Methods of Functional Analysis and Topology
Vol. 26 (2020), no. 3, pp. 272–282
https://doi.org/10.31392/MFAT-npu26_3.2020.08

WEAK SOLUTION FOR FRACTIONAL $p(x)$-LAPLACIAN PROBLEM WITH DIRICHLET-TYPE BOUNDARY CONDITION

ABDELALI SABRI, AHMED JAMEA, AND HAMAD TALIBI ALAOUI

Abstract. In the present paper, we prove the existence and uniqueness result of weak solutions to a class of fractional $p(x)$-Laplacian problem with Dirichlet-type boundary condition, the main tool used here is the varitional method combined with the theory of fractional Sobolev spaces with variable exponent.

Для одного класу задач із дробовим $p(x)$-лапласіаном з граничною умовою типу Дирихле доведено теорему про існування та єдиність слабкого розв'язку. Використовуються варіаційний метод і теорія дробових просторів Соболева змінного порядку.

1. Introduction

This paper is devoted to study the existence and uniqueness question of weak solutions for the fractional $p(x)$-Laplacian problem

$$\begin{cases} u + \left(-\Delta_{p(x)}\right)^{s}(u-\Theta(u)) + \alpha(u) = f \quad in\ \Omega, \\ u = 0 \quad on\ \ \partial\Omega, \end{cases} \tag{1.1}$$

where $(-\Delta)^{s}_{p(x)}$ is the fractional $p(x)$-Laplacian operator which can be defined as

$$\left(-\Delta_{p(x)}\right)^{s} u(x) = P\cdot V\cdot \int_{\Omega} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{N+sp(x,y)}}dy, for\ all\ x\in\Omega,$$

and $P.V.$ is a commonly used abbreviation in the principal value sense. Ω is a bounded open domain of $\mathbb{R}^N (N \geq 3)$. $p : \overline{\Omega}\times\overline{\Omega} \to (1,\infty)$ is a continuous function with $s\times p(x,y) < N$ for any $(x,y)\in\overline{\Omega}\times\overline{\Omega}$. s is a fixed number between 0 and 1. α is a non decreasing continuous real function defined on $\mathbb{R}$ and Θ is a continuous function defined from $\mathbb{R}$ to $\mathbb{R}$, the datum f is in L^∞.
Note that $\left(-\Delta_{p(x)}\right)^{s}$ is a generalized operator of fractional p-Laplacian operator $\left(-\Delta_{p}\right)^{s}$ (i.e., when $p(x,y) = p =$ constant) and it is the fractional version of the $p(x)$-Laplacian operator $\Delta_{p(x)}u = \operatorname{div}\left(|\nabla u|^{p(x)-2}u\right)$ which is associated with the variable exponent Sobolev space.

A very interesting area of nonlinear analysis lies in the study of elliptic equations involving fractional operators. Recently, great attention has been focused on these problems, both for pure mathematical research and in view of concrete real-world applications. Indeed, this type of operator arises in a quite natural way in different contexts, such as the description of several physical phenomena, optimization, population dynamics and mathematical finance. The fractional Laplacian operator $(-\Delta)^s$, $0 < s < 1$, also provides a simple model to describe some jump Lévy processes in probability theory (see for example [2], [8], [9], [11], [20] and the references therein).
As examples of applications of problem (1.1), we state the following two models:

2020 *Mathematics Subject Classification.* 34A08, 35A15, 34D30, 35J92.
Keywords. Existence, fractional $p(x)$-Laplacian, uniqueness, variable exponent Sobolev fractional space, weak solution.

• ***Model 1. Filtration in a porous medium*** . The filtration phenomena of fluids in porous media are modeled by the following equation,

$$\frac{\partial c(p)}{\partial t} = \nabla a[k(c(p))(\nabla p + e)], \tag{1.2}$$

where p is the unknown pressure, c volumetric moisture content, k the hydraulic conductivity of the porous medium, a the heterogeneity matrix and $-e$ is the direction of gravity.

• ***Model 2. Fluid flow through porous media.*** This model is governed by the following equation,

$$\frac{\partial \theta}{\partial t} - div\left(|\nabla\varphi(\theta) - K(\theta)e|^{p-2}(\nabla\varphi(\theta) - K(\theta)e)\right) = 0, \tag{1.3}$$

where θ is the volumetric content of moisture, $K(\theta)$ the hydraulic conductivity, $\varphi(\theta)$ the hydrostatic potential and e is the unit vector in the vertical direction.
In last years, a large number of papers are written on fractional Sobolev spaces and nonlocal problems driven by this operator (see for example [3], [7], [8], [9], [10], [22], [23] and [24] for further details). Specifically, we refer to Di Nezza, Palatucci and Valdinoci [10], for a full introduction to study the fractional Sobolev spaces and the fractional p-Laplacian operators.
On the other hand, attention has been paid to the study of partial differential equations involving the $p(x)$-Laplacian operators (see [13], [14], [15], [16], [18], [21] and the references therein). So the natural question that arises is to see which result can be obtained, if we replace the $p(x)$-Laplacian operator by its fractional version (the fractional $p(x)$-Laplacian operator). Currently, as far as we know, the only results for fractional Sobolev spaces with variable exponents and fractional $p(x)$-Laplacian operator are obtained by [4], [5], [12], [17] and [25]. In particular, the authors generalized the last operator to fractional case. Then, they introduced an appropriate functional space to study problems in which a fractional variable exponent operator is present.

In [6] and [26], the authors used the Browder-Minty Theorem to establish the existence of weak solutions, they proved the boundedness, the coerciveness, the hemi-continuity, and the monotonicity condition of the operator to achieve their work. Motiveted by the ideas in [6] and [26], we will show the existence and uniqueness of weak solutions for problem (1.1) in the fractional Sobolev space with variable exponent, using the variational method under the conditions on α, Θ and f (see $(H_1), (H_2)$ and (H_3) below) . In the particular case when $\Theta = 0$, the existence of weak solutions for problem (1.1) was treated by several authors (see for example [5] and [17]).
The plan of our paper is divided into three sections, organized as follows: In Section 2, we present some preliminaries on fractional Sobolev spaces with variable exponent and some basic tools to prove Theorem 3.2 . In Section 3, we introduce the assumptions and we give the definition of weak solution of problem (1.1), we finish this section by proving the main result.

2. Preliminaries and notations

In this section, we will recall some notations and definitions and we will state some results which will be used in this work.
We introduce the fractional Sobolev space with the variable exponent as it is defined in [17].
Let Ω be a smooth bounded open set in $\mathbb{R}^N$ and let $p : \overline{\Omega} \times \overline{\Omega} \longrightarrow (1, +\infty)$ and $q : \overline{\Omega} \rightarrow$

$(1, \infty)$ be two continuous functions such that

$$1 < p^- = \min_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x,y) \leqslant p(x,y) \leqslant p^+ = \max_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x,y) < +\infty \tag{2.1}$$

and

$$1 < q^- = \inf_{x\in\overline{\Omega}} q(x) \leq q^+ = \sup_{x\in\mathbb{R}} q(x) < \infty.$$

We set

$$C^+(\overline{\Omega}) = \{q : \overline{\Omega} \to \mathbb{R}^+ : \text{q is continuous and such that } 1 < q_- < q_+ < \infty\}$$

and

$$\overline{p}(x) = p(x,x) \text{ for all } x \in \overline{\Omega}.$$

We assume that

$$p \text{ is symmetric, that is, } p(x,y) = p(y,x) \text{ for all } (x,y) \in \overline{\Omega}\times\overline{\Omega}. \tag{2.2}$$

For $0 < s < 1$, we define the fractional Sobolev space with variable exponent via the Gagliardo approach as follows:

$$W^{s,q(x),p(x,y)}(\Omega) = \left\{ u \in L^{q(x)}(\Omega) : \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{sp(x,y)+N}} dxdy < +\infty, \text{ for some } \lambda > 0 \right\},$$

where $L^{q(x)}(\Omega)$ is the variable exponent Lebesgue space.
Let

$$[u]_{s,p(x,y)}(\Omega) = \inf\left\{ \lambda > 0 : \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}} dxdy \leq 1 \right\}.$$

It is the variable exponent seminorm. For simplicity, we omit the set Ω from the notation. The space $W^{s,q(x),p(x,y)}(\Omega)$ is a Banach space with the norm

$$\|u\|_{W^{s,q(x),p(x,y)}(\Omega)} = \|u\|_{L^{q(x)}(\Omega)} + [u]_{s,p(x,y)},$$

where

$$\|u\|_{L^{q(x)}(\Omega)} = inf\left\{ \lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{q(x)} dx \leqslant 1 \right\}.$$

By $W_0^{s,q(x),p(x,y)}(\Omega)$, we denote the subspace of $W^{s,q(x),p(x,y)}(\Omega)$ which is the closure of compactly supported functions in Ω with respect to norm $\|\cdot\|_{W^{s,q(x),p(x,y)}(\Omega)}$. In particular, if $q(x) = \overline{p}(x)$ for all $x \in \overline{\Omega}$, we denote $W^{s,q(x),p(x,y)}(\Omega)$ and $W_0^{s,q(x),p(x,y)}(\Omega)$ by $W^{s,p(x,y)}(\Omega)$ and $W_0^{s,p(x,y)}(\Omega)$ (see [17]), respectively.

Definition 2.1. Let $p : \overline{\Omega}\times\overline{\Omega} \longrightarrow]1, +\infty[$, be a continuous variable exponent and let $s \in (0,1)$. For any $u \in W^{s,q(x),p(x,y)}(\Omega)$, we define the modular $\rho_{p(x,y)} : W^{s,q(x),p(x,y)}(\Omega) \longrightarrow \mathbb{R}$ by

$$\rho_{p(x,y)}(u) = \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dxdy + \int_\Omega |u(x)|^{q(x)} dx$$

and

$$\|u\|_{\rho_{p(x,y)}} = inf\left\{ \lambda > 0 : \rho_{p(x,y)}\left(\frac{u}{\lambda}\right) \leqslant 1 \right\}.$$

It is easy to see that $\|\cdot\|_{\rho_{p(x,y)}}$ is a norm which is equivalent to norm $\|\cdot\|_{W^{s,q(x),p(x,y)}(\Omega)}$.

Lemma 2.2 ([25]). *$\left(W^{s,q(x),p(x,y)}(\Omega), \|\cdot\|_{\rho_{p(x,y)}}\right)$ is uniformly convex and the space $W^{s,q(x),p(x,y)}(\Omega)$ is a reflexive Banach space.*

We have the following properties:

Lemma 2.3 ([25], Lemma 2.1). . *Let $p : \overline{\Omega} \times \overline{\Omega} \longrightarrow (1, +\infty)$, be a continuous variable exponent and let $s \in (0,1)$. For any $u \in W_0^{s,q(x),p(x,y)}(\Omega)$, we have*

i): $1 \leq [u]_{s,p(x,y)} \Rightarrow [u]^{p^-}_{s,p(x,y)} \leq \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dxdy \leq [u]^{p^+}_{s,p(x,y)}$.

ii): $[u]_{s,p(x,y)} \leq 1 \Rightarrow [u]^{p^+}_{s,p(x,y)} \leq \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dxdy \leq [u]^{p-}_{s,p(x,y)}$.

We would like to mention that the continuous and compact embedding theorem is proved in [17] under the assumption $q(x) > \overline{p}(x)$ for all $x \in \overline{\Omega}$. Here, we give a slightly different version of compact embedding theorem assuming that $q(x) = \overline{p}(x)$ for all $x \in \overline{\Omega}$, which can be obtained by following the same discussions in [17].

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain and $s \in (0,1)$. Let $p(x,y)$ be continuous variable exponent with $s \times p(x,y) < N$ for all $(x,y) \in \overline{\Omega} \times \overline{\Omega}$. Let 2.1 and 2.2 be satisfied. Assume that $r : \overline{\Omega} \longrightarrow (1, +\infty)$ is a continuous variable exponent such that*

$$p_s^*(x) = \frac{N\overline{p}(x)}{N - s\overline{p}(x)} > r(x) \geqslant r^- = \min_{x \in \overline{\Omega}} r(x) > 1 \quad \textit{for all } x \in \overline{\Omega}.$$

Then, there exists a positive constant $C = C(N, s, p, r, \Omega)$ such that, for any $u \in W^{s,p(x,y)}(\Omega)$

$$\|u\|_{L^{r(x)}(\Omega)} \leqslant C\|u\|_{W^{s,p(x,y)}(\Omega)}.$$

Thus, the space $W^{s,p(x,y)}(\Omega)$ is continuously embedded in $L^{r(x)}(\Omega)$ for any $r \in (1, p_s^*)$. Moreover, this embedding is compact.

Remark 2.5. i) Theorem 2.4 remains true if we replace $W^{s,p(x,y)}(\Omega)$ by $W_0^{s,p(x,y)}(\Omega)$.

ii) Since $\frac{N\overline{p}(x)}{N-s\overline{p}(x)} > \overline{p}(x) \geq p_- > 1$ for all $x \in \overline{\Omega}$, then Theorem 2.4 implies that $[u]_{s,p(x,y)}$ is a norm on $W_0^{s,p(x,y)}(\Omega)$, which is equivalent to the norm $\|\cdot\|_{W^{s,p(x,y)}(\Omega)}$.

Let $q' \in C_+(\overline{\Omega})$ be the conjugate exponent of q, that is, $\frac{1}{q(x)} + \frac{1}{q'(x)} = 1$ for all $x \in \overline{\Omega}$, then we have the following Hölder-type inequality :

Lemma 2.6 ([16]). ***(Hölder-type inequality)****. If $u \in L^{q(x)}(\Omega)$ and $v \in L^{q'(x)}(\Omega)$, then*

$$\left|\int_\Omega uvdx\right| \leqslant \left(\frac{1}{q^-} + \frac{1}{q'^-}\right) \|u\|_{L^{q(x)}(\Omega)}\|v\|_{L^{q'(.)}(\Omega)} \leqslant 2\|u\|_{L^{q(x)}(\Omega)}\|v\|_{L^{q'(x)}(\Omega)}.$$

Definition 2.7 ([19]). Let Y be a reflexive Banach space and let P be an operator from Y to its dual Y'. We say that P is monotone if and only if

$$\langle Pu - Pv, u - v\rangle \geq 0, \quad \forall\, u, v \in Y.$$

Theorem 2.8 ([19]). *Let Y be a reflexive real Banach space and $P : Y \longrightarrow Y'$ be a bounded operator, hemi-continuous, coercive and monotone on space Y. Then, the equation $Pu = h$ has at least one solution $u \in Y$ for each $h \in Y'$.*

Lemma 2.9 ([1]). *For $\xi,\ \eta \in \mathbb{R}^N$ and $1 < p < \infty$, we have*

$$\frac{1}{p}|\xi|^p - \frac{1}{p}|\eta|^p \leq |\xi|^{p-2}\xi(\xi - \eta).$$

Lemma 2.10. *For $a \geq 0$, $b \geq 0$ and $1 \leq p < +\infty$, we have*

$$(a+b)^p \leq 2^{p-1}(a^p + b^p).$$

3. Assumptions and main result

In this section, we will introduce the concept of weak solutions for problem (1.1) and we will state the existence and uniqueness result for this type of solutions. Firstly, we cite the following assumptions

(H_1) : α is a non decreasing continuous real function defined on $\mathbb{R}$, surjective such that $\alpha(0)=0$ and there exists a positive constant λ_1 such that $|\alpha(z)| \leq \lambda_1 |z|^{\overline{p}(x)-1}$ for all $z \in \mathbb{R}$ and $x \in \overline{\Omega}$.

(H_2) : Θ is a continuous function from $\mathbb{R}$ to $\mathbb{R}$ such that for all real numbers x, y, we have $|\Theta(x)-\Theta(y)| \leq \lambda_2 |x-y|$, where λ_2 is a real constant such that $0<\lambda_2<\frac{1}{2}$.

(H_3) : $f \in L^{\infty}(\Omega)$.

Definition 3.1. A function $u \in W_0^{s,p(x,y)}(\Omega)$ is called a weak solution to problem (1.1) if and only if

$$\int_\Omega uv dx + \int_\Omega\int_\Omega \frac{|\psi^u_\Theta(x,y)|^{p(x,y)-2}\psi^u_\Theta(x,y)}{|x-y|^{N+sp(x,y)}}(v(x)-v(y))\,dx\,dy + \int_\Omega \alpha(u)v\,dx = \int_\Omega fv dx \tag{3.1}$$

for all $v \in W_0^{s,p(x,y)}(\Omega)$, where

$$\psi^u_\Theta(x,y) = u(x)-u(y)-\Theta(u(x))+\Theta(u(y)).$$

Our main result of this work is the following Theorem

Theorem 3.2. *Let $p : \overline{\Omega}\times\overline{\Omega} \longrightarrow (1,+\infty)$, be a continuous variable exponent satisfying* (2.1) *and* (2.2) *and let $s \in (0,1)$, with $s \times p(x,y) < N$ for all $(x,y) \in \overline{\Omega}\times\overline{\Omega}$. If hypotheses $(H_1),(H_2)$ and (H_3) hold, then, the problem* (1.1) *has a unique weak solution.*

Proof. **Existence part**. Let the operator $T : W_0^{s,p(x,y)}(\Omega) \longrightarrow (W_0^{s,p(x,y)}(\Omega))'$ (where $(W_0^{s,p(x,y)}(\Omega))'$ is the dual space of $(W_0^{s,p(x,y)}(\Omega))$ and let

$$T = A + L,$$

where for all $u, v \in W_0^{s,p(x,y)}(\Omega)$

$$\begin{aligned}\langle Au, v\rangle &= \int_\Omega\int_\Omega \frac{|\psi^u_\Theta(x,y)|^{p(x,y)-2}\psi^u_\Theta(x,y)}{|x-y|^{N+sp(x,y)}}(v(x)-v(y))dxdy + \int_\Omega \alpha(u)v\,dx \\ &:= \langle A_1u,v\rangle + \langle A_2 u, v\rangle\end{aligned}$$

and

$$\langle Lu, v\rangle = \int_\Omega uv\,dx - \int_\Omega fv\,dx.$$

The proof of existence part of Theorem 3.2 is divided into several steps.

• **Step 1. The operator T is bounded.**

On the one hand, we use Hölder-type inequality, hypothesis (H_2) and Lemma 2.10, we have for any $u, v \in W_0^{s,p(x,y)}(\Omega)$,

$$\begin{aligned}|\langle A_1 u, v\rangle| &\leq \int_\Omega\int_\Omega \frac{|\psi^u_\Theta(x,y)|^{p(x,y)-1}}{|x-y|^{N+sp(x,y)}}|v(x)-v(y)|\,dx\,dy \\ &\leq 2^{p^+-2}\int_\Omega\int_\Omega \Big(\frac{|u(x)-u(y)|^{p(x,y)-1}}{|x-y|^{N+sp(x,y)}} \\ &\qquad + \frac{|\Theta(u(x))-\Theta(u(y))|^{p(x,y)-1}}{|x-y|^{N+sp(x,y)}}\Big)|v(x)-v(y)|\,dx\,dy \\ &\leq 2^{p^+-2}(\lambda_2^{p^+-1}+1)\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^{p(x,y)-1}}{|x-y|^{N+sp(x,y)}}|v(x)-v(y)|\,dx\,dy\end{aligned}$$

$$
\begin{aligned}
&\leq C_0\left(\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy\right)^{\frac{p(x,y)-1}{p(x,y)}}\\
&\qquad\times\left(\int_\Omega\int_\Omega \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy\right)^{\frac{1}{p(x,y)}}\\
&\leq C_0\|u\|^{p(x,y)-1}_{W_0^{s,p(x,y)}(\Omega)}\|v\|_{W_0^{s,p(x,y)}(\Omega)}\\
&\leq C_0\max\left(\|u\|^{p^+-1}_{W_0^{s,p(x,y)}(\Omega)},\|u\|^{p^--1}_{W_0^{s,p(x,y)}(\Omega)}\right)\|v\|_{W_0^{s,p(x,y)}(\Omega)},
\end{aligned}
$$

where $C_0=2^{p^+-1}(\lambda_2^{p^+-1}+1)$. This implies that A_1 is bounded. On the other hand, using again Hölder-type inequality, hypothesis (H_1) and Theorem 2.4, we get

$$
\begin{aligned}
|\langle A_2u,v\rangle| &\leq \lambda_1\int_\Omega |u|^{\overline{p}(x)-1}|v|\,dx\\
&\leq 2\lambda_1\|u\|^{\overline{p}(x)-1}_{\overline{p}(x)}\|v\|_{\overline{p}(x)}\\
&\leq 2\lambda_1C_1C_2\|u\|^{\overline{p}(x)-1}_{W_0^{s,p(x,y)}(\Omega)}\|v\|_{W_0^{s,p(x,y)}(\Omega)}\\
&\leq 2\lambda_1C_1C_2\max\left(\|u\|^{p^+-1}_{W_0^{s,p(x,y)}(\Omega)},\|u\|^{p^--1}_{W_0^{s,p(x,y)}(\Omega)}\right)\|v\|_{W_0^{s,p(x,y)}(\Omega)},
\end{aligned}
$$

where C_1, C_2 are two constants of continuous embedding given by Theorem 2.4. Then A_2 is bounded. This allows us to deduce that A is bounded. Finally, by Hölder-type inequality, we get immediately the boundedness of operator L . Hence, the operator T is bounded.

• **Step 2. The operator T is hemi-continuous.**

Let $\{u_n\}_{n\in\mathbb{N}}\subset W_0^{s,p(x,y)}(\Omega)$ and $u\in W_0^{s,p(x,y)}(\Omega)$ such that u_n converges strongly to u in $W_0^{s,p(x,y)}(\Omega)$. Firstly, we will prove that A_1 is continuous on $W_0^{s,p(x,y)}(\Omega)$, indeed,

$$
\begin{aligned}
&\langle A_1u_n-A_1u,v\rangle\\
&\quad=\int_\Omega\int_\Omega\left(\frac{|\psi_\Theta^{u_n}(x,y)|^{p(x,y)-2}\psi_\Theta^{u_n}(x,y)-|\psi_\Theta^{u}(x,y)|^{p(x,y)-2}\psi_\Theta^{u}(x,y)}{|x-y|^{N+sp(x,y)}}\right)\\
&\qquad\times(v(x)-v(y))\,dx\,dy\\
&\quad=\int_\Omega\int_\Omega\left(\frac{|\psi_\Theta^{u_n}(x,y)|^{p(x,y)-2}\psi_\Theta^{u_n}(x,y)}{|x-y|^{(N+sp(x,y))\frac{p(x,y)-1}{p(x,y)}}}-\frac{|\psi_\Theta^{u}(x,y)|^{p(x,y)-2}\psi_\Theta^{u}(x,y)}{|x-y|^{(N+sp(x,y))\frac{p(x,y)-1}{p(x,y)}}}\right)\\
&\qquad\times\frac{(v(x)-v(y))}{|x-y|^{\frac{N+sp(x,y)}{p(x,y)}}}\,dx\,dy.
\end{aligned}
$$

Let us set

$$
F_{\theta,n}(x,y)=\frac{|\psi_\Theta^{u_n}(x,y)|^{p(x,y)-2}\psi_\Theta^{u_n}(x,y)}{|x-y|^{(N+sp(x,y))\frac{p(x,y)-1}{p(x,y)}}}\ \in\ L^{p'(x,y)}(\Omega\times\Omega),
$$

$$
F_{\theta}(x,y)=\frac{|\psi_\Theta^{u}(x,y)|^{p(x,y)-2}\psi_\Theta^{u}(x,y)}{|x-y|^{(N+sp(x,y))\frac{p(x,y)-1}{p(x,y)}}}\ \in\ L^{p'(x,y)}(\Omega\times\Omega),
$$

$$
\varphi(x,y)=\frac{(v(x)-v(y))}{|x-y|^{\frac{N+sp(x,y)}{p(x,y)}}}\ \in\ L^{p(x,y)}(\Omega\times\Omega),
$$

where $\frac{1}{p(x,y)}+\frac{1}{p'(x,y)}=1$, for all $x,y\in\overline{\Omega}\times\overline{\Omega}$.

Then, we have by Hölder-type inequality

$$
\langle A_1u_n-A_1u,v\rangle\leq 2\|F_{\theta,n}-F_\theta\|_{L^{p'(x,y)}(\Omega\times\Omega)}\|\varphi\|_{L^{p(x,y)}(\Omega\times\Omega)}.
$$

This implies that

$$\|A_1u_n - A_1u\|_{(W_0^{s,p(x,y)}(\Omega))'} = \sup_{\|\varphi\|_{L^{p(x,y)}(\Omega\times\Omega)}\leq 1} |\langle A_1u_n - A_1u, v\rangle| \leq 2\|F_{\theta,n} - F_\theta\|_{L^{p'(x,y)}(\Omega\times\Omega)}.$$

Now, we denote

$$Z_{\theta,n}(x,y) = \frac{\psi_\Theta^{u_n}(x,y)}{|x-y|^{\frac{N+sp(x,y)}{p(x,y)}}} \in L^{p(x,y)}(\Omega\times\Omega)$$

$$Z_\theta(x,y) = \frac{\psi_\Theta^{u}(x,y)}{|x-y|^{\frac{N+sp(x,y)}{p(x,y)}}} \in L^{p(x,y)}(\Omega\times\Omega)$$

Since u_n converges to u strongly in $W_0^{s,p(x,y)}(\Omega)$, then

$$Z_{\theta,n}(x,y) \longrightarrow Z_\theta(x,y) \text{ in } L^{p(x,y)}(\Omega\times\Omega).$$

Hence, for a subsequence of $Z_{\theta,n}(x,y)$, we get $Z_{\theta,n}(x,y) \longrightarrow Z_\theta(x,y)$ in $\Omega\times\Omega$ and there exists an $h \in L^{p(x,y)}(\Omega\times\Omega)$ such that $|Z_{\theta,n}(x,y)| \leq h(x,y)$.
So, we have

$$F_{\theta,n}(x,y) \longrightarrow F_\theta(x,y) \text{ a.e in } \Omega\times\Omega$$

and

$$|F_{\theta,n}(x,y)| = |Z_{\theta,n}(x,y)|^{p(x,y)-1} \leq |h(x,y)|^{p(x,y)-1}.$$

Then, by Dominated Convergence Theorem, we deduce that

$$F_{\theta,n}(x,y) \longrightarrow F_\theta(x,y) \text{ in } L^{p'(x,y)}(\Omega\times\Omega).$$

Consequently

$$A_1u_n \longrightarrow A_1u \text{ in } (W_0^{s,p(x,y)}(\Omega))'.$$

This implies that the operator A_1 is continuous on $W_0^{s,p(x,y)}(\Omega)$. Secondly, by application of hypothesis (H_1), we get immediately the continuity of operator A_2. Therefore, T is hemi-continuous on $W_0^{s,p(x,y)}(\Omega)$.

• **Step 3 . The operator T is coercive.**

For any $u \in W_0^{s,p(x,y)}(\Omega)$, we have

$$\begin{aligned}
&\langle Tu, u\rangle \\
&= \int_\Omega u^2\,dx + \int_\Omega\int_\Omega \frac{|\psi_\Theta^u(x,y)|^{p(x,y)-2}\psi_\Theta^u(x,y)}{|x-y|^{N+sp(x,y)}}(u(x)-u(y))\,dx\,dy \\
&\quad + \int_\Omega \alpha(u)u\,dx - \int_\Omega fu\,dx \\
&\geq \int_\Omega\int_\Omega \frac{|\psi_\Theta^u(x,y)|^{p(x,y)-2}\psi_\Theta^u(x,y)}{|x-y|^{N+sp(x,y)}}(u(x)-u(y))\,dx\,dy + \int_\Omega \alpha(u)u\,dx - \int_\Omega fu\,dx.
\end{aligned}$$

On the one hand, by application of hypothesis (H_1), we have

$$\int_\Omega \alpha(u)u\,dx \geq 0.$$

And, by Hölder-type inequality and Theorem 2.4, there exists a positive constant C_3 such that

$$\int_\Omega fu\,dx \leq 2C_3\|f\|_{L^{\overline{p}'(x)}(\Omega)}\|u\|_{W_0^{s,p(x,y)}(\Omega)},$$

where $\frac{1}{\overline{p}'(x)} + \frac{1}{\overline{p}(x)} = 1$.
This implies that

$$\langle Tu, u\rangle \geq \langle A_1u, u\rangle - 2C_3\|f\|_{L^{\overline{p}'(x)}(\Omega)}\|u\|_{W_0^{s,p(x,y)}(\Omega)}. \tag{3.2}$$

On the other hand, using Lemma 2.9, we obtain

$$\begin{aligned}
&\langle A_1 u, u\rangle \\
= &\int_\Omega\int_\Omega \frac{|\psi^u_\Theta(x,y)|^{p-2}\psi^u_\Theta(x,y)}{|x-y|^{N+sp(x,y)}}(u(x)-u(y))\,dx\,dy \\
\geq &\int_\Omega\int_\Omega \frac{|u(x)-u(y)-(\Theta(u(x))-\Theta(u(y)))|^{p(x,y)}-|\Theta(u(x))-\Theta(u(y))|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\,dx\,dy.
\end{aligned}$$

And Lemma 2.10 allows us to deduce that

$$\begin{aligned}
&\frac{1}{2^{p^+-1}}|u(x)-u(y)|^{p(x,y)} \\
= &\frac{1}{2^{p^+-1}}|u(x)-u(y)-(\Theta(u(x))-\Theta(u(y)))+(\Theta(u(x))-\Theta(y))|^{p(x,y)} \\
\leq &|u(x)-u(y)-(\Theta(u(x))-\Theta(u(y)))|^{p(x,y)}+|\Theta(u(x))-\Theta(u(y))|^{p(x,y)}.
\end{aligned}$$

Then

$$\begin{aligned}
&\frac{1}{2^{p^+-1}}|u(x)-u(y)|^{p(x,y)}-|\Theta(u(x))-\Theta(u(y))|^{p(x,y)} \\
&\qquad\qquad\leq |u(x)-u(y)-(\Theta(u(x))-\Theta(u(y)))|^{p(x,y)}.
\end{aligned}$$

Consequently

$$\begin{aligned}
&\langle A_1 u, u\rangle \\
\geq &\int_\Omega\int_\Omega \frac{1}{p(x,y)}\left[\frac{1}{2^{p^+-1}}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}-\frac{2|\Theta(u(x))-\Theta(u(y))|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\right]dx\,dy \\
\geq &\int_\Omega\int_\Omega \frac{1}{p(x,y)}\frac{1}{2^{p^+-1}}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}-\frac{2\lambda_2^{p(x,y)}}{p(x,y)}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy \\
\geq &\frac{1}{p^+}\Big(\frac{1}{2^{p^+-1}}-2\lambda_2^{p^+}\Big)\|u\|^{p(x,y)}_{W_0^{s,p(x,y)}(\Omega)} \\
\geq &\frac{1}{p^+}\Big(\frac{1}{2^{p^+-1}}-2\lambda_2^{p^+}\Big)\min\left(\|u\|^{p^+}_{W_0^{s,p(x,y)}(\Omega)},\|u\|^{p^-}_{W_0^{s,p(x,y)}(\Omega)}\right).
\end{aligned}$$

So, the choice of constant λ_2 in (H_2) gives the existence of a positive constant C_4 such that

$$\langle A_1 u, u\rangle \geq C_4\|u\|^\gamma_{W_0^{s,p(x,y)}(\Omega)},$$

where

$$\gamma=\begin{cases} p^- & \text{if } \|u\|_{W_0^{s,p(x,y)}(\Omega)}\geq 1 \\ p^+ & \text{if } \|u\|_{W_0^{s,p(x,y)}(\Omega)}\leq 1.\end{cases}$$

Then, inequality (3.2) becomes

$$\langle Tu, u\rangle \geq C_4\|u\|^\gamma_{W_0^{s,p(x,y)}(\Omega)}-2C_3\|f\|_{L^{\overline{p}'(x)}(\Omega)}\|u\|_{W_0^{s,p(x,y)}(\Omega)}.$$

Therefore

$$\frac{\langle Tu, u\rangle}{\|u\|_{W_0^{s,p(x,y)}(\Omega)}}\to+\infty \text{ as } \|u\|_{W_0^{s,p(x,y)}(\Omega)}\to+\infty.$$

Hence, the operator T is coercive.

• **Step 4. The operator T is monotone.**

For that, it suffices to prove that A is monotone. Firstly ,we have by application of hypothesis (H_1) that

$$\langle A_2u-A_2v, u-v\rangle=\int_\Omega\Big(\alpha(u)-\alpha(v)\Big)(u-v)\,dx\geq 0 \quad \text{for all } \ u,v\in W_0^{s,p(x,y)}(\Omega).$$

It remains to show that $\langle A_1 u - A_1 v, u - v\rangle \geq 0$. Indeed, we have

$$\begin{aligned}\langle A_1 u - A_1 v, u - v\rangle &= \langle A_1 u, u\rangle + \langle A_1 v, v\rangle - \langle A_1 u, v\rangle - \langle A_1 v, u\rangle \\ &\geq C_4 \Upsilon_1(u,v) - C_0 \Upsilon_2(u,v) \\ &\geq \min(C_0, C_4)(\Upsilon_1(u,v) - \Upsilon_2(u,v)),\end{aligned}$$

where C_0 and C_4 are the two constants getting in the proof of boundedness and coerciveness of operator T and

$$\Upsilon_1(u,v) = \|u\|^{p(x,y)}_{W_0^{s,p(x,y)}(\Omega)} + \|v\|^{p(x,y)}_{W_0^{s,p(x,y)}(\Omega)},$$

$$\Upsilon_2(u,v) = \|u\|^{p(x,y)-1}_{W_0^{s,p(x,y)}(\Omega)}\|v\|_{W_0^{s,p(x,y)}(\Omega)} + \|v\|^{p(x,y)-1}_{W_0^{s,p(x,y)}(\Omega)}\|u\|_{W_0^{s,p(x,y)}(\Omega)}.$$

This implies that

$$\begin{aligned}&\langle A_1 u - A_1 v, u - v\rangle \\ &\geq \min(C_0, C_4)\left[\left(\|u\|^{p(x,y)-1}_{W_0^{s,p(x,y)}(\Omega)} - \|v\|^{p(x,y)-1}_{W_0^{s,p(x,y)}(\Omega)}\right)\left(\|u\|_{W_0^{s,p(x,y)}(\Omega)} - \|v\|_{W_0^{s,p(x,y)}(\Omega)}\right)\right] \\ &\geq 0. \qquad (3.3)\end{aligned}$$

This implies that A_1 is monotone. Therefore T is monotone. Hence, the existence of weak solution for problem (1.1) follows from Theorem 2.8.

Uniqueness part. Let u and w be two weak solutions of problem (1.1). As a test function for the solution u, we take $v = u - w$ in equality (3.1) and for the solution w we take $v = w - u$ as a test function in (3.1), we have

$$\int_\Omega u(u-w)dx + \int_\Omega\int_\Omega \frac{|\psi^u_\Theta(x,y)|^{p(x,y)-2}\psi^u_\Theta(x,y)}{|x-y|^{N+sp(x,y)}}\Big(u(x) - u(y) - \big(w(x) - w(y)\big)\Big)\,dx\,dy$$
$$+\int_\Omega \alpha(u)(u-w)\,dx = \int_\Omega f(u-w)dx$$

and

$$\int_\Omega w(w-u)dx + \int_\Omega\int_\Omega \frac{|\psi^u_\Theta(x,y)|^{p(x,y)-2}\psi^u_\Theta(x,y)}{|x-y|^{N+sp(x,y)}}\Big(w(x) - w(y) - \big(u(x) - u(y)\big)\Big)\,dx\,dy$$
$$+\int_\Omega \alpha(w)(w-u)\,dx = \int_\Omega f(w-u)dx.$$

By summing up the two above equalities, we get

$$\int_\Omega (u-w)^2 dx + \langle A_1 u - A_1 w, u - w\rangle + \int_\Omega \Big(\alpha(u) - \alpha(w)\Big)(u-w)\,dx = 0. \qquad (3.4)$$

On the one hand, we have by application of hypothesis (H_1) that

$$\int_\Omega \Big(\alpha(u) - \alpha(w)\Big)(u-w)\,dx \geq 0.$$

On the other hand, by using (3.3), we deduce that

$$\langle A_1 u - A_1 w, u - w\rangle \geq 0.$$

Therefore, inequality (3.4) becomes

$$\int_\Omega (u-w)^2 dx \leq 0.$$

This implies that

$$u = w \text{ a.e in } \Omega.$$

References

[1] A. Abassi, A. El Hachimi, A. Jamea, Entropy solutions to nonlinear Neumann problems with L^1-data, International Journal of Mathematics and Statistics. 2, no. S08 (2008), pp. 4–17.

[2] D. Applebaum, Lévy processes and stochastic calculus, Second edition. Cambridge Studies in Advanced Mathematics, 116. Cambridge University Press, Cambridge, (2009).

[3] E. Azroul, A. Benkirane, M. Srati, Three solutions for a Kirchhoff type problem involving nonlocal fractional p-Laplacian, Adv. Oper. Theory, a-no. 4, 821- 835, (2019).

[4] A. Bahrouni, Comparison and sub-supersolution principles for the fractional $p(x)$-Laplacian, J. Math. Anal. Appl., 458, no. 2, 1363-1372, (2018).

[5] A. Bahrouni and V. Rădulescu, On a new fractional Sobolev space and applications to nonlocal variational problems with variable exponent, Discrete Contin. Dyn. Syst. Ser. S, 11 no. 3, 379-389, (2018).

[6] G. Benhamida, T. Moussaoui and D. O'Regan, Existence and uniqueness of solutions for fractional p-laplacian problem in $\mathbb{R}^N$, Discussiones Mathematicae, Differential Inclusions, Control and Optimization,38, 5-14, (2018).

[7] P. Bousquet; A. C. Ponce; J. V. Schaftingen, Density of smooth maps for fractional Sobolev spaces Ws,p into l simply connected manifolds when $s \geq 1$, Confluentes Mathematici, 5 (2013) no. 2, 3-24.

[8] C. Bucur and E. Valdinoci, Nonlocal diffusion and applications, Lecture Notes of the Unione Matematica Italiana, 20. Springer, [Cham], Unione Matematica Italiana, Bologna, (2016).

[9] L. Caffarelli, Nonlocal diffusions, drifts and games, Nonlinear partial differential equations, 37-52, Abel Symp., 7, Springer, Heidelberg, (2012).

[10] E. Di Nezza, G. Palatucci and E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, Bull. Sci. Math.,136, no. 5, 521-573, (2012).

[11] G. Devillanova and G. Carlo Marano, A free fractional viscous oscillator as a forced standard damped vibration, Fractional Calculus and Applied Analysis 19(2), 2016, pp. 319-356

[12] L. M. Del Pezzo, and J. D. Rossi, Traces for fractional Sobolev spaces with variable expo- nents, Adv. Oper. Theory 2, 435-446, (2017).

[13] X. L. Fan, Remarks on eigenvalue problems involving the $p(x)$-Laplacian, J. Math. Anal. Appl.,352, no. 1, 85-98, (2009).

[14] X. L. Fan and Q. H. Zhang, Existence of solutions for $p(x)$-Laplacian Dirichlet problem, Nonlinear Anal., 52, 1843-1852, (2003).

[15] X. L. Fan, Q. Zhang, and D. Zhao, Eigenvalues of $p(x)$-Laplacian Dirichlet problem, J. Math. Anal. Appl.,302, 306-317, (2005).

[16] X. L. Fan and D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m;p(x)}(\Omega)$, J. Math. Anal. Appl., 263, 424-446, (2001).

[17] U. Kaufmann, J. D. Rossi, and R. Vidal, Fractional Sobolev spaces with variable exponents and fractional $p(x)$-Laplacians, Electron. J. Qual. Theory Differ. Equ., 2017, no. 76, 1-10, (2017).

[18] O. Kováčik and J. Rákosník, On spaces $L^{p(x)}(\Omega)$ and $W^{m;p(x)}(\Omega)$, Czechoslovak Math. J., 41, no. 4, 592-618, (1991).

[19] J.L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod, Paris, (1969).

[20] G. Molica Bisci, V. Rădulescu, and R. Servadi, Variational methods for nonlocal fractional problems. With a foreword by Jean Mawhin, Encyclopedia of Mathematics and its Applications, 162. Cambridge University Press, Cambridge, (2016).

[21] V. D. Rădulescu and D. Repovš, Partial differential equations with variable exponents: Variational methods and qualitative analysis, Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL, (2015).

[22] R. Servadei and E. Valdinoci, Variational methods for nonlocal operators of elliptic type, Discrete Contin. Dyn. Syst.,33, no. 5, 2105-2137, (2013).

[23] R. Servadei and E. Valdinoci, Mountain pass solutions for nonlocal elliptic operators, J. Math. Anal. Appl., 389, no. 2, 887-898, (2012).

[24] R. Servadei and E. Valdinoci, Weak and viscosity solutions of the fractional Laplace equation, Publ. Mat.,58,133-154, (2014).

[25] C. Zhang and X. Zhang, Renormalized solutions for the fractional $p(x)$-Laplacian equation with L1 data, arXiv:1708.04481v1.

[26] J. Xu, D. O'Regan and W. Dong, Existence of weak solutions for a fractional p-Laplacian equation in Sci. China Math. 111 (2017) 1647-1660.

EMAPI, Faculté des Sciences, Université Chouaib Doukkali, El Jadida, Morocco

EMAPI, Faculté des Sciences, Université Chouaib Doukkali, El Jadida, Morocco; Centre RÃ©gional des MÃ©tiers de l'Education et de Formation Casablanca-Settat S.P. El Jadida, Morocco

EMAPI, Faculté des Sciences, Université Chouaib Doukkali, El Jadida, Morocco

Received 14/06/2020; Revised 02/07/2020