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STABILITY OF DUAL g-FUSION FRAMES IN HILBERT SPACES

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Abstract. We give a characterization of K-g-fusion frames and discuss the stability of dual g-fusion frames. We also present a necessary and sufficient condition for a quotient operator to be bounded.

Надається характерізація K-g фреймів злиття та розглядається стійкисть двоїстих g-фпеймів злиття. Також надаються необхідні та достатні умови обмеженності оператора факторизації.

1. Introduction

Frames in Hilbert spaces have many remarkable properties which make them very useful in processing of signals and images, filter bank theory, coding and communications, system modeling and many other fields. The notion of frame in Hilbert space was born in 1952 in the work of Duffin and Schaeffer [3] and their idea did not appear to make much general interest outside of non-harmonic Fourier series. Later on, after some innovative work of Daubechies, Grossman, Meyer [4], the theory of frames began to be studied more widely.

The theory of frames has been generalized rapidly and various generalizations of frames in Hilbert spaces namely, K-frames, G-frames, fusion frames etc. have been introduced in recent times. K-frames in Hilbert spaces were introduced by L. Gavruta [6] to study the atomic system relative to a bounded linear operator. Ramu and Johnson ([9]) obtained characterizations connecting K-frames and quotient operators. Sun [10] introduced a g-frame and a g-Riesz basis in complex Hilbert space and discussed several properties of them. g-frames were also defined by Kaftal, Larson, Zhang ([8]). Huang [7] began to study K-g-frame by combining K-frame and g-frame. General frame theory of subspaces were introduced by P. Casazza and G. Kutynoik [2] as a natural generalization of the frame theory in Hilbert spaces. Fusion frames and K-frames are the special case of generalized frames. Construction of K-g-fusion frames and their dual were presented by Sadri and Rahimi [12] to generalize the theory of K-frame, fusion frame and g-frame. In the theorey of frames, the stability of a frames is very important concept. The stability of g-frames and their dual g-frames have been studied by W. Sun ([11]) and proved that if two g-frames are closed to each other, so their dual g-frames are also closed to each other.

In this paper, we study the stability of dual g-fusion frames and see that dual g-fusion frames are stable under small perturbation. Also, we give a characterization of K-g-fusion frames and at the end, we establish that a quotient operator will be bounded if and only if a g-fusion frame becomes $U\,K$-g-fusion frame.

Throughout this paper, H is considered to be a separable Hilbert space with associated inner product $\langle \cdot, \cdot \rangle$ and I_H is the identity operator on H. We denote the collection of all bounded linear operators from H_1 to H_2 by $\mathcal{B}(H_1, H_2)$, where H_1, H_2 are two Hilbert spaces. In particular $\mathcal{B}(H)$ denote the space of all bounded linear operators on H. For $T \in \mathcal{B}(H)$, we denote $\mathcal{N}(T)$ and $\mathcal{R}(T)$ for null space and range of T, respectively. Also $P_V \in \mathcal{B}(H)$ is the orthogonal projection onto a closed subspace

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$V \subset H$. I, J will denote countable index sets and $\{H_j\}_{j \in J}$ is a sequence of Hilbert spaces. Define $l^2\left(\left\{H_j\right\}_{j \in J}\right)$ by

$$l^2(\{H_j\}_{j \in J}) = \left\{\{f_j\}_{j \in J} : f_j \in H_j, \sum_{j \in J} \|f_j\|^2 < \infty\right\}$$

with inner product is given by

$$\langle \{f_j\}_{j \in J}, \{g_j\}_{j \in J} \rangle = \sum_{j \in J} \langle f_j, g_j \rangle_{H_j}.$$

Clearly $l^2\left(\{H_j\}_{j \in J}\right)$ is a Hilbert space with the pointwise operations ([12]).

2. Preliminaries

In this section, we briefly recall some necessary definitions and results that will be needed later.

Theorem 2.1. *([5]) Let T be a bounded linear operator on H and V be a closed subspace of H. Then $P_V T^* = P_V T^* P_{\overline{TV}}$. Moreover, if T is an unitary operator then $P_{\overline{TV}} T = T P_V$.*

Theorem 2.2. *([1]) The set $\mathcal{S}(H)$ of all self-adjoint operators on H is a partially ordered set with respect to the partial order $\leq$ which is defined as for $T, S \in \mathcal{S}(H)$*

$$T \leq S \Leftrightarrow \langle Tf, f \rangle \leq \langle Sf, f \rangle \;\; \forall f \in H.$$

Definition 2.3. ([9]) Let $U, V \in \mathcal{B}(H)$ with $\mathcal{N}(V) \subset \mathcal{N}(U)$. Then the linear operator $T = [U/V] : \mathcal{R}(V) \to \mathcal{R}(U)$, defined by $T(Vf) = Uf$, $f \in H$ is called quotient operator on H. It can be verify that $\mathcal{R}(T) \subset \mathcal{R}(U)$ and $TV = U$.

Definition 2.4. ([1]) A frame for H is a sequence $\{f_j\}_{j \in J} \subseteq H$ such that

$$A \|f\|^2 \leq \sum_{j \in J} |\langle f, f_j \rangle|^2 \leq B \|f\|^2 \;\; \forall f \in H$$

for some positive constants A, B. The constants A and B are called frame bounds.

Definition 2.5. ([2]) Let $\{W_j\}_{j \in J}$ be a collection of closed subspaces of H and $\{v_j\}_{j \in J}$ be a collection of positive weights. A fusion frame for H is a family of weighted closed subspaces $\{(W_j, v_j) : j \in J\}$ such that

$$A \|f\|^2 \leq \sum_{j \in J} v_j^2 \left\|P_{W_j}(f)\right\|^2 \leq B \|f\|^2 \;\; \forall f \in H$$

for some $0 < A \leq B < \infty$. The constants A, B are called fusion frame bounds.

Definition 2.6. ([8, 10]) A generalized frame or g-frame for H with respect to $\{H_j\}_{j \in J}$ is a sequence of operators $\{\Lambda_j \in \mathcal{B}(H, H_j) : j \in J\}$ such that

$$A \|f\|^2 \leq \sum_{j \in J} \|\Lambda_j f\|^2 \leq B \|f\|^2 \;\; \forall f \in H$$

for some positive constants A and B. The constants A and B are called the lower and upper bounds, respectively.

Definition 2.7. ([13]) Let $\{W_j\}_{j \in J}$ be collection of closed subspaces of H and $\{v_j\}_{j \in J}$ be a collection of positive weights and let $\Lambda_j \in \mathcal{B}(H, H_j)$ for each $j \in J$. A generalized fusion frame or a g-fusion frame for H with respect to $\{H_j\}_{j \in J}$ is a family of the form $\Lambda = \{(W_j, \Lambda_j, v_j)\}_{j \in J}$ such that

$$A \|f\|^2 \leq \sum_{j \in J} v_j^2 \left\|\Lambda_j P_{W_j}(f)\right\|^2 \leq B \|f\|^2 \quad \forall f \in H, \tag{2.1}$$

for some constants $0 < A \leq B < \infty$. The constants A and B are called the lower and upper bounds of g-fusion frame, respectively. If $A = B$ then Λ is called tight g-fusion frame and if $A = B = 1$ then we say Λ is a Parseval g-fusion frame. If the family Λ satisfies

$$\sum_{j \in J} v_j^2 \left\|\Lambda_j P_{W_j}(f)\right\|^2 \leq B \|f\|^2 \quad \forall f \in H$$

then it is called a g-fusion Bessel sequence for H with a bound B.

Definition 2.8. ([13]) Let $\Lambda = \{(W_j, \Lambda_j, v_j)\}_{j \in J}$ be a g-fusion Bessel sequence for H. Then the operator $T_\Lambda : l^2\left(\{H_j\}_{j \in J}\right) \to H$ defined by

$$T_\Lambda\left(\{f_j\}_{j \in J}\right) = \sum_{j \in J} v_j P_{W_j} \Lambda_j^* f_j \quad \forall \{f_j\}_{j \in J} \in l^2\left(\{H_j\}_{j \in J}\right)$$

is called synthesis operator and the operator $T_\Lambda^* : H \to l^2\left(\{H_j\}_{j \in J}\right)$ defined by

$$T_\Lambda^*(f) = \left\{v_j \Lambda_j P_{W_j}(f)\right\}_{j \in J} \; \forall f \in H$$

is called analysis operator. The operator $S_\Lambda : H \to H$ defined by

$$S_\Lambda f = T_\Lambda T_\Lambda^* f = \sum_{j \in J} v_j^2 P_{W_j} \Lambda_j^* \Lambda_j P_{W_j}(f) \; \forall f \in H \tag{2.2}$$

is called g-fusion frame operator. It can be easily verify that

$$\langle S_\Lambda f, f \rangle = \sum_{j \in J} v_j^2 \left\|\Lambda_j P_{W_j}(f)\right\|^2 \quad \forall f \in H.$$

Furthermore, if Λ is a g-fusion frame with bounds A and B, then from (2.1),

$$\langle A f, f \rangle \leq \langle S_\Lambda f, f \rangle \leq \langle B f, f \rangle \quad \forall f \in H.$$

The operator S_Λ is bounded, self-adjoint, positive and invertible. Now, according to the Theorem (2.2), we can write, $A I_H \leq S_\Lambda \leq B I_H$ and this gives

$$B^{-1} I_H \leq S_\Lambda^{-1} \leq A^{-1} I_H.$$

Theorem 2.9. *([13]) $\Lambda = \{(W_j, \Lambda_j, v_j)\}_{j \in J}$ is a g-fusion Bessel sequence for H with bound B if and only if the synthesis operator T_Λ is a well-defined and bounded with $\|T_\Lambda\| \leq \sqrt{B}$.*

Definition 2.10. ([12]) Let $\{W_j\}_{j \in J}$ be collection of closed subspaces of H and $\{v_j\}_{j \in J}$ be a collection of positive weights and let $\Lambda_j \in \mathcal{B}(H, H_j)$ for each $j \in J$ and $K \in \mathcal{B}(H)$. Then K-g-fusion frame for H with respect to $\{H_j\}_{j \in J}$ is a family of the form $\Lambda = \{(W_j, \Lambda_j, v_j)\}_{j \in J}$ such that

$$A \|K^* f\|^2 \leq \sum_{j \in J} v_j^2 \left\|\Lambda_j P_{W_j}(f)\right\|^2 \leq B \|f\|^2 \quad \forall f \in H \tag{2.3}$$

for some constants $0 < A \leq B < \infty$. The constants A and B are called the lower and upper bounds of K-g-fusion frame, respectively. If $A = B$ then Λ is called a tight K-g-fusion frame. If $K = I_H$ then Λ is a g-fusion frame and if $K = I_H$ and $\Lambda_j = P_{W_j}$ for any $j \in J$, then Λ is a fusion frame for H.

3. Some properties of K-g-fusion frames

Theorem 3.1. *Let $U \in \mathcal{B}(H)$ be an invertible operator on H and $\Lambda = \{(W_j, \Lambda_j, v_j)\}_{j \in J}$ be a K-g-fusion frame for H for some $K \in \mathcal{B}(H)$. Then $\Gamma = \left\{\left(U W_j, \Lambda_j P_{W_j} U^*, v_j\right)\right\}_{j \in J}$ is a $U K U^*$-g-fusion frame for H.*

Proof. Since Λ is a K-g-fusion frame for H, $\exists\, A, B > 0$ such that

$$A \|K^* f\|^2 \leq \sum_{j \in J} v_j^2 \left\|\Lambda_j P_{W_j}(f)\right\|^2 \leq B \|f\|^2 \;\; \forall f \in H. \tag{3.4}$$

Also, U is an invertible bounded linear operator on H, so for any $j \in J$, $U W_j$ is closed in H. Now, for each $f \in H$, using Theorem (2.1), we obtain

$$\begin{aligned}\sum_{j \in J} v_j^2 \left\|\Lambda_j P_{W_j} U^* P_{U W_j}(f)\right\|^2 &= \sum_{j \in J} v_j^2 \left\|\Lambda_j P_{W_j} U^*(f)\right\|^2 \\ &\leq B \|U^* f\|^2 \leq B \|U\|^2 \|f\|^2 \;\; [\text{ by } (3.4)\,].\end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{A}{\|U\|^2} \left\|(U K U^*)^* f\right\|^2 &= \frac{A}{\|U\|^2} \|U K^* U^* f\|^2 \leq A \|K^* U^* f\|^2 \\ &\leq \sum_{j \in J} v_j^2 \left\|\Lambda_j P_{W_j}(U^* f)\right\|^2 \;\; [\text{ by } (3.4)\,] \\ &= \sum_{j \in J} v_j^2 \left\|\Lambda_j P_{W_j} U^* P_{U W_j}(f)\right\|^2 \;\; \forall f \in H.\end{aligned}$$

Therefore, Γ is a $U K U^*$-g-fusion frame for H. □

Theorem 3.2. *Let U be an invertible bounded linear operator on H and $\Gamma = \left\{\left(U W_j, \Lambda_j P_{W_j} U^*, v_j\right)\right\}_{j \in J}$ be a K-g-fusion frame for H for some $K \in \mathcal{B}(H)$. Then $\Lambda = \{(W_j, \Lambda_j, v_j)\}_{j \in J}$ is a $U^{-1} K U$-g-fusion frame for H.*

Proof. Since Γ is a K-g-fusion frame for H, for all $f \in H$, $\exists\, A, B > 0$ such that

$$A \|K^* f\|^2 \leq \sum_{j \in J} v_j^2 \left\|\Lambda_j P_{W_j} U^* P_{U W_j}(f)\right\|^2 \leq B \|f\|^2. \tag{3.5}$$

Now, for each $f \in H$, we have

$$\begin{aligned}\frac{A}{\|U\|^2} \left\|\left(U^{-1} K U\right)^* f\right\|^2 &= \frac{A}{\|U\|^2} \left\|U^* K^* (U^{-1})^* f\right\|^2 \\ &\leq A \left\|K^* \left(U^{-1}\right)^* f\right\|^2 \\ &\leq \sum_{j \in J} v_j^2 \left\|\Lambda_j P_{W_j} U^* P_{U W_j}\left(\left(U^{-1}\right)^* f\right)\right\|^2 \;\; [\text{ by } (3.5)\,] \\ &= \sum_{j \in J} v_j^2 \left\|\Lambda_j P_{W_j}\left(U^* \left(U^{-1}\right)^* f\right)\right\|^2 \;\; [\text{ by Theorem } (2.1)\,] \\ &= \sum_{j \in J} v_j^2 \left\|\Lambda_j P_{W_j}(f)\right\|^2.\end{aligned}$$

Also, for each $f \in H$, we have

$$\begin{aligned}\sum_{j \in J} v_j^2 \left\|\Lambda_j P_{W_j}(f)\right\|^2 &= \sum_{j \in J} v_j^2 \left\|\Lambda_j P_{W_j}\left(U^* \left(U^{-1}\right)^* f\right)\right\|^2 \\ &= \sum_{j \in J} v_j^2 \left\|\Lambda_j P_{W_j} U^* P_{U W_j}\left(\left(U^{-1}\right)^* f\right)\right\|^2\end{aligned}$$

$$\leq B \left\| \left(U^{-1} \right)^* f \right\|^2 \text{ [by (3.5)]}$$
$$\leq B \left\| U^{-1} \right\|^2 \| f \|^2.$$

Thus, Λ is a $U^{-1} K U$-g-fusion frame for H with bounds $\frac{A}{\| U \|^2}$ and $B \left\| U^{-1} \right\|^2$. □

Theorem 3.3. *Let K be an invertible bounded linear operator on H and $\Lambda = \{ (W_j, \Lambda_j, v_j) \}_{j \in J}$ be a g-fusion frame for H with frame bounds A, B and S_Λ be the associated g-fusion frame operator. Then $\left\{ \left(K S_\Lambda^{-1} W_j, \Lambda_j P_{W_j} S_\Lambda^{-1} K^*, v_j \right) \right\}_{j \in J}$ is a K-g-fusion frame for H with the corresponding g-fusion frame operator $K S_\Lambda^{-1} K^*$.*

Proof. Let $T = K S_\Lambda^{-1}$. Then T is invertible on H and $T^* = \left(K S_\Lambda^{-1} \right)^* = S_\Lambda^{-1} K^*$. For $f \in H$, we have

$$\| K^* f \|^2 = \left\| S_\Lambda S_\Lambda^{-1} K^* f \right\|^2 \leq \| S_\Lambda \|^2 \left\| S_\Lambda^{-1} K^* f \right\|^2$$
$$\leq B^2 \left\| S_\Lambda^{-1} K^* f \right\|^2. \tag{3.6}$$

Now, for each $f \in H$, using Theorem (2.1), we get

$$\sum_{j \in J} v_j^2 \left\| \Lambda_j P_{W_j} T^* P_{T W_j} (f) \right\|^2 = \sum_{j \in J} v_j^2 \left\| \Lambda_j P_{W_j} (T^* f) \right\|^2$$
$$= \sum_{j \in J} v_j^2 \left\| \Lambda_j P_{W_j} \left(S_\Lambda^{-1} K^* f \right) \right\|^2$$
$$\leq B \left\| S_\Lambda^{-1} K^* f \right\|^2 \text{ [since } \Lambda \text{ is a } g\text{-fusion frame]}$$
$$\leq B \| S_\Lambda^{-1} \|^2 \| K^* f \|^2$$
$$\leq \frac{B}{A^2} \| K \|^2 \| f \|^2 \text{ [using } B^{-1} I_H \leq S_\Lambda^{-1} \leq A^{-1} I_H \text{]}.$$

On the other hand,

$$\sum_{j \in J} v_j^2 \left\| \Lambda_j P_{W_j} T^* P_{T W_j} (f) \right\|^2 = \sum_{j \in J} v_j^2 \left\| \Lambda_j P_{W_j} \left(S_\Lambda^{-1} K^* f \right) \right\|^2$$
$$\geq A \left\| S_\Lambda^{-1} K^* f \right\|^2 \geq \frac{A}{B^2} \| K^* f \|^2 \text{ [by (3.6)]}.$$

Thus, $\left\{ \left(K S_\Lambda^{-1} W_j, \Lambda_j P_{W_j} S_\Lambda^{-1} K^*, v_j \right) \right\}_{j \in J}$ is a K-g-fusion frame for H. Furthermore, for each $f \in H$,

$$\sum_{j \in J} v_j^2 P_{T W_j} \left(\Lambda_j P_{W_j} T^* \right)^* \left(\Lambda_j P_{W_j} T^* \right) P_{T W_j} (f)$$
$$= \sum_{j \in J} v_j^2 \left(P_{T W_j} T P_{W_j} \right) \Lambda_j^* \Lambda_j \left(P_{W_j} T^* P_{T W_j} \right) (f)$$
$$= \sum_{j \in J} v_j^2 \left(P_{W_j} T^* P_{T W_j} \right)^* \Lambda_j^* \Lambda_j (P_{W_j} T^* P_{T W_j}) (f)$$
$$= \sum_{j \in J} v_j^2 \left(P_{W_j} T^* \right)^* \Lambda_j^* \Lambda_j P_{W_j} T^* (f) \text{ [using Theorem (2.1)]}$$
$$= \sum_{j \in J} v_j^2 T P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} T^* f = T S_\Lambda T^* f \text{ [by (2.2)]}$$
$$= \left(K S_\Lambda^{-1} \right) S_\Lambda \left(S_\Lambda^{-1} K^* f \right) = K S_\Lambda^{-1} K^* f \quad \forall f \in H.$$

This implies that $K S_\Lambda^{-1} K^*$ is the associated g-fusion frame operator. □

Corollary 3.4. *Let* $\Lambda = \{ (W_j, \Lambda_j, v_j) \}_{j \in J}$ *be a g-fusion frame for* H *with g-fusion frame operator* S_Λ. *If* P_V *is the orthogonal projection onto closed subspace* $V \subset H$ *then* $\left\{ \left(P_V S_\Lambda^{-1} W_j, \Lambda_j P_{W_j} S_\Lambda^{-1} P_V, v_j \right) \right\}_{j \in J}$ *is a* P_V*-g-fusion frame for* H *with the corresponding g-fusion frame operator* $P_V S_\Lambda^{-1} P_V$.

Proof. Proof of this Corollary directly follows from that of the Theorem (3.3), by putting $K = P_V$. □

Theorem 3.5. *Let* $\Lambda = \{ (W_j, \Lambda_j, v_j) \}_{j \in J}$ *be a K-g-fusion frame for* H *with bounds* A, B *and for each* $j \in J, T_j \in \mathcal{B}(H_j)$ *be invertible operator. Suppose*

$$0 < m = \inf_{j \in J} \frac{1}{\left\| T_j^{-1} \right\|} \leq \sup_{j \in J} \| T_j \| = M. \tag{3.7}$$

If $T \in \mathcal{B}(H)$ *is an invertible operator on* H *with* $KT = TK$ *then* $\Gamma = \left\{ \left(T W_j, T_j \Lambda_j P_{W_j} T^*, v_j \right) \right\}_{j \in J}$ *is a K-g-fusion frame for* H.

Proof. Since T and T_j (for each $j \in J$) are invertible, so

$$\| K^* f \|^2 = \left\| \left(T^{-1} \right)^* T^* K^* f \right\|^2 \leq \left\| T^{-1} \right\|^2 \| T^* K^* f \|^2, \quad \& \tag{3.8}$$

$$\| f \|^2 = \left\| T_j^{-1} T_j f \right\|^2 \leq \left\| T_j^{-1} \right\|^2 \| T_j f \|^2. \tag{3.9}$$

By Theorem (2.1), for each $f \in H$, we have

$$\begin{aligned}
\sum_{j \in J} v_j^2 \left\| T_j \Lambda_j P_{W_j} T^* P_{T W_j}(f) \right\|^2 &= \sum_{j \in J} v_j^2 \left\| T_j \Lambda_j P_{W_j} (T^* f) \right\|^2 \\
&\geq \sum_{j \in J} \frac{1}{\left\| T_j^{-1} \right\|^2} v_j^2 \left\| \Lambda_j P_{W_j} (T^* f) \right\|^2 \quad [\text{ using } (3.9)] \\
&\geq m^2 \sum_{j \in J} v_j^2 \left\| \Lambda_j P_{W_j} (T^* f) \right\|^2 \quad [\text{ using } (3.7)] \\
&\geq m^2 A \| K^* T^* f \|^2 \quad [\text{ since } \Lambda \text{ is } K\text{-}g\text{-fusion frame}] \\
&= m^2 A \| T^* K^* f \|^2 \quad [\text{ because } KT = TK] \\
&\geq m^2 A \left\| T^{-1} \right\|^{-2} \| K^* f \|^2 \quad [\text{ using } (3.8)].
\end{aligned}$$

On the other hand, for each $f \in H$, we have

$$\begin{aligned}
\sum_{j \in J} v_j^2 \left\| T_j \Lambda_j P_{W_j} T^* P_{T W_j}(f) \right\|^2 &= \sum_{j \in J} v_j^2 \left\| T_j \Lambda_j P_{W_j} (T^* f) \right\|^2 \\
&\leq \sum_{j \in J} \| T_j \|^2 v_j^2 \left\| \Lambda_j P_{W_j} (T^* f) \right\|^2 \\
&\leq M^2 \sum_{j \in J} v_j^2 \left\| \Lambda_j P_{W_j} (T^* f) \right\|^2 \quad [\text{ using } (3.7)] \\
&\leq M^2 B \| T \|^2 \| f \|^2 \quad [\text{ since } \Lambda \text{ is } K\text{-}g\text{-fusion frame}].
\end{aligned}$$

Thus, Γ is a K-g-fusion frame with bounds $m^2 A \left\| T^{-1} \right\|^{-2}$ and $M^2 B \| T \|^2$. □

Remark 3.6. We further notice that the g-fusion frame operator S_Γ of Γ satisfies the followings

(I) By (2.2),

$$S_{\Gamma} f = \sum_{j \in J} v_j^2 P_{T W_j} \left(T_j \Lambda_j P_{W_j} T^* \right)^* \left(T_j \Lambda_j P_{W_j} T^* \right) P_{T W_j} (f)$$
$$= \sum_{j \in J} v_j^2 \left(P_{W_j} T^* P_{T W_j} \right)^* \Lambda_j^* T_j^* T_j \Lambda_j \left(P_{W_j} T^* P_{T W_j} \right) (f)$$
$$= \sum_{j \in J} v_j^2 \left(P_{W_j} T^* \right)^* \Lambda_j^* T_j^* T_j \Lambda_j P_{W_j} T^* (f) \text{ [by Theorem (2.1)]}$$
$$= \sum_{j \in J} v_j^2 T P_{W_j} \Lambda_j^* T_j^* T_j \Lambda_j P_{W_j} T^* (f). \tag{3.10}$$

(II) Moreover, if $K = I_H$, i.e, if Λ is a g-fusion frame then Γ is also g-fusion frame. Then S_{Γ} is invertible on H and by Theorem (2.2), we can write

$$\frac{1}{M^2 B \| T \|^2} I_H \leq S_{\Gamma}^{-1} \leq \frac{1}{m^2 A \| T^{-1} \|^{-2}} I_H. \tag{3.11}$$

Remark 3.7. Let us now denote $U = T^* S_{\Gamma}^{-1} T$ and for each $j \in J$, $L_j = T_j^* T_j$ and $\Delta_j = L_j \Lambda_j P_{W_j} U$, where T, T_j, Λ and Γ are all as in the Theorem (3.5). Now it is easy to verify the following:

(I) $U \in \mathcal{B}(H)$,
(II) for all $j \in J$, $L_j \in \mathcal{B}(H_j)$ and $\Delta_j \in \mathcal{B}(H, H_j)$.
(III) U and L_j are self-adjoint and invertible.
(IV) From (3.11), it can be obtained

$$\| U \| \leq \| T^* \| \left\| S_{\Gamma}^{-1} \right\| \| T \| \leq \frac{\| T \|^2}{m^2 A \| T^{-1} \|^{-2}}. \tag{3.12}$$

(V) For each $j \in J$, using (3.7),

$$\| L_j \| = \left\| T_j^* T_j \right\| = \| T_j \|^2 \leq M^2. \tag{3.13}$$

Theorem 3.8. *Let Λ be a g-fusion frame for H with bounds A and B. Then $\Delta = \{ (T W_j, \Delta_j, v_j) \}_{j \in J}$ is a g-fusion frame for H. Furthermore,*

$$f = \sum_{j \in J} v_j^2 P_{W_j} \Lambda_j^* \Delta_j P_{T W_j} (f) = \sum_{j \in J} v_j^2 P_{T W_j} \Delta_j^* \Lambda_j P_{W_j} (f) \quad \forall f \in H.$$

Proof. For each $f \in H$, we have

$$\sum_{j \in J} v_j^2 \left\| \Delta_j P_{T W_j} (f) \right\|^2 = \sum_{j \in J} v_j^2 \left\| L_j \Lambda_j P_{W_j} U P_{T W_j} (f) \right\|^2$$
$$= \sum_{j \in J} v_j^2 \left\| L_j \Lambda_j P_{W_j} (U f) \right\|^2 \text{ [by Theorem (2.1)]}$$
$$\leq \sum_{j \in J} v_j^2 \| L_j \|^2 \left\| \Lambda_j P_{W_j} (U f) \right\|^2$$
$$\leq M^4 \sum_{j \in J} v_j^2 \left\| \Lambda_j P_{W_j} (U f) \right\|^2 \text{ [using (3.13)]}$$
$$\leq B M^4 \| U f \|^2 \text{ [since } \Lambda \text{ is } g\text{-fusion frame]}$$
$$\leq \frac{B M^4 \| T \|^4}{m^4 A^2 \| T^{-1} \|^{-4}} \| f \|^2 \text{ [by (3.12)]}.$$

Since for all $j \in J$, L_j is invertible so

$$\sum_{j \in J} v_j^2 \left\| \Delta_j P_{T W_j} (f) \right\|^2 = \sum_{j \in J} v_j^2 \left\| L_j \Lambda_j P_{W_j} U P_{T W_j} (f) \right\|^2$$

$$\geq \sum_{j \in J} \frac{1}{\|L_j^{-1}\|^2} v_j^2 \left\| \Lambda_j P_{W_j}(U f) \right\|^2$$
$$\geq m_1^2 A \|U f\|^2 \;\left[\text{ taking } m_1 = \inf_{j \in J} \frac{1}{\|L_j^{-1}\|}\right]$$
$$\geq \frac{m_1^2 A}{\|U^{-1}\|^2} \|f\|^2 \;[\text{since } U \text{ is invertible}].$$

Therefore, Δ is a g-fusion frame for H. Furthermore, for each $f \in H$, we have

$$\sum_{j \in J} v_j^2 P_{W_j} \Lambda_j^* \Delta_j P_{T W_j}(f) = \sum_{j \in J} v_j^2 P_{W_j} \Lambda_j^* \left(L_j \Lambda_j P_{W_j} U\right) P_{T W_j}(f)$$
$$= \sum_{j \in J} v_j^2 P_{W_j} \Lambda_j^* T_j^* T_j \Lambda_j \left(P_{W_j} U P_{T W_j}(f)\right)$$
$$= \sum_{j \in J} v_j^2 P_{W_j} \Lambda_j^* T_j^* T_j \Lambda_j P_{W_j} U(f) \;[\text{by Theorem (2.1)}]$$
$$= \sum_{j \in J} v_j^2 P_{W_j} \Lambda_j^* T_j^* T_j \Lambda_j P_{W_j} \left(T^* S_\Gamma^{-1} T f\right)$$
$$= T^{-1}\Big(\sum_{j \in J} v_j^2 T P_{W_j} \Lambda_j^* T_j^* T_j \Lambda_j P_{W_j} T^* \left(S_\Gamma^{-1} T f\right)\Big)$$
$$= T^{-1} S_\Gamma \left(S_\Gamma^{-1} T f\right) = f \;[\text{using } (3.10)].$$

According to the preceding procedure, we also get

$$f = \sum_{j \in J} v_j^2 P_{T W_j} \Delta_j^* \Lambda_j P_{W_j}(f) \quad \forall f \in H.$$

This completes the proof. □

4. Stability of dual g-fusion frame

We know that if $\Lambda = \{(W_j, \Lambda_j, v_j)\}_{j \in J}$ is a g-fusion frame for H with associated frame operator S_Λ then $\Lambda^\circ = \left\{\left(S_\Lambda^{-1} W_j,\, \Lambda_j P_{W_j} S_\Lambda^{-1},\, v_j\right)\right\}_{j \in J}$ is called as the canonical dual g-fusion frame of Λ. For each $f \in H$, the frame operator S_{Λ° of Λ° is described by

$$S_{\Lambda^\circ}(f) = \sum_{j \in J} v_j^2 P_{S_\Lambda^{-1} W_j} \left(\Lambda_j P_{W_j} S_\Lambda^{-1}\right)^* \left(\Lambda_j P_{W_j} S_\Lambda^{-1}\right) P_{S_\Lambda^{-1} W_j}(f) \;[\text{by } (2.2)]$$
$$= \sum_{j \in J} v_j^2 P_{S_\Lambda^{-1} W_j} S_\Lambda^{-1} P_{W_j} \Lambda_j^* \Lambda_j \left(P_{W_j} S_\Lambda^{-1} P_{S_\Lambda^{-1} W_j}\right)(f)$$
$$= \sum_{j \in J} v_j^2 \left(P_{W_j} S_\Lambda^{-1} P_{S_\Lambda^{-1} W_j}\right)^* \Lambda_j^* \Lambda_j \left(P_{W_j} S_\Lambda^{-1} P_{S_\Lambda^{-1} W_j}\right)(f)$$
$$= \sum_{j \in J} v_j^2 \left(P_{W_j} S_\Lambda^{-1}\right)^* \Lambda_j^* \Lambda_j P_{W_j} S_\Lambda^{-1}(f) \;[\text{by Theorem (2.1)}]$$
$$= \sum_{j \in J} v_j^2 S_\Lambda^{-1} P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} S_\Lambda^{-1}(f)$$
$$= S_\Lambda^{-1} \sum_{j \in J} v_j^2 P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} S_\Lambda^{-1}(f)$$
$$= S_\Lambda^{-1}\left(S_\Lambda\left(S_\Lambda^{-1} f\right)\right) = S_\Lambda^{-1}(f). \tag{4.14}$$

In this section, we shall study the stability of dual g-fusion frames and at the end, a necessary and sufficient condition for some $K \in \mathcal{B}(H)$, for some invertible operator

$U \in \mathcal{B}(H)$, a quotient operator will be bounded if and only if g-fusion frame becomes $U K$-g-fusion frame.

Theorem 4.1. *Let $\Lambda = \{(W_j, \Lambda_j, v_j)\}_{j \in J}$ and $\Gamma = \{(V_j, \Gamma_j, v_j)\}_{j \in J}$ be two g-fusion frames for H. If the condition*

$$\sum_{j \in J} v_j^2 \left\| \left(\Lambda_j P_{W_j} - \Gamma_j P_{V_j} \right)(f) \right\|^2 \leq D \| f \|^2$$

holds for each $f \in H$ and for some $D > 0$ then $\sum_{j \in J} v_j \left(P_{W_j} \Lambda_j^* - P_{V_j} \Gamma_j^* \right) f_j$ converges unconditionally for each $\{f_j\}_{j \in J} \in l^2\left(\{H_j\}_{j \in J}\right)$.

Proof. Let I be any finite subset of J. Then by Cauchy-Schwarz inequality for each $\{f_j\}_{j \in J} \in l^2\left(\{H_j\}_{j \in J}\right)$, we have

$$\begin{aligned}
\left\| \sum_{j \in I} v_j \left(P_{W_j} \Lambda_j^* - P_{V_j} \Gamma_j^* \right) f_j \right\| &= \sup_{\|g\| = 1} \left| \left\langle \sum_{j \in I} v_j \left(P_{W_j} \Lambda_j^* - P_{V_j} \Gamma_j^* \right) f_j \,,\, g \right\rangle \right| \\
&= \sup_{\|g\| = 1} \left| \sum_{j \in I} \left\langle f_j \,,\, v_j \left(\Lambda_j P_{W_j} - \Gamma_j P_{V_j} \right)(g) \right\rangle \right| \\
&\leq \left(\sum_{j \in I} \| f_j \|^2 \right)^{\frac{1}{2}} \sup_{\|g\| = 1} \left(\sum_{j \in I} v_j^2 \left\| \left(\Lambda_j P_{W_j} - \Gamma_j P_{V_j} \right)(g) \right\|^2 \right)^{\frac{1}{2}} \\
&\leq D^{\frac{1}{2}} \left(\sum_{j \in I} \| f_j \|^2 \right)^{\frac{1}{2}} < \infty
\end{aligned}$$

and therefore $\sum_{j \in J} v_j \left(P_{W_j} \Lambda_j^* - P_{V_j} \Gamma_j^* \right) f_j$ is unconditionally convergent in H. □

Theorem 4.2. *Let $\Lambda = \{(W_j, \Lambda_j, v_j)\}_{j \in J}$ and $\Gamma = \{(V_j, \Gamma_j, v_j)\}_{j \in J}$ be two g-fusion frames for H with frame bounds (A_1, B_1) and (A_2, B_2), respectively. Take $\Lambda^\circ = \left\{ \left(W_j^\circ, \Lambda_j^\circ, v_j \right) \right\}_{j \in J}$ and $\Gamma^\circ = \left\{ \left(V_j^\circ, \Gamma_j^\circ, v_j \right) \right\}_{j \in J}$ be the corresponding canonical dual g-fusion frames for Λ and Γ, respectively. Then the following statements hold:*

(I) *If the condition*

$$\sum_{j \in J} v_j^2 \left\| \left(\Lambda_j P_{W_j} - \Gamma_j P_{V_j} \right) f \right\|^2 \leq D \| f \|^2$$

holds for each $f \in H$ and for some $D > 0$ then for all $f \in H$,

$$\sum_{j \in J} v_j^2 \left\| \left(\Lambda_j^\circ P_{W_j^\circ} - \Gamma_j^\circ P_{V_j^\circ} \right) f \right\|^2 \leq D \left(\frac{A_1 + B_1 + B_1^{\frac{1}{2}} B_2^{\frac{1}{2}}}{A_1 A_2} \right)^2 \| f \|^2.$$

(II) *If the condition*

$$\left| \sum_{j \in J} v_j^2 \left\| \Lambda_j P_{W_j}(f) \right\|^2 - \sum_{j \in J} v_j^2 \left\| \Gamma_j P_{V_j}(f) \right\|^2 \right| \leq D \| f \|^2$$

holds for each $f \in H$ and for some $D > 0$ then for all $f \in H$,

$$\left| \sum_{j \in J} v_j^2 \left\| \Lambda_j^\circ P_{W_j^\circ}(f) \right\|^2 - \sum_{j \in J} v_j^2 \left\| \Gamma_j^\circ P_{V_j^\circ}(f) \right\|^2 \right| \leq \frac{D}{A_1 A_2} \| f \|^2.$$

Proof. (I) Let S_Λ and S_Γ be the corresponding g-fusion frame operators for Λ and Γ, then for each $f \in H$, we have

$$S_\Lambda f = \sum_{j \in J} v_j^2 P_{W_j} \Lambda_j^* \Lambda_j P_{W_j}(f), \quad S_\Gamma f = \sum_{j \in J} v_j^2 P_{V_j} \Gamma_j^* \Gamma_j P_{V_j}(f), \ \&$$

$$B_1^{-1} I_H \leq S_\Lambda^{-1} \leq A_1^{-1} I_H, \quad B_2^{-1} I_H \leq S_\Gamma^{-1} \leq A_2^{-1} I_H. \tag{4.15}$$

Since Λ° and Γ° are canonical dual g-fusion frames of Λ and Γ, so

$$W_j^\circ = S_\Lambda^{-1} W_j, \ \Lambda_j^\circ = \Lambda_j P_{W_j} S_\Lambda^{-1} \text{ and } V_j^\circ = S_\Gamma^{-1} V_j, \ \Gamma_j^\circ = \Gamma_j P_{V_j} S_\Gamma^{-1}.$$

Then for any $f \in H$, using Theorem (2.9) and the proof of Theorem (4.1), we get

$$\begin{aligned}
\| S_\Lambda f - S_\Gamma f \| &= \Big\| \sum_{j \in J} v_j^2 \left(P_{W_j} \Lambda_j^* \Lambda_j P_{W_j}(f) - P_{V_j} \Gamma_j^* \Gamma_j P_{V_j}(f) \right) \Big\| \\
&= \Big\| \sum_{j \in J} v_j^2 \big(P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} - P_{W_j} \Lambda_j^* \Gamma_j P_{V_j} \\
&\qquad + P_{W_j} \Lambda_j^* \Gamma_j P_{V_j} - P_{V_j} \Gamma_j^* \Gamma_j P_{V_j} \big)(f) \Big\| \\
&\leq \Big\| \sum_{j \in J} v_j P_{W_j} \Lambda_j^* v_j \left(\Lambda_j P_{W_j} - \Gamma_j P_{V_j} \right)(f) \Big\| \\
&\qquad + \Big\| \sum_{j \in J} v_j \left(P_{W_j} \Lambda_j^* - P_{V_j} \Gamma_j^* \right) v_j \Gamma_j P_{V_j}(f) \Big\| \\
&\leq B_1^{\frac{1}{2}} \Big(\sum_{j \in J} v_j^2 \left\| \left(\Lambda_j P_{W_j} - \Gamma_j P_{V_j} \right)(f) \right\|^2 \Big)^{\frac{1}{2}} \\
&\qquad + D^{\frac{1}{2}} \Big(\sum_{j \in J} v_j^2 \left\| \Gamma_j P_{V_j}(f) \right\|^2 \Big)^{\frac{1}{2}} \\
&\leq B_1^{\frac{1}{2}} D^{\frac{1}{2}} \| f \| + D^{\frac{1}{2}} B_2^{\frac{1}{2}} \| f \| = D^{\frac{1}{2}} \left(B_1^{\frac{1}{2}} + B_2^{\frac{1}{2}} \right) \| f \|.
\end{aligned}$$

Therefore,

$$\| S_\Lambda - S_\Gamma \| = \sup_{\| f \| = 1} \| S_\Lambda f - S_\Gamma f \| \leq D^{\frac{1}{2}} \left(B_1^{\frac{1}{2}} + B_2^{\frac{1}{2}} \right).$$

On the other hand,

$$\begin{aligned}
\left\| S_\Lambda^{-1} - S_\Gamma^{-1} \right\| &= \left\| S_\Lambda^{-1} (S_\Lambda - S_\Gamma) S_\Gamma^{-1} \right\| \\
&\leq \left\| S_\Lambda^{-1} \right\| \| S_\Lambda - S_\Gamma \| \left\| S_\Gamma^{-1} \right\| \\
&\leq \frac{D^{\frac{1}{2}}}{A_1 A_2} \left(B_1^{\frac{1}{2}} + B_2^{\frac{1}{2}} \right) [\text{ by } (4.15)].
\end{aligned}$$

Since Λ is a g-fusion frame, for $f \in H$, we have

$$\begin{aligned}
\sum_{j \in J} v_j^2 \left\| \Lambda_j P_{W_j} \left(S_\Lambda^{-1} - S_\Gamma^{-1} \right) f \right\|^2 &\leq B_1 \left\| \left(S_\Lambda^{-1} - S_\Gamma^{-1} \right) f \right\|^2 \\
&\leq \frac{B_1}{A_1^2 A_2^2} D \left(B_1^{\frac{1}{2}} + B_2^{\frac{1}{2}} \right)^2 \| f \|^2.
\end{aligned} \tag{4.16}$$

Also, by given condition, we obtain

$$\sum_{j \in J} v_j^2 \left\| \left(\Lambda_j P_{W_j} - \Gamma_j P_{V_j} \right) S_\Gamma^{-1} f \right\|^2 \leq D \left\| S_\Gamma^{-1} f \right\|^2 \leq \frac{D}{A_2^2} \| f \|^2. \tag{4.17}$$

Now, by Minkowski inequality, for each $f \in H$, we have

$$\begin{aligned}
&\sum_{j \in J} v_j^2 \left\| \left(\Lambda_j^{\circ} P_{W_j^{\circ}} - \Gamma_j^{\circ} P_{V_j^{\circ}} \right) (f) \right\|^2 \\
&= \sum_{j \in J} v_j^2 \left\| \left(\Lambda_j P_{W_j} S_{\Lambda}^{-1} P_{S_{\Lambda}^{-1} W_j} - \Gamma_j P_{V_j} S_{\Gamma}^{-1} P_{S_{\Gamma}^{-1} V_j} \right) (f) \right\|^2 \\
&= \sum_{j \in J} v_j^2 \left\| \left(\Lambda_j P_{W_j} S_{\Lambda}^{-1} - \Gamma_j P_{V_j} S_{\Gamma}^{-1} \right) (f) \right\|^2 \quad [\text{by Theorem (2.1)}] \\
&= \sum_{j \in J} \left\| v_j \Lambda_j P_{W_j} \left(S_{\Lambda}^{-1} - S_{\Gamma}^{-1} \right) (f) + v_j \left(\Lambda_j P_{W_j} - \Gamma_j P_{V_j} \right) S_{\Gamma}^{-1} (f) \right\|^2 \\
&\leq \left(\left(\sum_{j \in J} v_j^2 \left\| \Lambda_j P_{W_j} \left(S_{\Lambda}^{-1} - S_{\Gamma}^{-1} \right) f \right\|^2 \right)^{\frac{1}{2}} \right. \\
&\qquad \left. + \left(\sum_{j \in J} v_j^2 \left\| \left(\Lambda_j P_{W_j} - \Gamma_j P_{V_j} \right) S_{\Gamma}^{-1} f \right\|^2 \right)^{\frac{1}{2}} \right)^2 \\
&\leq \left(\frac{B_1^{\frac{1}{2}}}{A_1 A_2} D^{\frac{1}{2}} \left(B_1^{\frac{1}{2}} + B_2^{\frac{1}{2}} \right) + \frac{D^{\frac{1}{2}}}{A_2} \right)^2 \| f \|^2 \quad [\text{using (4.16) and (4.17)}] \\
&= D \left(\frac{B_1^{\frac{1}{2}}}{A_1 A_2} \left(B_1^{\frac{1}{2}} + B_2^{\frac{1}{2}} \right) + \frac{1}{A_2} \right)^2 \| f \|^2 \\
&= D \left(\frac{A_1 + B_1 + B_1^{\frac{1}{2}} B_2^{\frac{1}{2}}}{A_1 A_2} \right)^2 \| f \|^2.
\end{aligned}$$

This completes the proof of (I).

Proof of (II). Since $S_{\Lambda} - S_{\Gamma}$ is self-adjoint so

$$\begin{aligned}
\| S_{\Lambda} - S_{\Gamma} \| &= \sup_{\| f \| = 1} | \langle (S_{\Lambda} - S_{\Gamma}) f , f \rangle | = \sup_{\| f \| = 1} | \langle S_{\Lambda} f , f \rangle - \langle S_{\Gamma} f , f \rangle | \\
&= \sup_{\| f \| = 1} \left| \sum_{j \in I} v_j^2 \left\| \Lambda_j P_{W_j} (f) \right\|^2 - \sum_{j \in I} v_j^2 \left\| \Gamma_j P_{V_j} (f) \right\|^2 \right| \leq D.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left\| S_{\Lambda}^{-1} - S_{\Gamma}^{-1} \right\| &\leq \left\| S_{\Lambda}^{-1} \right\| \| S_{\Lambda} - S_{\Gamma} \| \left\| S_{\Gamma}^{-1} \right\| \\
&\leq \frac{1}{A_1} D \frac{1}{A_2} = \frac{D}{A_1 A_2}. \qquad (4.18)
\end{aligned}$$

Now, for each $f \in H$, we have

$$\begin{aligned}
\sum_{j \in J} v_j^2 \left\| \Lambda_j^{\circ} P_{W_j^{\circ}} (f) \right\|^2 &= \sum_{j \in J} v_j^2 \left\| \Lambda_j P_{W_j} S_{\Lambda}^{-1} P_{S_{\Lambda}^{-1} W_j} (f) \right\|^2 \\
&= \sum_{j \in J} v_j^2 \left\| \Lambda_j P_{W_j} \left(S_{\Lambda}^{-1} f \right) \right\|^2 \quad [\text{by Theorem (2.1)}] \\
&= \sum_{j \in J} \left\langle v_j^2 \Lambda_j P_{W_j} \left(S_{\Lambda}^{-1} f \right) , \Lambda_j P_{W_j} \left(S_{\Lambda}^{-1} f \right) \right\rangle \\
&= \left\langle \sum_{j \in J} v_j^2 P_{W_j} \Lambda_j^{*} \Lambda_j P_{W_j} \left(S_{\Lambda}^{-1} f \right) , S_{\Lambda}^{-1} f \right\rangle
\end{aligned}$$

$$= \left\langle\, S_{\Lambda}\left(S_{\Lambda}^{-1} f\right),\, S_{\Lambda}^{-1} f \,\right\rangle = \left\langle\, f,\, S_{\Lambda}^{-1} f \,\right\rangle.$$

Similarly it can be shown that

$$\sum_{j \in J} v_j^2 \left\| \Gamma_j^{\circ} P_{V_j^{\circ}}(f) \right\|^2 = \left\langle\, f,\, S_{\Gamma}^{-1} f \,\right\rangle \quad \forall\, f \in H.$$

Then for each $f \in H$, we have

$$\left| \sum_{j \in J} v_j^2 \left\| \Lambda_j^{\circ} P_{W_j^{\circ}}(f) \right\|^2 - \sum_{j \in J} v_j^2 \left\| \Gamma_j^{\circ} P_{V_j^{\circ}}(f) \right\|^2 \right|$$
$$= \left| \left\langle f,\, S_{\Lambda}^{-1} f \right\rangle - \left\langle f,\, S_{\Gamma}^{-1} f \right\rangle \right| = \left| \left\langle f,\, \left(S_{\Lambda}^{-1} - S_{\Gamma}^{-1}\right) f \right\rangle \right|$$
$$\leq \left\| S_{\Lambda}^{-1} - S_{\Gamma}^{-1} \right\| \|f\|^2 \leq \frac{D}{A_1 A_2} \|f\|^2 \; [\text{ by } (4.18)\,].$$

This completes the proof. □

Remark 4.3. Another representation of the statement (II) is given by, if the condition

$$\left\| \sum_{j \in J} v_j^2 \left(P_{W_j} \Lambda_j^* \Lambda_j P_{W_j}(f) - P_{V_j} \Gamma_j^* \Gamma_j P_{V_j}(f) \right) \right\| \leq D \|f\|$$

holds for each $f \in H$ and for some $D > 0$ then for all $f \in H$,

$$\left\| \sum_{j \in J} v_j^2 \left(P_{W_j^{\circ}} (\Lambda_j^{\circ})^* \Lambda_j^{\circ} P_{W_j^{\circ}}(f) - P_{V_j^{\circ}} (\Gamma_j^{\circ})^* \Gamma_j^{\circ} P_{V_j^{\circ}}(f) \right) \right\| \leq \frac{D}{A_1 A_2} \|f\|.$$

Proof. In this case, we also find that

$$\|S_{\Lambda} - S_{\Gamma}\| = \sup_{\|f\| = 1} \|S_{\Lambda} f - S_{\Gamma} f\|$$
$$= \sup_{\|f\| = 1} \left\| \sum_{j \in J} v_j^2 \left(P_{W_j} \Lambda_j^* \Lambda_j P_{W_j}(f) - P_{V_j} \Gamma_j^* \Gamma_j P_{V_j}(f) \right) \right\|$$
$$\leq \sup_{\|f\| = 1} D \|f\| = D.$$

Then for each $f \in H$,

$$\left\| \sum_{j \in I} v_j^2 \left(P_{W_j^{\circ}} (\Lambda_j^{\circ})^* \Lambda_j^{\circ} P_{W_j^{\circ}}(f) - P_{V_j^{\circ}} (\Gamma_j^{\circ})^* \Gamma_j^{\circ} P_{V_j^{\circ}}(f) \right) \right\|$$
$$= \|S_{\Lambda^{\circ}}(f) - S_{\Gamma^{\circ}}(f)\| = \left\| S_{\Lambda}^{-1} f - S_{\Gamma}^{-1} f \right\| \; [\text{ using } (4.14)\,]$$
$$\leq \left\| S_{\Lambda}^{-1} - S_{\Gamma}^{-1} \right\| \|f\| \leq \frac{D}{A_1 A_2} \|f\| \; [\text{ using } (4.18)\,].$$

□

Theorem 4.4. *Let* $K \in \mathcal{B}(H)$ *and* $\Lambda = \{(W_j, \Lambda_j, v_j)\}_{j \in J}$ *be a K-g-fusion frame for* H *with frame operator* S_{Λ}. *Let* $U \in \mathcal{B}(H)$ *be an invertible operator on* H. *Then the following statements are equivalent:*

(I) $\Gamma = \left\{ \left(U W_j,\, \Lambda_j P_{W_j} U^*,\, v_j \right) \right\}_{j \in J}$ *is a* UK*-g-fusion frame.*

(II) *The quotient operator* $\left[(UK)^* \,/\, S_{\Lambda}^{\frac{1}{2}} U^* \right]$ *is bounded.*

(III) *The quotient operator* $\left[(UK)^* \,/\, (U S_{\Lambda} U^*)^{\frac{1}{2}} \right]$ *is bounded.*

Proof. $(I) \Rightarrow (II)$ Since Γ is a UK-g-fusion frame, $\exists\, A, B > 0$ such that for all $f \in H$, we have

$$A \left\| (UK)^* f \right\|^2 \leq \sum_{j \in J} v_j^2 \left\| \Lambda_j P_{W_j} U^* P_{UW_j}(f) \right\|^2 \leq B \|f\|^2. \tag{4.19}$$

By Theorem (2.1), for each $f \in H$, we obtain

$$\sum_{j \in J} v_j^2 \left\| \Lambda_j P_{W_j} U^* P_{UW_j}(f) \right\|^2 = \sum_{j \in J} v_j^2 \left\| \Lambda_j P_{W_j} (U^* f) \right\|^2$$
$$= \langle S_\Lambda (U^* f), U^* f \rangle = \left\| S_\Lambda^{\frac{1}{2}} (U^* f) \right\|^2$$

and therefore from (4.19),

$$A \left\| (UK)^* f \right\|^2 \leq \left\| S_\Lambda^{\frac{1}{2}} (U^* f) \right\|^2. \tag{4.20}$$

Let us define the operator $T : \mathcal{R}\left(S_\Lambda^{\frac{1}{2}} U^* \right) \to \mathcal{R}\left((UK)^* \right)$ by

$$T\left(S_\Lambda^{\frac{1}{2}} U^* f \right) = (UK)^* f \quad \forall\, f \in H.$$

Then it can be easily verify that T is a linear operator and $\mathcal{N}\left(S_\Lambda^{\frac{1}{2}} U^* \right) \subset \mathcal{N}\left((UK)^* \right)$. Thus T is well-defined quotient operator. Also for each $f \in H$,

$$\left\| T\left(S_\Lambda^{\frac{1}{2}} U^* f \right) \right\| = \|(UK)^* f\| \leq \frac{1}{\sqrt{A}} \left\| S_\Lambda^{\frac{1}{2}} (U^* f) \right\| \quad [\text{ using } (4.20)\,]$$

and hence T is bounded.

$(II) \Rightarrow (III)$ Suppose the quotient operator $\left[(UK)^* \,/\, S_\Lambda^{\frac{1}{2}} U^* \right]$ is bounded. Then for each $f \in H, \exists\, B > 0$ such that

$$\left\| (UK)^* f \right\|^2 \leq B \left\| S_\Lambda^{\frac{1}{2}} (U^* f) \right\|^2 = B \langle S_\Lambda (U^* f), U^* f \rangle$$
$$= B \langle U S_\Lambda U^* f, f \rangle$$
$$= B \left\| (U S_\Lambda U^*)^{\frac{1}{2}} f \right\|^2 \quad [\text{ since } U S_\Lambda U^* \text{ is self-adjoint }].$$

Hence, the quotient operator $\left[(UK)^* \,/\, (U S_\Lambda U^*)^{\frac{1}{2}} \right]$ is bounded.

$(III) \Rightarrow (I)$ Suppose that the quotient operator $\left[(UK)^* \,/\, (U S_\Lambda U^*)^{\frac{1}{2}} \right]$ is bounded. Then for each $f \in H$, $\exists\, B > 0$ such that

$$\left\| (UK)^* f \right\|^2 \leq B \left\| (U S_\Lambda U^*)^{\frac{1}{2}} f \right\|^2. \tag{4.21}$$

Now, by Theorem (2.1), for each $f \in H$, we have

$$\sum_{j \in J} v_j^2 \left\| \Lambda_j P_{W_j} U^* P_{UW_j}(f) \right\|^2 = \sum_{j \in J} v_j^2 \left\| \Lambda_j P_{W_j} (U^* f) \right\|^2$$
$$= \langle S_\Lambda (U^* f), U^* f \rangle = \left\| (U S_\Lambda U^*)^{\frac{1}{2}} f \right\|^2 \geq \frac{1}{B} \left\| (UK)^* f \right\|^2 \quad [\text{ by } (4.21)\,].$$

Also, since Λ is a K-g-fusion frame, $\exists\; C > 0$ such that

$$\sum_{j \in J} v_j^2 \left\| \Lambda_j P_{W_j} U^* P_{UW_j}(f) \right\|^2 = \sum_{j \in J} v_j^2 \left\| \Lambda_j P_{W_j} (U^* f) \right\|^2$$
$$\leq C \left\| U^* f \right\|^2 \leq C \|U\|^2 \|f\|^2 \quad \forall\, f \in H.$$

Hence, Γ is a UK-g-fusion frame. This completes the proof. □

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