{2.3}$$

Also, using Lemmas 2.2 and 2.3, we have

$$\left|\left|\left|f\left(\frac{m^2-m}{2}(\operatorname{Re}C_{ij})^2\right)\right|\right|\right| \leq \left|\left|\left|f\left(\frac{m^2-m}{2}|C_{ij}|^2\right)\right|\right|\right| \tag{2.4}$$

and

$$\left|\left|\left|f\left(\frac{m^2-m}{2}(\operatorname{Im}C_{ij})^2\right)\right|\right|\right| \leq \left|\left|\left|f\left(\frac{m^2-m}{2}|C_{ij}|^2\right)\right|\right|\right|. \tag{2.5}$$

Now,

$$\begin{aligned}
&\left|\left|\left|f\left(\sum_{i<j}|T_{ij}|^2\right)+f\left(\sum_{i<j}\left|T_{ji}^*\right|^2\right)\right|\right|\right| \\
\leq\ & \left|\left|\left|f\left(\sum_{i<j}\left(|T_{ij}|^2+\left|T_{ji}^*\right|^2\right)\right)\right|\right|\right| \qquad \text{(by Lemma 2.5(c))} \\
=\ & \left|\left|\left|f\left(\sum_{i<j}\left(|A_{ij}+iB_{ij}|^2+|A_{ij}-iB_{ij}|^2\right)\right)\right|\right|\right| \\
=\ & \left|\left|\left|f\left(2\sum_{i<j}\left(|A_{ij}|^2+|B_{ij}|^2\right)\right)\right|\right|\right|
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{2}\left|\left|\left| f\left(4\sum_{i<j}\left(|A_{ij}|^2\right)\right) + f\left(4\sum_{i<j}\left(|B_{ij}|^2\right)\right)\right|\right|\right| \qquad \text{(by Lemma 2.5(a))}\\
&\le \frac{1}{2}\left|\left|\left| f\left(4\sum_{i<j}\left(|A_{ij}|^2\right)\right)\right|\right|\right| + \frac{1}{2}\left|\left|\left| f\left(4\sum_{i<j}\left(|B_{ij}|^2\right)\right)\right|\right|\right|\\
&\le \frac{1}{m^2-m}\left(\left|\left|\left|\sum_{i<j} f\left((2m^2-2m)\,|A_{ij}|^2\right)\right|\right|\right| + \left|\left|\left|\sum_{i<j} f\left((2m^2-2m)\,|B_{ij}|^2\right)\right|\right|\right|\right)\\
&\qquad \text{(by Lemma 2.5(a))}\\
&\le \frac{1}{m^2-m}\sum_{i<j}\left(\left|\left|\left| f\left((2m^2-2m)\,|A_{ij}|^2\right)\right|\right|\right| + \left|\left|\left| f\left((2m^2-2m)\,|B_{ij}|^2\right)\right|\right|\right|\right)\\
&\le \frac{1}{m^2-m}\sum_{i<j}\left(\left|\left|\left| f\left(\frac{m^2-m}{2}(\mathrm{Re}\, C_{ij})^2\right)\right|\right|\right| + \left|\left|\left| f\left(\frac{m^2-m}{2}(\mathrm{Im}\, C_{ij})^2\right)\right|\right|\right|\right)\\
&\qquad \text{(by the inequalities (2.2) and (2.3))}\\
&\le \frac{2}{m^2-m}\sum_{i<j}\left|\left|\left| f\left(\frac{m^2-m}{2}\,|C_{ij}|^2\right)\right|\right|\right| \qquad \text{(by the inequalities (2.4) and (2.5)).}
\end{aligned}
$$

So, we have

$$
\left|\left|\left| f\left(\sum_{i<j}|T_{ij}|^2\right) + f\left(\sum_{i<j}|T_{ji}^*|^2\right)\right|\right|\right| \le \frac{2}{m^2-m}\sum_{i<j}\left|\left|\left| f\left(\frac{m^2-m}{2}\,|C_{ij}|^2\right)\right|\right|\right|. \tag{2.6}
$$

Since C_{ij} is a principal submatrix of T, it can be inferred from Lemmas 2.4 and 2.2 that

$$
\left|\left|\left| f\left(\frac{m^2-m}{2}\,|C_{ij}|^2\right)\right|\right|\right| \le \left|\left|\left| f\left(\frac{m^2-m}{2}\,|T|^2\right)\right|\right|\right|. \tag{2.7}
$$

Now, the result follows from the inequalities (2.6) and (2.7). □

Note that the inequality (1.3) follows by taking $m = 2$ in the inequality (2.1). So, the inequality (2.1) gives a generalization to the inequality (1.3).

Applications of Theorem 2.6 will be given in the following corollaries.

Corollary 2.7. *Let $T \in \mathbb{M}_{mn}(\mathbb{C})$ be a partitioned accretive-dissipative matrix as given in (1.1), and let f be a function that is convex and increasing on $[0,\infty)$ where $f(0) = 0$. For all $p \ge 2$, we have*

$$
\left|\left|\left| f\left(\left(\sum_{i<j}|T_{ij}|^2\right)^{p/2}\right) + f\left(\left(\sum_{i<j}|T_{ji}^*|^2\right)^{p/2}\right)\right|\right|\right| \le \left|\left|\left| f\left(\left(\frac{m^2-m}{2}\right)^{p/2}|T|^p\right)\right|\right|\right|. \tag{2.8}
$$

In particular, when $m = 2$, we have

$$
|||f\left(|T_{12}|^p\right) + f\left(|T_{21}^*|^p\right)||| \le |||f\left(|T|^p\right)|||.
$$

Proof. The inequality (2.8) follows by applying the inequality (2.1) to the convex function $f(t^{p/2})$. □

Corollary 2.8. *Let $T \in \mathbb{M}_{mn}(\mathbb{C})$ be a partitioned accretive-dissipative matrix as given in (1.1). Then*

$$\left|\left|\left| e^{\left(\sum_{i<j} |T_{ij}|^2\right)} + e^{\left(\sum_{i<j} \left|T_{ji}^*\right|^2\right)} - 2I_n \right|\right|\right| \leq \left|\left|\left| e^{\left(\frac{m^2-m}{2}|T|^2\right)} - I_{mn} \right|\right|\right|.$$

Proof. The proof follows by applying the inequality (2.1) to the function $f(t) = e^t - 1$ which is a convex function that is increasing on $[0, \infty)$ with $f(0) = 0$. □

Corollary 2.9. *Let $T \in \mathbb{M}_{mn}(\mathbb{C})$ be a partitioned accretive-dissipative matrix as given in (1.1). Then*

$$\left|\left|\left| e^{\left(\sum_{i<j} |T_{ij}|^2\right)} + e^{\left(\sum_{i<j} \left|T_{ji}^*\right|^2\right)} \right|\right|\right| \leq \left|\left|\left| e^{\left(\frac{m^2-m}{2}|T|^2\right)} + I_{mn} \right|\right|\right|. \tag{2.9}$$

Proof. Applying Corollary 2.8 to the Ky Fan $k-$norms, we have

$$\left\|\left(e^{\left(\sum_{i<j} |T_{ij}|^2\right)} + e^{\left(\sum_{i<j} \left|T_{ji}^*\right|^2\right)} - 2I_n \right) \oplus 0 \oplus ... \oplus 0\right\|_{(k)} \leq \left\| e^{\left(\frac{m^2-m}{2}|T|^2\right)} - I_{mn} \right\|_{(k)}$$

for $k = 1, ..., mn$. Thus, for $k = 1, ..., n$, we have

$$\begin{aligned}
&\left\|\left(e^{\left(\sum_{i<j} |T_{ij}|^2\right)} + e^{\left(\sum_{i<j} \left|T_{ji}^*\right|^2\right)} \right) \oplus 0 \oplus ... \oplus 0\right\|_{(k)} -2k \\
&= \left\|\left(e^{\left(\sum_{i<j} |T_{ij}|^2\right)} + e^{\left(\sum_{i<j} \left|T_{ji}^*\right|^2\right)} - 2I_n \right) \oplus 0 \oplus ... \oplus 0\right\|_{(k)} \\
&\leq \left\| e^{\left(\frac{m^2-m}{2}|T|^2\right)} - I_{mn} \right\|_{(k)} \\
&= \left\| e^{\left(\frac{m^2-m}{2}|T|^2\right)} \right\|_{(k)} - k
\end{aligned} \tag{2.10}$$

and for $k = n+1, ..., mn$, we have

$$\begin{aligned}
&\left\|\left(e^{\left(\sum_{i<j} |T_{ij}|^2\right)} + e^{\left(\sum_{i<j} \left|T_{ji}^*\right|^2\right)} \right) \oplus 0 \oplus ... \oplus 0\right\|_{(k)} -2k \\
&\leq \left\|\left(e^{\left(\sum_{i<j} |T_{ij}|^2\right)} + e^{\left(\sum_{i<j} \left|T_{ji}^*\right|^2\right)} \right) \oplus 0 \oplus ... \oplus 0\right\|_{(k)} - 2n \\
&= \left\|\left(e^{\left(\sum_{i<j} |T_{ij}|^2\right)} + e^{\left(\sum_{i<j} \left|T_{ji}^*\right|^2\right)} - 2I_n \right) \oplus 0 \oplus ... \oplus 0\right\|_{(k)} \\
&\leq \left\| e^{\left(\frac{m^2-m}{2}|T|^2\right)} - I_{mn} \right\|_{(k)} \\
&= \left\| e^{\left(\frac{m^2-m}{2}|T|^2\right)} \right\|_{(k)} - k.
\end{aligned} \tag{2.11}$$

From the inequalities (2.10) and (2.11), we have

$$\begin{aligned}
&\left\|\left(e^{\left(\sum\limits_{i<j}|T_{ij}|^2\right)}+e^{\left(\sum\limits_{i<j}|T_{ji}^*|^2\right)}\right)\oplus 0\oplus ...\oplus 0\right\|_{(k)}\\
&\qquad\le \left\|e^{\left(\frac{m^2-m}{2}|T|^2\right)}\right\|_{(k)}+k\\
&\qquad= \left\|e^{\left(\frac{m^2-m}{2}|T|^2\right)}+I_{mn}\right\|_{(k)} \qquad (2.12)
\end{aligned}$$

for $k=1,...,mn$. Now, the inequality (2.9) follows from the inequality (2.12) and the Ky Fan dominance principle. □

Our second main result in this section can be stated as follows.

Theorem 2.10. *Let $T\in\mathbb{M}_{mn}(\mathbb{C})$ be a partitioned accretive-dissipative matrix as given in (1.1), and let f be a function that is concave and increasing on $[0,\infty)$ where $f(0)=0$. Then*

$$\left|\left|\left|f\left(\sum_{i<j}|T_{ij}|^2\right)+f\left(\sum_{i<j}|T_{ji}^*|^2\right)\right|\right|\right|\le(2m^2-2m)\left|\left|\left|f\left(\frac{|T|^2}{4}\right)\right|\right|\right|. \qquad (2.13)$$

Proof. Let $C_{ij}=\begin{bmatrix}T_{ii} & T_{ij}\\ T_{ji} & T_{jj}\end{bmatrix}\in\mathbb{M}_{2n}(\mathbb{C})$, then C_{ij} is a principal submatrix of T, it follows that C_{ij} is accretive-dissipative with Cartesian decomposition $C_{ij}=\begin{bmatrix}A_{ii} & A_{ij}\\ A_{ij}^* & A_{jj}\end{bmatrix}+i\begin{bmatrix}B_{ii} & B_{ij}\\ B_{ij}^* & B_{jj}\end{bmatrix}$. Using Lemma 2.1 and Lemma 2.2, we get that

$$\left|\left|\left|f\left(|A_{ij}|^2\right)\right|\right|\right|\le\left|\left|\left|f\left(\frac{(\mathrm{Re}\,C_{ij})^2}{4}\right)\right|\right|\right| \qquad (2.14)$$

and

$$\left|\left|\left|f\left(|B_{ij}|^2\right)\right|\right|\right|\le\left|\left|\left|f\left(\frac{(\mathrm{Im}\,C_{ij})^2}{4}\right)\right|\right|\right|. \qquad (2.15)$$

And by Lemma 2.2 and Lemma 2.3, we have

$$\left|\left|\left|f\left(\frac{(\mathrm{Re}\,C_{ij})^2}{4}\right)\right|\right|\right|\le\left|\left|\left|f\left(\frac{|C_{ij}|^2}{4}\right)\right|\right|\right| \qquad (2.16)$$

and

$$\left|\left|\left|f\left(\frac{(\mathrm{Im}\,C_{ij})^2}{4}\right)\right|\right|\right|\le\left|\left|\left|f\left(\frac{|C_{ij}|^2}{4}\right)\right|\right|\right|. \qquad (2.17)$$

Now,

$$\begin{aligned}
&\left|\left|\left|f\left(\sum_{i<j}|T_{ij}|^2\right)+f\left(\sum_{i<j}|T_{ji}^*|^2\right)\right|\right|\right|\\
&= \left|\left|\left|f\left(\sum_{i<j}|A_{ij}+iB_{ij}|^2\right)+f\left(\sum_{i<j}|A_{ij}-iB_{ij}|^2\right)\right|\right|\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq 2\left|\left|\left| f\left(\frac{\sum_{i<j}\left(|A_{ij}+iB_{ij}|^2+|A_{ij}-iB_{ij}|^2\right)}{2}\right)\right|\right|\right| \qquad \text{(by Lemma 2.5(b))}\\
&= 2\left|\left|\left| f\left(\sum_{i<j}\left(|A_{ij}|^2+|B_{ij}|^2\right)\right)\right|\right|\right|\\
&\leq 2\left|\left|\left| \sum_{i<j} f\left(|A_{ij}|^2+|B_{ij}|^2\right)\right|\right|\right| \qquad \text{(by Lemma 2.5(d))}\\
&\leq 2\sum_{i<j}\left|\left|\left| f\left(|A_{ij}|^2+|B_{ij}|^2\right)\right|\right|\right|\\
&\leq 2\sum_{i<j}\left|\left|\left| f\left(|A_{ij}|^2\right)+f\left(|B_{ij}|^2\right)\right|\right|\right| \qquad \text{(by Lemma 2.5(d))}\\
&\leq 2\sum_{i<j}\left|\left|\left| f\left(|A_{ij}|^2\right)\right|\right|\right| + 2\sum_{i<j}\left|\left|\left| f\left(|B_{ij}|^2\right)\right|\right|\right|\\
&\leq 2\sum_{i<j}\left|\left|\left| f\left(\frac{(\operatorname{Re} C_{ij})^2}{4}\right)\right|\right|\right| + 2\sum_{i<j}\left|\left|\left| f\left(\frac{(\operatorname{Im} C_{ij})^2}{4}\right)\right|\right|\right|\\
&\qquad\qquad \text{(by the inequalities (2.14) and (2.15))}\\
&\leq 4\sum_{i<j}\left|\left|\left| f\left(\frac{|C_{ij}|^2}{4}\right)\right|\right|\right| \qquad \text{(by the inequalities (2.16) and (2.17)).}
\end{aligned}
$$

So, we have

$$
\left|\left|\left| f\left(\sum_{i<j}|T_{ij}|^2\right)+f\left(\sum_{i<j}|T_{ji}^*|^2\right)\right|\right|\right| \leq 4\sum_{i<j}\left|\left|\left| f\left(\frac{|C_{ij}|^2}{4}\right)\right|\right|\right|. \tag{2.18}
$$

Since C_{ij} is a principal submatrix of T, it can be inferred from Lemmas 2.4 and 2.2 that

$$
\left|\left|\left| f\left(\frac{|C_{ij}|^2}{4}\right)\right|\right|\right| \leq \left|\left|\left| f\left(\frac{|T|^2}{4}\right)\right|\right|\right|. \tag{2.19}
$$

Now, the result follows from the inequalities (2.18) and (2.19). □

Note that the inequality (1.4) follows by taking $m = 2$ in the inequality (2.13). So, the inequality (2.13) gives a generalization to the inequality (1.4).

Corollary 2.11. *Let $T \in \mathbb{M}_{mn}(\mathbb{C})$ be a partitioned accretive-dissipative matrix as given in (1.1) and let f be a function that is concave and increasing on $[0,\infty)$ where $f(0) = 0$. For all $0 < p \leq 2$, we have*

$$
\left|\left|\left| f\left(\left(\sum_{i<j}|T_{ij}|^2\right)^{p/2}\right)+f\left(\left(\sum_{i<j}|T_{ji}^*|^2\right)^{p/2}\right)\right|\right|\right| \leq (2m^2-2m)\left|\left|\left| f\left(\frac{|T|^p}{2^p}\right)\right|\right|\right|. \tag{2.20}
$$

In particular, when $m = 2$, we have

$$
|||f\left(|T_{12}|^p\right)+f\left(|T_{21}^*|^p\right)||| \leq 4\left|\left|\left| f\left(\frac{|T|^p}{2^p}\right)\right|\right|\right|.
$$

Proof. The inequality (2.20) follows by applying the inequality (2.13) to the concave function $f(t^{p/2})$. □

Corollary 2.12. *Let $T \in \mathbb{M}_{mn}(\mathbb{C})$ be a partitioned accretive-dissipative matrix as given in (1.1). Then*

$$\left|\left|\left|\log\left(\left(\sum_{i<j}|T_{ij}|^2\right)^{1/2}+I_n\right)+\log\left(\left(\sum_{i<j}|T_{ji}^*|^2\right)^{1/2}+I_n\right)\right|\right|\right|$$

$$\leq (2m^2-2m)\,|||\log\left(|T|+2I_{mn}\right)-(\log 2)\,I_{mn}|||\,.$$

Proof. The proof follows by taking $p=1$ and applying the inequality (2.20) to the function $f(t)=\log(t+1)$ which is a concave and increasing function on $[0,\infty)$ and satisfies that $f(0)=0$. □

The following corollary can be obtained by applying Theorems 2.6 and 2.10 to the function $f(t)=t^{p/2}$.

Corollary 2.13. *Let $T \in \mathbb{M}_{mn}(\mathbb{C})$ be a partitioned accretive-dissipative matrix as given in (1.1). Then*

$$\left|\left|\left|\left(\sum_{i<j}|T_{ij}|^2\right)^{p/2}+\left(\sum_{i<j}|T_{ji}^*|^2\right)^{p/2}\right|\right|\right| \leq \left(\frac{m^2-m}{2}\right)^{p/2}|||\,|T|^p\,||| \quad \text{for all } p\geq 2$$

and

$$\left|\left|\left|\left(\sum_{i<j}|T_{ij}|^2\right)^{p/2}+\left(\sum_{i<j}|T_{ji}^*|^2\right)^{p/2}\right|\right|\right| \leq \frac{m^2-m}{2^{p-1}}\,|||\,|T|^p\,||| \quad \text{for all } 0<p\leq 2.$$

3. Unitarily invariant norm inequalities involving a special class of functions

In this section, we give unitarily invariant norm inequalities including functions belongs to the class ζ.

We start this section with the following lemma (see [8]).

Lemma 3.1. *Let $A=\begin{bmatrix} X & B \\ B^* & Y\end{bmatrix} \in \mathbb{M}_{2n}(\mathbb{C})$ be positive semidefinite, and let $f\in\zeta$ be submultiplicative function. If p and q are positive real numbers with $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\left|\left|\left|f(|B|^2)\right|\right|\right| \leq |||f^p(X)|||^{1/p}\,|||f^q(Y)|||^{1/q}\,. \tag{3.1}$$

Our first result in this section is the following theorem.

Theorem 3.2. *Let $T \in \mathbb{M}_{mn}(\mathbb{C})$ be a partitioned accretive-dissipative matrix as given in (1.1), and let $f\in\zeta$ be a function that is convex and submultiplicative and satisfies that $f(0)=0$. If $p,q>0$ satisfying $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\left|\left|\left|f\left(\sum_{i<j}|T_{ij}|^2\right)+f\left(\sum_{i<j}|T_{ji}^*|^2\right)\right|\right|\right|$$

$$\leq \frac{2}{m^2-m}\sum_{i<j}\left(\left|\left|\left|f^p\left(\sqrt{2m^2-2m}\,|T_{ii}|\right)\right|\right|\right|^{1/p}\left|\left|\left|f^q\left(\sqrt{2m^2-2m}\,|T_{jj}|\right)\right|\right|\right|^{1/q}\right). \tag{3.2}$$

Proof. Since $\begin{bmatrix} A_{ii} & A_{ij} \\ A_{ij}^* & A_{jj} \end{bmatrix}$ and $\begin{bmatrix} B_{ii} & B_{ij} \\ B_{ij}^* & B_{jj} \end{bmatrix}$ are positive semidefinite matrices, and by Lemma 3.1, we have

$$\left|\left|\left| f\left((2m^2-2m)\left|A_{ij}\right|^2\right)\right|\right|\right| \leq \left|\left|\left| f^p\left(\sqrt{2m^2-2m}A_{ii}\right)\right|\right|\right|^{1/p} \left|\left|\left| f^q\left(\sqrt{2m^2-2m}A_{jj}\right)\right|\right|\right|^{1/q} \tag{3.3}$$

and

$$\left|\left|\left| f\left((2m^2-2m)\left|B_{ij}\right|^2\right)\right|\right|\right| \leq \left|\left|\left| f^p\left(\sqrt{2m^2-2m}B_{ii}\right)\right|\right|\right|^{1/p} \left|\left|\left| f^q\left(\sqrt{2m^2-2m}B_{jj}\right)\right|\right|\right|^{1/q} \tag{3.4}$$

Now,

$$\begin{aligned}
&\left|\left|\left| f\left(\sum_{i<j}\left|T_{ij}\right|^2\right) + f\left(\sum_{i<j}\left|T_{ji}^*\right|^2\right)\right|\right|\right| \\
&\leq \left|\left|\left| f\left(\sum_{i<j}\left(\left|T_{ij}\right|^2 + \left|T_{ji}^*\right|^2\right)\right)\right|\right|\right| \qquad \text{(by Lemma 2.5(c))} \\
&= \left|\left|\left| f\left(\sum_{i<j}\left(\left|A_{ij}+iB_{ij}\right|^2 + \left|A_{ij}-iB_{ij}\right|^2\right)\right)\right|\right|\right| \\
&= \left|\left|\left| f\left(2\sum_{i<j}\left(\left|A_{ij}\right|^2 + \left|B_{ij}\right|^2\right)\right)\right|\right|\right| \\
&\leq \frac{1}{2}\left|\left|\left| f\left(4\sum_{i<j}\left(\left|A_{ij}\right|^2\right)\right) + f\left(4\sum_{i<j}\left(\left|B_{ij}\right|^2\right)\right)\right|\right|\right| \qquad \text{(by Lemma 2.5(a))} \\
&\leq \frac{1}{2}\left|\left|\left| f\left(4\sum_{i<j}\left(\left|A_{ij}\right|^2\right)\right)\right|\right|\right| + \frac{1}{2}\left|\left|\left| f\left(4\sum_{i<j}\left(\left|B_{ij}\right|^2\right)\right)\right|\right|\right| \\
&\leq \frac{1}{m^2-m}\left(\left|\left|\left|\sum_{i<j} f\left((2m^2-2m)\left|A_{ij}\right|^2\right)\right|\right|\right| + \left|\left|\left|\sum_{i<j} f\left((2m^2-2m)\left|B_{ij}\right|^2\right)\right|\right|\right|\right) \\
&\qquad \text{(by Lemma 2.5(a))} \\
&\leq \frac{1}{m^2-m}\sum_{i<j}\left(\left|\left|\left| f\left((2m^2-2m)\left|A_{ij}\right|^2\right)\right|\right|\right| + \left|\left|\left| f\left((2m^2-2m)\left|B_{ij}\right|^2\right)\right|\right|\right|\right) \\
&\leq \frac{1}{m^2-m}\sum_{i<j}\left(\begin{array}{c} \left|\left|\left| f^p\left(\sqrt{2m^2-2m}A_{ii}\right)\right|\right|\right|^{1/p}\left|\left|\left| f^q\left(\sqrt{2m^2-2m}A_{jj}\right)\right|\right|\right|^{1/q} + \\ \left|\left|\left| f^p\left(\sqrt{2m^2-2m}B_{ii}\right)\right|\right|\right|^{1/p}\left|\left|\left| f^q\left(\sqrt{2m^2-2m}B_{jj}\right)\right|\right|\right|^{1/q} \end{array}\right) \\
&\qquad \text{(by the inequalities (3.3) and (3.4))} \\
&\leq \frac{1}{m^2-m}\sum_{i<j}\left(\begin{array}{c} \left(\left|\left|\left| f^p\left(\sqrt{2m^2-2m}A_{ii}\right)\right|\right|\right| + \left|\left|\left| f^p\left(\sqrt{2m^2-2m}B_{ii}\right)\right|\right|\right|\right)^{1/p} \times \\ \left(\left|\left|\left| f^q\left(\sqrt{2m^2-2m}A_{jj}\right)\right|\right|\right| + \left|\left|\left| f^q\left(\sqrt{2m^2-2m}B_{jj}\right)\right|\right|\right|\right)^{1/q} \end{array}\right) \\
&\qquad \text{(by Hölder's inequality)}
\end{aligned}$$

$$= \frac{1}{m^2-m}\sum_{i<j}\left(\begin{array}{c}\left(\begin{array}{c}\left|\left|\left|f^p\left(\sqrt{2m^2-2m}\mathrm{Re}\,T_{ii}\right)\right|\right|\right|+\\ \left|\left|\left|f^p\left(\sqrt{2m^2-2m}\mathrm{Im}\,T_{ii}\right)\right|\right|\right|\end{array}\right)^{1/p}\times\\ \left(\begin{array}{c}\left|\left|\left|f^q\left(\sqrt{2m^2-2m}\mathrm{Re}\,Tjj\right)\right|\right|\right|+\\ \left|\left|\left|f^q\left(\sqrt{2m^2-2m}\mathrm{Im}\,Tjj\right)\right|\right|\right|\end{array}\right)^{1/q}\end{array}\right). \quad (3.5)$$

Since the matrices T_{ii} are accretive-dissipative for $i=1,..,m$, it follows from Lemmas 2.2 and 2.3 that

$$\left|\left|\left|f^p\left(\sqrt{2m^2-2m}\mathrm{Re}\,T_{ii}\right)\right|\right|\right| \leq \left|\left|\left|f^p\left(\sqrt{2m^2-2m}\,|T_{ii}|\right)\right|\right|\right| \quad (3.6)$$

$$\left|\left|\left|f^p\left(\sqrt{2m^2-2m}\mathrm{Im}\,T_{ii}\right)\right|\right|\right| \leq \left|\left|\left|f^p\left(\sqrt{2m^2-2m}\,|T_{ii}|\right)\right|\right|\right| \quad (3.7)$$

$$\left|\left|\left|f^q\left(\sqrt{2m^2-2m}\mathrm{Re}\,T_{ii}\right)\right|\right|\right| \leq \left|\left|\left|f^q\left(\sqrt{2m^2-2m}\,|T_{ii}|\right)\right|\right|\right| \quad (3.8)$$

and

$$\left|\left|\left|f^q\left(\sqrt{2m^2-2m}\mathrm{Im}\,T_{ii}\right)\right|\right|\right| \leq \left|\left|\left|f^q\left(\sqrt{2m^2-2m}\,|T_{ii}|\right)\right|\right|\right| \quad (3.9)$$

Now, the result follows from the inequalities (3.5)-(3.9). □

Note that the inequality (1.5) follows by taking $m=2$ in the inequality (3.2). So, the inequality (3.2) gives a generalization to the inequality (1.5).

Theorem 3.3. *Let $T\in\mathbb{M}_{mn}(\mathbb{C})$ be a partitioned accretive-dissipative matrix as given in (1.1), and let $f\in\zeta$ be a function that is concave and submultiplicative and satisfying that $f(0)=0$. If $p,q>0$ satisfying $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\left|\left|\left|f\left(\sum_{i<j}|T_{ij}|^2\right)+f\left(\sum_{i<j}\left|T_{ji}^*\right|^2\right)\right|\right|\right| \leq 4\sum_{i<j}\left(|||f^p\left(|T_{ii}|\right)|||^{1/p}\,|||f^q\left(|T_{jj}|\right)|||^{1/q}\right). \quad (3.10)$$

Proof.

$$\begin{aligned}
&\left|\left|\left|f\left(\sum_{i<j}|T_{ij}|^2\right)+f\left(\sum_{i<j}\left|T_{ji}^*\right|^2\right)\right|\right|\right| \\
&= \left|\left|\left|f\left(\sum_{i<j}|A_{ij}+iB_{ij}|^2\right)+f\left(\sum_{i<j}|A_{ij}-iB_{ij}|^2\right)\right|\right|\right| \\
&\leq 2\left|\left|\left|f\left(\frac{\sum_{i<j}\left(|A_{ij}+iB_{ij}|^2+|A_{ij}-iB_{ij}|^2\right)}{2}\right)\right|\right|\right| \quad \text{(by Lemma 2.5(b))} \\
&= 2\left|\left|\left|f\left(\sum_{i<j}\left(|A_{ij}|^2+|B_{ij}|^2\right)\right)\right|\right|\right| \\
&\leq 2\left|\left|\left|\sum_{i<j}f\left(|A_{ij}|^2+|B_{ij}|^2\right)\right|\right|\right| \quad \text{(by Lemma 2.5(d))} \\
&\leq 2\sum_{i<j}\left|\left|\left|f\left(|A_{ij}|^2+|B_{ij}|^2\right)\right|\right|\right| \\
&\leq 2\sum_{i<j}\left|\left|\left|f\left(|A_{ij}|^2\right)+f\left(|B_{ij}|^2\right)\right|\right|\right| \quad \text{(by Lemma 2.5(d))}
\end{aligned}$$

$$\begin{aligned}
&\le 2\sum_{i<j}\left(\left|\left|\left|f\left(|A_{ij}|^2\right)\right|\right|\right|+\left|\left|\left|f\left(|B_{ij}|^2\right)\right|\right|\right|\right)\\
&\le 2\sum_{i<j}\left(|||f^p(A_{ii})|||^{1/p}\,|||f^q(A_{jj})|||^{1/q}+|||f^p(B_{ii})|||^{1/p}\,|||f^q(B_{jj})|||^{1/q}\right)\\
&\qquad\text{(by Lemma 3.1)}\\
&\le 2\sum_{i<j}\left((|||f^p(A_{ii})|||+|||f^p(B_{ii})|||)^{1/p}(|||f^q(A_{jj})|||+|||f^q(B_{jj})|||)^{1/q}\right)\\
&\qquad\text{(by Hölder's inequality)}\\
&\le 2\sum_{i\neq j}\left((|||f^p(|T_{ii}|)|||+|||f^p(|T_{ii}|)|||)^{1/p}(|||f^q(|T_{jj}|)|||+|||f^q(|T_{jj}|)|||)^{1/q}\right)\\
&\qquad\text{(by Lemmas 2.2 and 2.3)}\\
&\le 4\sum_{i<j}\left(|||f^p(|T_{ii}|)|||^{1/p}\,|||f^q(|T_{jj}|)|||^{1/q}\right).
\end{aligned}$$

□

Note that the inequality (1.6) follows by taking $m = 2$ in the inequality (3.10). So, the inequality (3.10) gives a generalization to the inequality (1.6).

4. Some results for 2×2 block matrices

In this section, our results consider the case when T partitioned as in (1.1) with $m = 2$. Our first result in this section is the following theorem.

Theorem 4.1. *Let $T \in \mathbb{M}_{2n}(\mathbb{C})$ be a partitioned accretive-dissipative matrix as given in (1.1), and let f be a function that is increasing on $[0,\infty)$ with $f(0) = 0$ such that $f(\sqrt{t})$ is convex Then*

$$|||f(|T_{12}|+|T_{21}^*|)+f(||T_{12}|-|T_{21}^*||)||| \le \left|\left|\left|f\left(\sqrt{2}\,|T|\right)\right|\right|\right|. \tag{4.1}$$

Proof. Let $g(t) = f(\sqrt{t}), t \in [0,\infty)$. Then g is an increasing convex function on $[0,\infty)$.

Since $T = A+iB$ with $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{bmatrix}$ and $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{bmatrix}$ are positive semidefinite, it follows from Lemma 2.1 and Lemma 2.2 that

$$\left|\left|\left|g\left(8\,|A_{12}|^2\right)\right|\right|\right| \le \left|\left|\left|g\left(2\,(\mathrm{Re}\,T)^2\right)\right|\right|\right| \ \text{ and } \ \left|\left|\left|g\left(8\,|B_{12}|^2\right)\right|\right|\right| \le \left|\left|\left|g\left(2\,(\mathrm{Im}\,T)^2\right)\right|\right|\right|. \tag{4.2}$$

Also, using Lemmas 2.2 and 2.3, we have

$$\left|\left|\left|g\left(2\,(\mathrm{Re}\,T)^2\right)\right|\right|\right| \le \left|\left|\left|g\left(2\,|T|^2\right)\right|\right|\right| \ \text{ and } \ \left|\left|\left|g\left(2\,(\mathrm{Im}\,T)^2\right)\right|\right|\right| \le \left|\left|\left|g\left(2\,|T|^2\right)\right|\right|\right|. \tag{4.3}$$

Now,

$$\begin{aligned}
&|||f(|T_{12}|+|T_{21}^*|)+f(||T_{12}|-|T_{21}^*||)|||\\
&\qquad= \left|\left|\left|g\left((|T_{12}|+|T_{21}^*|)^2\right)+g\left(||T_{12}|-|T_{21}^*||^2\right)\right|\right|\right|\\
&\qquad\le \left|\left|\left|g\left((|T_{12}|+|T_{21}^*|)^2+||T_{12}|-|T_{21}^*||^2\right)\right|\right|\right| \qquad \text{(by Lemma 2.5(c))}\\
&\qquad= \left|\left|\left|g\left(2\,|T_{12}|^2+2\,|T_{21}^*|^2\right)\right|\right|\right|\\
&\qquad= \left|\left|\left|g\left(4\,|A_{12}|^2+4\,|B_{12}|^2\right)\right|\right|\right|\\
&\qquad\le \frac{1}{2}\left|\left|\left|g\left(8\,|A_{12}|^2\right)+g\left(8\,|B_{12}|^2\right)\right|\right|\right| \qquad \text{(by Lemma 2.5(a))}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2}\left|\left|\left|g\left(8\left|A_{12}\right|^2\right)\right|\right|\right| + \frac{1}{2}\left|\left|\left|g\left(8\left|B_{12}\right|^2\right)\right|\right|\right| \\
&\leq \frac{1}{2}\left|\left|\left|g\left(2\left(\operatorname{Re} T\right)^2\right)\right|\right|\right| + \frac{1}{2}\left|\left|\left|g\left(2\left(\operatorname{Im} T\right)^2\right)\right|\right|\right| \qquad \text{(by the inequalities (4.2))} \\
&\leq \left|\left|\left|g\left(2\left|T\right|^2\right)\right|\right|\right| \qquad \text{(by the inequalities (4.3))} \\
&= \left|\left|\left|f\left(\sqrt{2}\left|T\right|\right)\right|\right|\right|.
\end{aligned}$$

□

The following example asserts that the convexity of the function $f(\sqrt{t})$ given in Theorem 4.1 is essential and can not be replaced by $f(t)$ to be convex.

Example 4.2. Consider $T = A + iB = \begin{bmatrix} 1+i & 2i \\ 0 & 1+i \end{bmatrix}$. Then $A = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ are positive semidefinite matrices. Take $f(t) = t$, then for the spectral norm $\|\cdot\|$, the right hand side of the inequality (4.1) equals 4 and the left hand side of the inequality (4.1) equals $\sqrt{8+4\sqrt{3}}$, but $4 \nleq \sqrt{8+4\sqrt{3}}$.

The following corollary gives an application on Theorem 4.1.

Corollary 4.3. *Let $T \in \mathbb{M}_{2n}(\mathbb{C})$ be a partitioned accretive-dissipative matrix as given in (1.1). Then*

$$|||(|T_{12}| + |T_{21}^*|)^p + ||T_{12}| - |T_{21}^*||^p||| \leq 2^{p/2}\, ||||T|^p||| \quad \textit{for all } p \geq 2.$$

Proof. The proof follows by applying the inequality (4.1) to the function $f(t) = t^p, p \geq 2$. □

Our second result in this section is given in the following theorem.

Theorem 4.4. *Let $T \in \mathbb{M}_{2n}(\mathbb{C})$ be a partitioned accretive-dissipative matrix as given in (1.1), and let f be a function that is increasing on $[0,\infty)$ with $f(0) = 0$ such that $f(\sqrt{t})$ is concave Then*

$$|||f\left(|T_{12}| + |T_{21}^*|\right) + f\left(||T_{12}| - |T_{21}^*||\right)||| \leq 4\left|\left|\left|f\left(\frac{|T|}{\sqrt{2}}\right)\right|\right|\right|. \tag{4.4}$$

Proof. Let $g(t) = f(\sqrt{t}), t \in [0,\infty)$. Then g is a concave and increasing function on $[0,\infty)$.

Since $T = A+iB$ with $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{bmatrix}$ and $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{bmatrix}$ are positive semidefinite, using Lemma 2.1 and Lemma 2.2, we get that

$$\left|\left|\left|g\left(2\left|A_{12}\right|^2\right)\right|\right|\right| \leq \left|\left|\left|g\left(\frac{(\operatorname{Re} T)^2}{2}\right)\right|\right|\right| \text{ and } \left|\left|\left|g\left(2\left|B_{12}\right|^2\right)\right|\right|\right| \leq \left|\left|\left|g\left(\frac{(\operatorname{Im} T)^2}{2}\right)\right|\right|\right|. \tag{4.5}$$

Also, using Lemmas 2.2 and 2.3, we have

$$\left|\left|\left|g\left(\frac{(\operatorname{Re} T)^2}{2}\right)\right|\right|\right| \leq \left|\left|\left|g\left(\frac{|T|^2}{2}\right)\right|\right|\right| \text{ and } \left|\left|\left|g\left(\frac{(\operatorname{Im} T)^2}{2}\right)\right|\right|\right| \leq \left|\left|\left|g\left(\frac{|T|^2}{2}\right)\right|\right|\right|. \tag{4.6}$$

Now,

$$\begin{aligned}
&|||f\left(|T_{12}| + |T_{21}^*|\right) + f\left(||T_{12}| - |T_{21}^*||\right)||| \\
&= \left|\left|\left|g\left(\left(|T_{12}| + |T_{21}^*|\right)^2\right) + g\left(||T_{12}| - |T_{21}^*||^2\right)\right|\right|\right|
\end{aligned}$$

$$
\begin{aligned}
&\le 2\left|\left|\left|g\left(\frac{(|T_{12}|+|T_{21}^*|)^2+||T_{12}|-|T_{21}^*||^2}{2}\right)\right|\right|\right| \qquad \text{(by Lemma 2.5(b))}\\
&= 2\left|\left|\left|g\left(|T_{12}|^2+|T_{21}^*|^2\right)\right|\right|\right|\\
&= 2\left|\left|\left|g\left(2\,|A_{12}|^2+2\,|B_{12}|^2\right)\right|\right|\right|\\
&\le 2\left(\left|\left|\left|g\left(2\,|A_{12}|^2\right)\right|\right|\right|+\left|\left|\left|g\left(2\,|B_{12}|^2\right)\right|\right|\right|\right)\\
&\le 2\left(\left|\left|\left|g\left(\frac{(\mathrm{Re}\,T)^2}{2}\right)\right|\right|\right|+\left|\left|\left|g\left(\frac{(\mathrm{Im}\,T)^2}{2}\right)\right|\right|\right|\right) \qquad \text{(by the inequalities (4.5))}\\
&\le 4\left|\left|\left|g\left(\frac{|T|^2}{2}\right)\right|\right|\right| \qquad \text{(by the inequalities (4.6))}\\
&= 4\left|\left|\left|f\left(\frac{|T|}{\sqrt{2}}\right)\right|\right|\right|.
\end{aligned}
$$

□

We conclude this paper by the following corollary.

Corollary 4.5. *Let $T \in \mathbb{M}_{2n}(\mathbb{C})$ be partitioned accretive-dissipative matrix as given in (1.1). Then*

$$|||(|T_{12}|+|T_{21}^*|)^p+||T_{12}|-|T_{21}^*||^p||| \le 2^{2-p/2}\,||||T|^p||| \quad \textit{for all } 0<p\le 2.$$

Proof. The proof follows by applying the inequality (4.4) to the function $f(t)=t^p, 0<p\le 2$. □

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