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THE QUENCHED CENTRAL LIMIT THEOREM FOR A MODEL OF RANDOM WALK IN RANDOM ENVIRONMENT

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Abstract. In the present paper we provide a proof of the quenched central limit theorem for the random walk in random environment model introduced by Boldrighini, Minlos, and Pellegrinotti in [3].

У цій статті дано доведення квенч-центральної граничної теореми для випадкових блукань у моделі з випадковим середовищем, запропонованій Болдрігіні, Мінлосом і Пеллегринотті [3].

1. Introduction

In this article we prove the quenched Central Limit Theorem (CLT) for a model of random walk in random environment, as it has been introduced by Boldrighini, Minlos, and Pellegrinotti, see in particular [3, 4, 6]. At each site the transition probability kernel is affected by the current state of the environment at this site. A more detailed description can be found in Section 2. For a nice overview of the literature on the subject, we refer to [5], and a survey on the recent progress in the related models can be found in [10] or [2]. In [7] the anomalous behavior of the corrections to the CLT in low dimensions has been studied. Moreover, related models are considered in [1] and [9].

We underline that the novelty of this work is that the proofs are essentially based on the the multidimensional martingale CLT by Küchler and Sørensen, see [8].

The paper is organized as follows: in Section 2 we describe the model and give the statement; while in Section 3 we provide all the proofs and some further comments.

2. Model, conditions and results

Consider a particle moving in a n-dimensional infinite lattice and denote by X_t is position at time t. On the lattice, a dynamical random environment is considered. It is described by the random field

$$\xi = \left\{ \xi_t(x) : x \in \mathbb{Z}^n, t \in \mathbb{Z}^+ \right\}$$

Note that the time is discrete. We assume that ξ is the result of independent copies of the same random variable taking values in some finite space $\mathbb{S}$. The space of configurations is given by $\tilde{\Omega} = \mathbb{S}^{\mathbb{Z}^n \times \mathbb{Z}^+}$. Thus, $\left\{ \xi_t(x) \right\}_{(x,t) \in \mathbb{Z}^n \times \mathbb{Z}^+}$ is a collection of i.i.d random variables, distributed according to a given probability measure on $\mathbb{S}$ denoted by π. We denote by Π the distribution of ξ in $\tilde{\Omega}$.

The one step transition probability from position x at time t to position y at the subsequent time step $t+1$ is given by

$$\mathbb{P}\left\{ X_{t+1} = y | X_t = x, \xi \right\} = P(y-x, \xi_t(x)) = P_0(y-x) + c(y-x, \xi_t(x))$$

where P_0 is the transition probability of a free random walk and c is the function which provides the influence of the environment on the particle's dynamic. We note that in the original work [3] there was a small factor ε before the function c.

Keywords. Random walk in random environment; quenched central limit theorem; multidimensional martingale central limit theorem.

In order for the probability P to be well-defined, the following conditions must be fulfilled:

- $0 \le P(u,s) = P_0(u) + c(u,s) \le 1 \quad \forall s \in \mathbb{S} \quad \forall u \in \mathbb{Z}^n$;
- $\sum_{u\in\mathbb{Z}^n} c(u,s) = 0 \quad \forall s \in \mathbb{S}$.

Without loss of generality we assume that the random environment has the following property:

$$\sum_{s\in\mathbb{S}} c(u,s)\pi(s) = 0 \text{ for any } u \in \mathbb{Z}^n \,, \tag{2.1}$$

which means that P_0 is the mean transition probability. Indeed, if (2.1) is not satisfied, then we can replace $P_0(u)$ with $\tilde{P}_0(u) := P_0(u) + \sum_{s\in\mathbb{S}} c(u,s)\pi(s)$ and c with $\tilde{c}(u,s) = c(u,s) - \tilde{P}_0(u) + P_0(u)$. This replacement wouldn't change the law of the random walk and (2.1) would hold.

Additionally, let P_0 and c be of bounded range. We denote by $\mathbb{P}_\xi$ the conditional probability with respect to the environment ξ.

We will also assume that $\sum_{u\in\mathbb{Z}^n} uc(u,s_1) = \sum_{u\in\mathbb{Z}^n} uc(u,s_2)$ for $s_1, s_2 \in \mathbb{S}$. It then follows from (2.1) that in fact

$$\sum_{u\in\mathbb{Z}^n} uc(u,s) = 0 \quad \forall s \in \mathbb{S} \,. \tag{2.2}$$

Let $Y = \{Y_t\}_{t\in\mathbb{Z}^+}$ be the stochastic processes defined by $Y_t = X_t - tb$, where $b = \sum_{u\in\mathbb{Z}^n} uP_0(u)$. Note that

$$\sum_{u\in\mathbb{Z}^n} (u-b)P_0(u) = 0. \tag{2.3}$$

For a vector $u \in \mathbb{R}^n$, u_i denotes its i-th coordinate.

Theorem 1. *For almost every realization ξ of the random environment we have*

$$\frac{1}{\sqrt{t}} Y_t \Rightarrow \eta^2 U, \tag{2.4}$$

$\mathbb{P}_\xi$ *-a.s., where U is a standard normal vector and η^2 is the positive semidefinite matrix with entries*

$$(\eta^2)_{ij} = \sum_{u\in\mathbb{Z}^n} (u_i - b_i)(u_j - b_j)P_0(u) \,. \tag{2.5}$$

3. Proofs

Lemma 1. *For every $\xi \in \tilde{\Omega}$, the process Y is a martingale under $\mathbb{P}_\xi$.*

Proof. This is a consequence of the definition of Y along with the condition (2.2). Indeed, by (2.3), we have

$$\mathbb{E}\left[Y_{t+1} | Y_t, Y_{t-1}, ..., Y_0\right] = \mathbb{E}\left[Y_t + (Y_{t+1} - Y_t) | Y_t\right]$$

$$= Y_t + \sum_{u\in\mathbb{Z}^n} (u-b)\left[P_0(u) + c(u,\xi_t)\right] = Y_t \,.$$

□

Let us define the following $n \times n$ matrices: $H_t = \mathbb{E}\left(Y_t Y_t'\right)$, where Y_t' the transposed matrix, the matrix $[Y]_t = ([Y^i, Y^j]_t)_{1\le i,j\le n}$, and $H_t^\xi = \mathbb{E}_\xi(Y_t Y_t')$ (here and below we treat Y_t as a column-vector). Let also $K_t = \frac{1}{\sqrt{t}} I_n$, where I_n is the $n \times n$ identity matrix. Then, the following result holds true.

Lemma 2. *We have*

$$\begin{aligned}&\mathbb{E}_\xi\left[(Y^i_{r+1}-Y^i_r)(Y^j_{r+1}-Y^j_r)\right]\\&=\sum_{y\in\mathcal{Y}}\mathbb{P}\left\{Y_r=y\mid\xi\right\}\sum_{u\in\mathbb{Z}^n}(u_i-b_i)(u_j-b_j)\left[P_0(u)+c(u,\xi_r(y))\right]\ .\end{aligned}\tag{3.6}$$

The above sum by y is taken over the countable set

$$\mathcal{Y}:=\{z_1+z_2b\mid z_1,z_2\in\mathbb{Z}\}\ .$$

(Note that $\mathbb{P}\left\{Y_t\in\mathcal{Y}\text{ for all }t\in\mathbb{N}\right\}=1$).

Proof. By definition of Y and $\mathbb{P}_\xi$, we have

$$\begin{aligned}&\mathbb{E}_\xi\left[(Y^i_{r+1}-Y^i_r)(Y^j_{r+1}-Y^j_r)\right]\\&=\mathbb{E}\left[(Y^i_{r+1}-Y^i_r)(Y^j_{r+1}-Y^j_r)\Big|\xi\right]=\mathbb{E}\left[\mathbb{E}\left\{(Y^i_{r+1}-Y^i_r)(Y^j_{r+1}-Y^j_r)\Big|\xi,Y_r\right\}\Big|\xi\right]\\&=\mathbb{E}\left[\sum_u(u_i-b_i-Y^i_r)(u_j-b_j-Y^j_r)[P_0(u-Y_r)+c(u-Y_r,\xi(Y_r)]\Bigg|\xi\right]\\&=\mathbb{E}\left[\sum_u(u_i-b_i)(u_j-b_j)[P_0(u)+c(u,\xi(Y_r)]\Bigg|\xi\right]\\&=\sum_{y\in\mathcal{Y}}\mathbb{P}\{Y_r=y\mid\xi\}\sum_{u\in\mathbb{Z}^n}(u_i-b_i)(u_j-b_j)\left[P_0(u)+c(u,\xi_r(y))\right]\ .\end{aligned}$$

□

Lemma 3. *For Π-a.a. ξ, we have $\mathbb{P}_\xi$-a.s.*

$$\frac{\#\{r:r\le t,\xi_r(Y_r)=s\}}{t}\to\pi(s),\quad t\to\infty.\tag{3.7}$$

Proof. Recall that Π is defined on Page 311. The events $\{Y_r=y\}$ and $\{\xi_r(y)=s\}$ are independent, so (3.7) holds $\mathbb{P}$-a.s. by the law of large numbers. Hence (3.7) also holds $\mathbb{P}_\xi$-a.s for Π-a.a. ξ, otherwise, denoting the event in (3.7) by A, we would have

$$\mathbb{P}(A)=\int\mathbb{P}_\xi(A)\Pi(d\xi)<1\ .$$

Lemma 4. *We have*

$$\frac{1}{t}[Y]_t\to\eta^2,\tag{3.8}$$

$\mathbb{P}_\xi$-a.s. for Π-a.a. ξ.

Proof. Note that for $1\le i,j\le n$,

$$([Y]_t)_{ij}=\sum_{0\le r<t}\Delta_{r,ij},$$

where

$$\Delta_{r,ij}=\left[Y^i_{r+1}-Y^i_r\right]\left[Y^j_{r+1}-Y^j_r\right]\ .$$

Under $\mathbb{P}_\xi$ conditionally on $\{Y_t=y\}$ the distribution of $Y_{t+1}-Y_t$ is $P_0(u)+c(u,\xi_t(y))$. Since under $\mathbb{P}_\xi$ the random vectors $Y_{t+1}-Y_t$ are independent of each other for different t, the statement of the lemma follows from the law of large numbers. Indeed, by (3.7) and the law of large numbers $\mathbb{P}_\xi$-a.s.

$$\sum_{\{r:r\le t,\xi_r(Y_r)=s\}}\frac{\Delta_{r,ij}}{\#\{r:r\le t,\xi_r(Y_r)=s\}}\to\sum_{u\in\mathbb{Z}^n}(u_i-b_i)(u_j-b_j)P(u,s)\,,$$

and hence by (3.7) we obtain

$$\sum_{\{r:r\leq t,\xi_r(Y_r)=s\}} \frac{\Delta_{r,ij}}{\pi(s)t} \to \sum_{u\in\mathbb{Z}^n} (u_i-b_i)(u_j-b_j)P(u,s)\ . \tag{3.9}$$

Therefore, $\mathbb{P}_\xi$-a.s.

$$\lim_{t\to\infty} \sum_{\{r:r\leq t\}} \frac{\Delta_{r,ij}}{t} = \lim_{t\to\infty} \sum_{s\in\mathbb{S}} \sum_{\{r:r\leq t,\xi_r(Y_r)=s\}} \frac{\Delta_{r,ij}}{t} = \lim_{t\to\infty} \sum_{s\in\mathbb{S}} \pi(s) \sum_{\{r:r\leq t,\xi_r(Y_r)=s\}} \frac{\Delta_{r,ij}}{\pi(s)t}$$

$$\sum_{s\in\mathbb{S}} \pi(s) \sum_{u\in\mathbb{Z}^n} (u_i-b_i)(u_j-b_j)P(u,s) = \sum_{u\in\mathbb{Z}^n} (u_i-b_i)(u_j-b_j)P_0(u)\ .$$

□

Corollary 1. *The convergence in Lemma 4 also holds $\mathbb{P}$-a.s.*

Lemma 5.

(*i*) *We have*

$$(H_{r+1})_{ij} - (H_r)_{ij} = \sum_{s\in\mathbb{S}} \pi(s) \sum_{u\in\mathbb{Z}^n} (u_i-b_i)(u_j-b_j)\left[P_0(u)+c(u,s)\right]\ . \tag{3.10}$$

(*ii*) *We also have*

$$(H^\xi_{r+1})_{ij} - (H^\xi_r)_{ij}$$
$$= \sum_{y\in\mathcal{Y}} \mathbb{P}_\xi\{Y_r=y\} \sum_{u\in\mathbb{Z}^n} (u_i-b_i)(u_j-b_j)\left[P_0(u)+c(u,\xi_r(y))\right]\ . \tag{3.11}$$

Proof. (*i*) We start by noting that for $i,j\in\{1,...,n\}$,

$$\mathbb{E}\left((Y^i_{t+1}-Y^i_t)Y^j_t\right) = 0\ . \tag{3.12}$$

Indeed,

$$\mathbb{E}\left((Y^i_{t+1}-Y^i_t)Y^j_t\right) = \mathbb{E}\mathbb{E}\left[(Y^i_{t+1}-Y^i_t)Y^j_t \middle| Y_t\right]$$

$$= \sum_{y\in\mathcal{Y}} \mathbb{P}\{Y_t=y\} \sum_{u\in\mathbb{Z}^n} (y_i+u_i-b_i-y_i)y_jP_0(u) = \sum_{y\in\mathcal{Y}} \mathbb{P}\{Y_t=y\}y_j \sum_{u\in\mathbb{Z}^n} (u_i-b_i)P_0(u) = 0$$

by (2.3). Moreover, by (3.12), we have

$$(H_{r+1})_{ij} - (H_r)_{ij} = \mathbb{E}\left(Y^i_{t+1}Y^j_{t+1} - Y^i_tY^j_t\right)$$

$$= \mathbb{E}\left((Y^i_{t+1}-Y^i_t)(Y^j_{t+1}-Y^j_t)\right) + \mathbb{E}\left((Y^i_{t+1}-Y^i_t)Y^j_t\right) + \mathbb{E}\left(Y^i_t(Y^j_{t+1}-Y^j_t)\right)$$

$$= \mathbb{E}\left((Y^i_{t+1}-Y^i_t)(Y^j_{t+1}-Y^j_t)\right)\ .$$

Conditioning on Y_t, we get

$$(H_{r+1})_{ij} - (H_{r+1})_{ij} = \sum_y P\{Y_t=y\} \sum_u (u_i-b_i)(u_j-b_j)\left[P_0(u) + \mathbb{E}\left[c(u,\xi_r(y))|Y_t=y\right]\right]$$

$$= \sum_u (u_i-b_i)(u_j-b_j)P_0(u).$$

(*ii*) (3.12) holds for $\mathbb{E}_\xi$ too, since

$$\mathbb{E}\mathbb{E}\left[(Y^i_{t+1}-Y^i_t)Y^j_t \middle| Y_t,\xi\right]$$

$$= \sum_{y\in\mathcal{Y}} \mathbb{P}\{Y_t=y\} \sum_{u\in\mathbb{Z}^n} (y_i+u_i-b_i-y_i)y_j[P_0(u)+c(u,\xi_t(y))]$$

$$= \sum_{y\in\mathcal{Y}} \mathbb{P}\{Y_t = y\} y_j \sum_{u\in\mathbb{Z}^n} (u_i - b_i) P_0(u) + \sum_{y\in\mathcal{Y}} \mathbb{P}\{Y_t = y\} y_j \sum_{u\in\mathbb{Z}^n} u_i c(u, \xi_t(y))$$
$$-b_i \sum_{y\in\mathcal{Y}} \mathbb{P}\{Y_t = y\} y_j \sum_{u\in\mathbb{Z}^n} c(u, \xi_t(y)) = 0$$

and the proof continues as in (i). □

Lemma 6. *We have*

$$\frac{1}{t} H_t \to \eta^2, \quad \frac{1}{t} H_t^{\xi} \to \eta^2, \tag{3.13}$$

where η^2 *is as in* (2.5), $\mathbb{P}_\xi$*-a.s for* Π*-a.a.* ξ .

Proof. Let us only prove the second convergence in (3.13). By Lemma 5,

$$\begin{aligned}(H_t^{\xi})_{ij} &= \sum_{r=0}^{t-1} \sum_{y\in\mathcal{Y}} \mathbb{P}_\xi\{Y_r = y\} \sum_{u\in\mathbb{Z}^n} (u_i - b_i)(u_j - b_j) \left[P_0(u) + c(u, \xi_r(y))\right] \\ &= t \sum_{u\in\mathbb{Z}^n} (u_i - b_i)(u_j - b_j) P_0(u) \\ &+ \sum_{u\in\mathbb{Z}^n} (u_i - b_i)(u_j - b_j) \sum_{r=0}^{t-1} \sum_{y\in\mathcal{Y}} \mathbb{P}_\xi\{Y_r = y\} c(u, \xi_r(y)) .\end{aligned} \tag{3.14}$$

Since $\sum_{y\in\mathcal{Y}} \mathbb{P}_\xi\{Y_r = y\} c(u, \xi_r(y)) \stackrel{(d)}{=} c(u, \xi_r(\mathbf{0}))$ under $\mathbb{P}$ (the symbol $\stackrel{(d)}{=}$ means here 'equal in distribution'), where $\mathbf{0}$ is the origin, and

$$\sum_{y\in\mathcal{Y}} \mathbb{P}_\xi\{Y_r = y\} c(u, \xi_r(y)), \quad r \in \mathbb{N},$$

constitute a sequence of independent random variables indexed by $r \in \mathbb{N}$, by the law of large numbers for Π-a.a. ξ

$$\frac{1}{t} \sum_{r=0}^{t-1} \sum_{y\in\mathcal{Y}} \mathbb{P}_\xi\{Y_r = y\} c(u, \xi_r(y)) \to \sum_{s\in\mathbb{S}} \pi(s) c(u, s) . \tag{3.15}$$

Combining (3.14) and (3.15) and recalling the definition of η^2, we get the desired result. □

Recall that we defined $K_t = \frac{1}{\sqrt{t}} I_n$.

Proof of Theorem 1. Theorem 2.1 in [8] and Lemmas 4 and 6 imply that P_ξ -a.s.

$$\frac{1}{\sqrt{t}} Y_t \Rightarrow \eta^2 U, \tag{3.16}$$

where U is a standard n-dimensional Gaussian vector. The theorem in [8] is formulated for continuous time processes, so to apply it we define Y_t, H_t, etc. for $t \in (1, \infty)$ by $Y_t = Y_{\lfloor t \rfloor}$, $H_t = H_{\lfloor t \rfloor}$, etc. □

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