

UNIQUE SOLVABILITY OF A DIRICHLET PROBLEM FOR A FRACTIONAL PARABOLIC EQUATION USING ENERGY-INEQUALITY METHOD

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ABSTRACT. In this paper, we establish sufficient conditions for the existence and uniqueness of the solution in fractional functional space for a class of initial boundary-value problems for a class of partial fractional parabolic differential equations that include a fractional derivative of Caputo. The results are established by the application of the method based on a priori estimate "energy inequality" and the density of the range of the operator generated by the problem considered.

Встановлені достатні умови існування та єдиності розв'язку з дробового функціонального простору для одного класу початково-крайових задач для деяких дробово-параболічних диференціальних рівнянь із дробовою похідною Капуто. Результати отримано шляхом застосування методу енергетичних нерівностей. Доведена щільність образу оператора, що відповідає задачі.

1. INTRODUCTION

Fractional differential equations (FDEs) are obtained by generalizing differential equations to an arbitrary order. Since fractional differential equations are used to model complex phenomena, they play a crucial role in engineering, physics and applied mathematics. Therefore they have been generating increasing interest from engineers and scientist in recent years. Since FDEs have memory, nonlocal relations in space and time, complex phenomena can be modeled by using these equations. Due to this fact, materials with memory and hereditary effects, through strongly anomalous media. Indeed, we can find numerous applications in viscoelasticity, electro-chemistry, signal processing, control theory, porous media, fluid flow, rheology, diffusive transport, electrical networks, electromagnetic theory and probability, signal processing, and many other physical processes are diverse applications of FDEs [1 – 7].

Recently, there has been a significant development in fractional differential and partial differential equations; see the monographs of Kilbas et al. [8], Miller and Ross [9], Samko et al. [10] and the papers of Agarwal et al. [11], Anguraj A. and Karthikeyan P. [12], Belmekki et al. [13], Daftardar-Gejji and Jafari [18], Furati and Tatar [25,26], Kaufmann and Mboumi [27], Kilbas and Marzan [28], Yu and Gao [32], Oussaeif [33], and also the general references in Baleanu et al. [34], and the references therein. However, many phenomena can better be described by integral boundary conditions. Integral boundary conditions are encountered in various applications such as population dynamics, blood flow models, chemical engineering and cellular systems.

The study of existence and uniqueness, periodicity, asymptotic behavior, stability, and methods of analytic and numerical solutions of fractional differential equations have been studied extensively in a large cycle works. But there are not many works in the fractional field of partial differential equation, this is due to the difficulty of applying classical theories and methods to a field of fractional partial differential equation. Motivated by this, we conducted a detailed and thorough study in this field to see the behavior of the solution to these problems using the classic energy estimat method. Then, the present

Keywords. Partial fractional differential equation, energy inequality; existence; uniqueness.

paper is devoted to the study of initial-boundary value problem for a parabolic equation with time-fractional derivative with Dirichlet condition, which has not been studied so far.

2. PRELIMINARIES AND FUNCTIONAL SPACES

Let $\Omega = [0, T]$ be a finite interval of the real numbers \mathbb{R} and $\Gamma(\cdot)$ denote the gamma function. For any positive integer $0 < \alpha < 1$, the Caputo derivative and the Riemann Liouville derivative are, respectively, defined as follows: Let $\Gamma(\cdot)$ denote the gamma function. For any positive integer $0 < \alpha < 1$, the Caputo derivative and the Riemann Liouville derivative are, respectively, defined as follows:

(1) The left Caputo derivatives:

$${}^C D_t^\alpha u(x, t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x, \tau)}{\partial \tau} \frac{1}{(t-\tau)^\alpha} d\tau. \quad (2.1)$$

(2) The left Riemann-Liouville derivatives:

$${}^R D_t^\alpha u(x, t) := \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{u(x, \tau)}{(t-\tau)^\alpha} d\tau. \quad (2.2)$$

(3) The right Caputo derivatives:

$${}^C D_t^\alpha u(x, t) := \frac{-1}{\Gamma(1-\alpha)} \int_t^T \frac{\partial u(x, \tau)}{\partial \tau} \frac{1}{(\tau-t)^\alpha} d\tau.$$

(4) The right Riemann-Liouville derivatives:

$${}^R D_t^\alpha u(x, t) = \frac{-1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_t^T \frac{u(x, \tau)}{(\tau-t)^\alpha} d\tau,$$

Many authors think that the Caputo's version is more natural because it allows the handling of inhomogeneous initial conditions in a easier way. Then the two definitions (2.1) and (2.2) are linked by the following relationship, which can be verified by a direct calculation:

$${}^R D_t^\alpha u(x, t) = {}^C D_t^\alpha u(x, t) + \frac{u(x, 0)}{\Gamma(1-\alpha)t^\alpha}. \quad (2.3)$$

Definition 2.1. [37] For any real $\sigma > 0$ and finite interval $[a, b]$ of the real axis \mathbb{R} , we define the semi-norm:

$$|u|_{{}^l H^\sigma(\Omega)}^2 := \|{}^R D_t^\sigma u\|_{L_2(\Omega)}^2,$$

and norm:

$$\|u\|_{{}^l H^\sigma(\Omega)} := \left(\|u\|_{L_2(\Omega)}^2 + |u|_{{}^l H^\sigma(\Omega)}^2 \right)^{\frac{1}{2}}, \quad (2.4)$$

we then define ${}^l H_0^\sigma(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{{}^l H^\sigma(\Omega)}$.

Definition 2.2. [37] For any real $\sigma > 0$, we define the semi-norm:

$$|u|_{{}^r H^\sigma(\Omega)}^2 := \|{}^R D_t^\sigma u\|_{L_2(\Omega)}^2,$$

and norm:

$$\|u\|_{{}^r H^\sigma(\Omega)} := \left(\|u\|_{L_2(\Omega)}^2 + |u|_{{}^r H^\sigma(\Omega)}^2 \right)^{\frac{1}{2}}, \quad (2.5)$$

we then define ${}^R H_0^\sigma(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{{}^r H^\sigma(\Omega)}$.

Definition 2.3. For any real $\sigma > 0$, we define the semi-norm:

$$|u|_{cH^\sigma(\Omega)} = \left| \frac{({}^R D_t^\sigma u, {}^R D_t^\sigma u)_{L^2(\Omega)}}{\cos(\sigma\pi)} \right|^{1/2}$$

and norm:

$$\|u\|_{cH^\sigma(\Omega)} = (\|u\|_{L^2(\Omega)}^2 + |u|_{cH^\sigma(\Omega)}^2)^{1/2}.$$

Lemma 2.4. [36, 37] For any real $\sigma \in \mathbb{R}_+$, if $u \in {}^L H^\alpha(\Omega)$ and $v \in C_0^\infty(\Omega)$, then

$$({}^R D_t^\sigma u(t), v(t))_{L^2(\Omega)} = (u(t), {}^R D_t^\sigma v(t))_{L^2(\Omega)}.$$

Lemma 2.5. [36, 37] For $0 < \sigma < 2$, $\sigma \neq 1$, $u \in H_0^{\frac{\sigma}{2}}(\Omega)$, on a :

$${}^R D_t^\sigma u(t) = {}^R D_t^{\frac{\sigma}{2}} {}^R D_t^{\frac{\sigma}{2}} u(t).$$

Lemma 2.6. [36, 37] For $\sigma \in \mathbb{R}_+$, $\sigma \neq n + \frac{1}{2}$, the semi-norms $|\cdot|_{L^{\sigma}(\Omega)}$, $|\cdot|_{rH^\sigma(\Omega)}$ and $|\cdot|_{cH^\sigma(\Omega)}$ are equivalent. Then we pose

$$|\cdot|_{L^{\sigma}(\Omega)} \cong |\cdot|_{rH^\sigma(\Omega)} \cong |\cdot|_{cH^\sigma(\Omega)}.$$

Lemma 2.7. [36, 37] For any real $\sigma > 0$, the space ${}^R H_0^\sigma(\Omega)$ with respect to the norm (2.5) is complete.

Definition 2.8. We denote by $L_2(0, T, L_2(0, 1)) := L_2(Q)$ the space of functions which are square integrable in the Bochner sense, with the scalar product

$$(u, w)_{L_2(0, T, L_2(0, 1))} = \int_0^T ((u, \cdot), (w, \cdot))_{L_2(0, 1)} dt. \quad (2.6)$$

Since the space $L_2(0, 1)$ is a Hilbert space, it can be shown that $L_2(0, T, L_2(0, 1))$ is a Hilbert space as well. Let $C^\infty(0, T)$ denote the space of infinitely differentiable functions on $(0, T)$ and $C_0^\infty(0, T)$ denote the space of infinitely differentiable functions with compact support in $(0, T)$.

3. STATEMENT OF PROBLEM

In the rectangular domain $Q = (0, d) \times (0, T)$, with $d, T < \infty$ and $0 < \alpha < 1$, we shall study the existence and uniqueness of solutions $u = u(x, t)$ to the following fractional parabolic problem :

$$\begin{cases} {}^C D_t^\alpha u(x, t) - \frac{\partial}{\partial x} \left(a(x, t) \frac{\partial u(x, t)}{\partial x} \right) + b(x, t)u(x, t) = \tilde{f}(x, t) & \text{in } Q; \\ u(x, 0) = \varphi(x) & \forall x \in (0, d), \\ u(0, t) = u(d, t) = 0 & \forall t \in (0, T). \end{cases} \quad (P)$$

We consider the following fractional parabolic equation of the type:

$$\mathcal{L}u = {}^C D_t^\alpha u - \frac{\partial}{\partial x} \left(a \frac{\partial u}{\partial x} \right) + bu = \tilde{f},$$

with the initial condition :

$$\ell u = u(x, 0) = \varphi(x), \quad \forall x \in (0, d),$$

and Dirichlet condition :

$$u(0, t) = u(d, t) = 0, \quad \forall t \in (0, T).$$

Where a, b, \tilde{f} and φ are known functions.

We shall assume that the function φ satisfies a compatibility conditions, i.e.,

$$\varphi(0) = \varphi(d) = 0.$$

And the functions a, b verify:

$$\begin{aligned} 0 &< a_0 \leq a(x, t) \leq a_1, \\ 0 &< b_0 \leq b(x, t) \leq b_1. \end{aligned}$$

Now, we shall introduce a new function :

$$v(x, t) = u(x, t) - U(x) \implies u(x, t) = v(x, t) + U(x),$$

where

$$\varphi(x) = U(x).$$

So, we get

$$\begin{cases} {}^C D_t^\alpha v(x, t) - \frac{\partial}{\partial x} \left(a(x, t) \frac{\partial v(x, t)}{\partial x} \right) + b(x, t)v(x, t) = \tilde{f}(x, t) - \mathcal{L}\varphi(x) = f(x, t) & \text{in } Q, \\ v(x, 0) = 0 & \forall x \in (0, d), \\ v(0, t) = v(d, t) = 0 & \forall t \in (0, T). \end{cases}$$

Such that

$${}^C D_t^\alpha v(x, t) - \frac{\partial}{\partial x} \left(a(x, t) \frac{\partial v(x, t)}{\partial x} \right) + b(x, t)v(x, t) = f(x, t) \tag{3.7}$$

with the initial condition

$$\ell v = v(x, 0) = 0, \quad \forall x \in (0, d), \tag{3.8}$$

the boundary condition of Dirichlet type

$$v(0, t) = v(d, t) = 0, \quad \forall t \in (0, T) \tag{3.9}$$

where

$$f(x, t) = \tilde{f}(x, t) + \frac{\partial}{\partial x} \left(a(x, t) \frac{\partial \varphi(x)}{\partial x} \right) - b(x, t)\varphi(x).$$

4. ESTIMATION À PRIORI

The method used here is one of the most efficient functional analysis methods and important techniques for solving partial differential equations with integral conditions, which has been successfully used in investigating the existence, uniqueness, and continuous dependence of the solutions of PDE's, the so-called a priori estimate method or the energy-inequality method. This method is essentially based on the construction of multipliers for each specific given problem, which provides the a priori estimate from which it is possible to establish the solvability of the posed problem. More precisely, the proof is based on an energy inequality and the density of the range of the operator generated by the abstract formulation of the stated problem, so to investigate the posed problem, we introduce the needed function spaces. In this paper, we prove the existence and the uniqueness for solution of the problem (3.1) – (3.3) as a solution of the operator equation

$$Lv = \mathcal{F}, \tag{4.10}$$

Where $L = (\mathcal{L}, \ell)$, with domain of definition B consisting of functions $v \in L^2(Q)$, such that $v, {}^C D_t^\alpha v, \frac{\partial v}{\partial x} \in L^2(Q)$ and v satisfies the condition (3.3).

The operator L is considered from B to F , where B is the Banach space consisting of all functions $v(x, t)$ having a finite norm

$$\|v\|_B^2 = \left\| {}^C D_t^{\frac{\alpha}{2}} v \right\|_{L^2(Q)}^2 + \|v\|_{L^2(Q)}^2 + \left\| \frac{\partial v}{\partial x} \right\|_{L^2(Q)}^2,$$

and F is the Hilbert space consisting of all elements $\mathcal{F} = (f, 0)$ for which the norm $L^2(Q)$ is finite.

Theorem 4.1. *For any function $u \in B$, we have the inequality*

$$\|v\|_B \leq k \|Lv\|_{L^2(Q)} \quad (4.11)$$

where k is a positive constant independent of v .

Proof. Multiplying the equation (3.1) by the following function:

$$Mv = v(x, t),$$

and integrating over $Q = (0, d) \times (0, T)$, we get

$$\begin{aligned} \int_Q \mathcal{L}v \cdot Mv dxdt &= \int_Q {}^C D_t^\alpha v(x, t) \cdot v(x, t) dxdt - \int_Q \frac{\partial}{\partial x} \left(a(x, t) \frac{\partial v(x, t)}{\partial x} \right) v(x, t) dxdt \\ &\quad + \int_Q b(x, t) \cdot v^2(x, t) dxdt \\ &= \int_Q f(x, t) \cdot v(x, t) dxdt. \end{aligned}$$

As $v(x, 0) = 0$, we have ${}^C D_t^\alpha v(x, t) = {}^R D_t^\alpha v(x, t)$, so by applying Lemmas 1,2 et 3, we find

$$\begin{aligned} \int_Q {}^C D_t^\alpha v(x, t) \cdot v(x, t) dxdt &= ({}^C D_t^\alpha v(x, t), v(x, t))_{L^2(Q)} \\ &= ({}^C D_t^{\frac{\alpha}{2}} {}^C D_t^{\frac{\alpha}{2}} v(x, t), v(x, t))_{L^2(Q)} \quad (\text{According to Lemma 2}) \\ &= ({}^C D_t^{\frac{\alpha}{2}} v(x, t), {}^C D_t^{\frac{\alpha}{2}} v(x, t))_{L^2(Q)} \quad (\text{According to Lemma 1}) \\ &= \cos\left(\frac{\alpha}{2}\pi\right) |v|_{L^2 H^{\frac{\alpha}{2}}(Q)}^2 \quad (\text{According to Definition 3}) \end{aligned}$$

then, according to Lemma 3, we get that

$$\begin{aligned} \int_Q {}^C D_t^\alpha v(x, t) \cdot v(x, t) dxdt &= \cos\left(\frac{\alpha}{2}\pi\right) |v|_{L^2 H^{\frac{\alpha}{2}}(Q)}^2 \cong \cos\left(\frac{\alpha}{2}\pi\right) |v|_{L^2 H^{\frac{\alpha}{2}}(Q)}^2 \\ &= \cos\left(\frac{\alpha}{2}\pi\right) \left\| {}^C D_t^{\frac{\alpha}{2}} v \right\|_{L^2(Q)}^2. \end{aligned}$$

By integration by parts over $(0, d)$ we get

$$\begin{aligned} - \int_Q \frac{\partial}{\partial x} \left(a(x, t) \frac{\partial v(x, t)}{\partial x} \right) v(x, t) dxdt &= - \int_0^T \int_0^d \frac{\partial}{\partial x} \left(a(x, t) \frac{\partial v(x, t)}{\partial x} \right) v(x, t) dxdt \\ &= - \int_0^T a(x, t) \frac{\partial v(x, t)}{\partial x} v(x, t) \Big|_{x=0}^{x=d} dt + \int_Q a(x, t) \left(\frac{\partial v(x, t)}{\partial x} \right)^2 dxdt \\ &= \int_Q a(x, t) \left(\frac{\partial v(x, t)}{\partial x} \right)^2 dxdt. \end{aligned}$$

By using the Cauchy inequality with ε , for $\varepsilon < 2b_0$; and because of the equivalent of the semi-norms $|\cdot|_{L^2 H^\sigma(Q)}$ and $|\cdot|_{L^2 H^\sigma(Q)}$, there is a positive constant m such that

$$m |\cdot|_{L^2 H^\sigma(Q)} \leq |\cdot|_{L^2 H^\sigma(Q)}.$$

So, we have

$$m^2 \|\cdot\|_{H^\sigma(Q)}^2 \leq \|\cdot\|_{C^{\frac{\alpha}{2}}(Q)}^2,$$

which gives that

$$\begin{aligned} \int_Q {}^C D_t^\alpha v(x, t) \cdot v(x, t) dx dt &= \cos\left(\frac{\alpha}{2}\pi\right) \|v\|_{C^{\frac{\alpha}{2}}(Q)}^2 \geq m^2 \cos\left(\frac{\alpha}{2}\pi\right) \|v\|_{H^{\frac{\alpha}{2}}(Q)}^2 \\ &= m^2 \cos\left(\frac{\alpha}{2}\pi\right) \left\| {}^C D_t^{\frac{\alpha}{2}} v \right\|_{L^2(Q)}^2. \end{aligned}$$

Hence, we get

$$\begin{aligned} m^2 \cos\left(\frac{\alpha}{2}\pi\right) \left\| {}^C D_t^{\frac{\alpha}{2}} v \right\|_{L^2(Q)}^2 &+ a_0 \int_Q \left(\frac{\partial v(x, t)}{\partial x}\right)^2 dx dt + b_0 \int_Q v^2(x, t) dx dt \\ &\leq \int_Q {}^C D_t^\alpha v(x, t) \cdot v(x, t) dx dt + \int_Q a(x, t) \left(\frac{\partial v(x, t)}{\partial x}\right)^2 dx dt \\ &\quad + \int_Q b(x, t) v^2(x, t) dx dt \\ &\leq \frac{1}{2\varepsilon} \int_Q |f(x, t)|^2 dx dt + \frac{\varepsilon}{2} \int_Q |v(x, t)|^2 dx dt. \end{aligned}$$

So, we obtain

$$\begin{aligned} m^2 \cos\left(\frac{\alpha}{2}\pi\right) \left\| {}^C D_t^{\frac{\alpha}{2}} v \right\|_{L^2(Q)}^2 &+ \int_Q a_0 \left(\frac{\partial v(x, t)}{\partial x}\right)^2 dx dt + \int_Q \left(b_0 - \frac{\varepsilon}{2}\right) v^2(x, t) dx dt \\ &\leq \frac{1}{2\varepsilon} \int_Q |f(x, t)|^2 dx dt. \end{aligned}$$

As all the terms are positive, we have

$$\begin{aligned} \left\| {}^C D_t^{\frac{\alpha}{2}} v \right\|_{L^2(Q)}^2 &+ \left\| \frac{\partial v}{\partial x} \right\|_{L^2(Q)}^2 + \|v\|_{L^2(Q)}^2 \\ &\leq \frac{1}{2\varepsilon} \left(\frac{1}{\min\{a_0, m^2 \cos\left(\frac{\alpha}{2}\pi\right), (b_0 - \frac{\varepsilon}{2})\}} \right) \|f\|_{L^2(Q)}^2. \end{aligned}$$

Finally, it follows that

$$\left\| {}^C D_t^{\frac{\alpha}{2}} v \right\|_{L^2(Q)}^2 + \left\| \frac{\partial v}{\partial x} \right\|_{L^2(Q)}^2 + \|v\|_{L^2(Q)}^2 \leq C \|f\|_{L^2(Q)}^2,$$

with

$$C = \frac{1}{2\varepsilon} \left(\frac{1}{\min\{a_0, m^2 \cos\left(\frac{\alpha}{2}\pi\right), (b_0 - \frac{\varepsilon}{2})\}} \right).$$

Therefore, we obtain that

$$\|v\|_B \leq k \|Lv\|_F, \quad \text{where } k = \sqrt{C}.$$

Hence the uniqueness of the solution. \square

Remark 4.2. This inequality $\|v\|_B \leq k \|Lv\|_F$ gives the uniqueness of the solution, indeed.

Let v_1 and v_2 two solutions, so

$$\begin{cases} Lv_1 = \mathcal{F} \\ Lv_2 = \mathcal{F} \end{cases} \implies L(v_1 - v_2) = 0$$

then

$$\|v_1 - v_2\|_B \leq k \|0\|_F \implies \|v_1 - v_2\|_B \leq 0 \implies v_1 - v_2 = 0$$

which gives the uniqueness of the solution.

Proposition 4.3. *The operator L from B to F admits a closure.*

Proof. Let $(v_n)_{n \in \mathbb{N}} \subset D(L)$ a sequence such that

$$v_n \rightarrow 0 \quad \text{in } B, \quad (4.12)$$

and

$$Lv_n \rightarrow \mathcal{F} \quad \text{dans } F, \quad (4.13)$$

it must be shown that

$$f \equiv 0.$$

The convergence of v_n toward 0 in B entails that

$$v_n \rightarrow 0 \quad \text{in } (C_0^\infty(Q))'. \quad (4.14)$$

As the continuity of the fractional derivation and the derivation of the first order (as a particular case of the fractional derivative) of $(C_0^\infty(Q))'$ in $(C_0^\infty(Q))'$, then (4.8) implies

$$\mathcal{L}u_n \rightarrow 0 \quad \text{in } (C_0^\infty(Q))'. \quad (4.15)$$

On the other hand the convergence of $\mathcal{L}v_n$ to f in $F = L^2(Q)$ implies that

$$\mathcal{L}u_n \rightarrow f \quad \text{in } (C_0^\infty(Q))'. \quad (4.16)$$

By virtue of the uniqueness of the limit in $(C_0^\infty(Q))'$, we conclude from (4.9) and (4.10) that

$$f \equiv 0.$$

Hence, the operator L is closable. \square

Definition 4.4. Let \bar{L} the closure of L and $D(\bar{L})$ the definition domain of \bar{L} . The solution of the equation

$$\bar{L}v = \mathcal{F}$$

is called generalized strong solution of the problem (3.1) – (3.3).

Theorem 1 is valid for a generalized strong solution, ie we have the following inequality:

$$\|v\|_B \leq k \|\bar{L}v\|_F, \quad \forall v \in D(\bar{L}). \quad (4.17)$$

Consequently, this last inequality entails the following corollaries:

Corollary 4.5. *The strong solution of the problem (3.1) – (3.3) is unique, if it exists and depends continuously on $f \in F$.*

Corollary 4.6. *The range $R(\bar{L})$ of the operator \bar{L} is equal to the closure $\overline{R(L)}$ of $R(L)$.*

Proof. Let $z \in \overline{R(L)}$, then there exists a Cauchy sequence $(z_n)_n$ in F consists of the elements of the set $R(L)$ such that

$$\lim_{n \rightarrow +\infty} z_n = z.$$

So there is a corresponding sequence $(v_n)_n \subset D(L)$ such that

$$Lv_n = z_n.$$

From the estimate (4.2), we obtain

$$\|v_p - v_q\|_B \leq k \|Lv_p - Lv_q\|_F \rightarrow 0, \quad \text{lorsque } p, q \rightarrow +\infty.$$

We can deduce that $(v_n)_n$ is a Cauchy sequence in B , so there is $v \in B$:

$$\lim_{n \rightarrow +\infty} v_n = v \text{ in } B.$$

By virtue of the denition of \bar{L} ($\lim_{n \rightarrow +\infty} v_n = v$ in B ; if $\lim_{n \rightarrow +\infty} Lv_n = \lim_{n \rightarrow +\infty} z_n = z$, so $\lim_{n \rightarrow +\infty} \bar{L}v_n = z$ and as \bar{L} is closed so $\bar{L}v = z$), the function v verify that

$$v \in D(\bar{L}), \quad \bar{L}v = z.$$

thus $z \in R(\bar{L})$, then

$$\overline{R(L)} \subset R(\bar{L}).$$

So we conclude here that $R(\bar{L})$ is closed because it is complete (any complete subspace of a metric space (not necessarily complete) is closed).

It remains to show the opposite inclusion.

Let $z \in R(\bar{L})$, then there is a sequence of $(z_n)_n$ in F consists of the elements of the set $R(\bar{L})$ such that

$$\lim_{n \rightarrow +\infty} z_n = z.$$

where $z \in R(\bar{L})$, because $R(\bar{L})$ is closed subset of a complete space F , then $R(\bar{L})$ is complete.

So there is a corresponding sequence $(v_n)_n \subset D(\bar{L})$ such that

$$\bar{L}v_n = z_n.$$

From the estimate (4.11), we obtain

$$\|v_p - v_q\|_B \leq k \|\bar{L}v_p - \bar{L}v_q\|_F \rightarrow 0, \text{ if } p, q \rightarrow +\infty.$$

We can deduce that $(v_n)_n$ is a Cauchy sequence in B , so there is $v \in B$:

$$\lim_{n \rightarrow +\infty} v_n = v \text{ in } B.$$

Once more, there is a corresponding sequence $(L(v_n))_n \in R(L)$ such that

$$Lv_n = \bar{L}v_n \text{ over } R(L), \forall n.$$

Then

$$\lim_{n \rightarrow +\infty} Lv_n = z,$$

Consequently $z \in \overline{R(L)}$, and then we conclude that

$$R(\bar{L}) \subset \overline{R(L)}. \quad \square$$

5. EXISTENCE OF SOLUTION

To show the existance of solutions, we prove that $R(L)$ is dense in F for all $u \in B$ and for arbitrary $\mathcal{F} = (f, 0) \in F$.

Theorem 5.1. *For $w \in L^2(Q)$ and for all $v \in B$, we have*

$$\int_Q \mathcal{L}v \cdot w dxdt = 0, \tag{5.18}$$

then w vanishes almost everywhere in Q , this implies that the problem (3.1)–(3.3) admits a unique solution.

Proof. The idea of the proof of the theorem is choose $w \in R(L)^\perp$ (exactly $w \in R(L)^\perp \subset L^2(Q)$) and for all $v \in B$, we have and demonstrate that $R(L)^\perp = \{0\}$ which give $\overline{R(L)} = F$.

The scalar product of F is defined by

$$(Lv, W)_F = \int_Q \mathcal{L}v \cdot w dx dt, \quad \text{where } W = (w, 0) \in D(L). \quad (5.19)$$

The equality (5.1) can be written as follows:

$$\int_Q \left({}^C D_t^\alpha v(x, t) - \frac{\partial}{\partial x} \left(a(x, t) \frac{\partial v(x, t)}{\partial x} \right) + b(x, t)v(x, t) \right) \cdot w(x, t) dx dt = 0 \quad (5.20)$$

where ${}^C D_t^\alpha v$, $\frac{\partial v}{\partial x}$, $v \in L^2(Q)$, with v satisfies the boundary conditions of (3.3). From (5.3), we get the equality

$$\int_Q \left[{}^C D_t^\alpha v(x, t) \cdot w(x, t) - \frac{\partial}{\partial x} \left(a(x, t) \frac{\partial v(x, t)}{\partial x} \right) \cdot w(x, t) + b(x, t)v(x, t) \cdot w(x, t) \right] dx dt = 0 \quad (5.21)$$

And from the equality (5.4), we give the function w in terms of v as follows:

$$w = v, \quad w(x, 0) = 0; \quad (5.22)$$

then $w \in L^2(Q)$.

Replacing w in (5.4) by its representation (5.5) and integrating by parts each term of (5.4) and by taking the condition of v , we obtain

$$\int_Q {}^C D_t^\alpha v(x, t) \cdot v(x, t) dx dt + \int_Q b(x, t)v^2(x, t) dx dt + \int_Q a(x, t) \left(\frac{\partial v(x, t)}{\partial x} \right)^2 dx dt = 0. \quad (5.23)$$

According to Lemma 1,2,3 and definition 3, follow the same steps in the previous section, we have that (5.6) becomes

$$\begin{aligned} m^2 \cos\left(\frac{\alpha}{2}\pi\right) \int_Q \left({}^C D_t^{\frac{\alpha}{2}} v(x, t) \right)^2 dx dt \\ + \int_Q b(x, t)v^2(x, t) dx dt + \int_Q a(x, t) \left(\frac{\partial v(x, t)}{\partial x} \right)^2 dx dt \leq 0. \end{aligned}$$

So, we obtain

$$\begin{aligned} m^2 \cos\left(\frac{\alpha}{2}\pi\right) \left\| {}^C D_t^{\frac{\alpha}{2}} v \right\|_{L^2(Q)}^2 + \int_Q b(x, t)v^2(x, t) dx dt \\ \leq - \int_Q a(x, t) \left(\frac{\partial v(x, t)}{\partial x} \right)^2 dx dt \\ \leq -a_0 \int_Q \left(\frac{\partial v(x, t)}{\partial x} \right)^2 dx dt \\ \leq 0, \end{aligned}$$

then

$$m^2 \cos\left(\frac{\alpha}{2}\pi\right) \left\| {}^C D_t^{\frac{\alpha}{2}} v \right\|_{L^2(Q)}^2 + b_0 \|v\|_{L^2(Q)}^2 \leq 0,$$

Hence

$$\|v\|_{L^2(Q)} = 0$$

And thus $v = 0$ in Q which gives $w = 0$ in Q . So $R(L)^\perp = \{0\}$. This proves Theorem 2. So $\overline{R(L)} = F$. \square

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