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ANALYSIS RESULTS FOR DYNAMIC CONTACT PROBLEM WITH FRICTION IN THERMO-VISCOELASTICITY

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ABSTRACT. We present a mathematical model which describes the dynamic frictional contact between a thermo-viscoelastc body and a conductive foundation. The contact is modeled using the normal compliance condition, the quasistatic version of Coulomb's law of fry friction. We derive the weak formulation and we prove the existence and uniqueness result. The proofs are based on the theory of first-order and second-order evolution inequalities and Banach fixed point theorem. We introduce a new problem on perturbation of the contact boundary condition and we establish its continuous dependence result.

Представлено математичну модель, що описує динамічний контакт тертя між термов'язкопружним тілом і основою-провідником. Контакт моделюється з використанням умови нормальної еластичності, квазістатичної версії закону фрикційного тертя Кулона. Пропонується означення слабкого розв'язку, доведена теорема існування та єдності. Доведення базуються на теорії еволюційних нерівностей першого та другого порядків і теоремі Банаха про нерухому точку. Ставиться нова задача про збурення контактної граничної умови; отримана теорема про неперервну залежність.

1. INTRODUCTION

The contact of deformable elastic bodies, with or without friction, is the relationship between the normal forces (contact pressure) and the relative movement between the body and the foundation. Contact problems are present in many industrial processes and daily life.

Dynamic frictional contact problems between a viscoelastic body and an obstacle with thermal effect was made in [6, 3, 7] and more recently in [1, 21]. The result in [14, 15, 16, 17], we can find several classes of dynamic frictional contact problems by hemivariational inequalities with or without thermal effects. For more information of the quasistatic and dynamic problem without thermal effects refer to [18, 9, 10]. A number of papers investigating quasistatic thermo-viscoelastic contact problems with friction can be found in [2, 13, 19].

Essoufi et.al [12, 11] study two dynamic frictional contact models between a piezoelectric and electro-viscoelastic body and a rigid, non conductive obstacle, using Tresca and Coulomb model for the contact. The authors in [4] introduced an evolution quasistatic contact problem of a viscoelastic body with a foundation, they modeled the contact with normal damped response and a local friction law. They established the existence of a unique solution and the continuous dependence of the solution on the contact boundary conditions.

In this paper, we study the process of dynamical frictional contact between a thermo-viscoelastic body and a rigid foundation which, is thermally conducting. We assume that the process is dynamic and the contact is modeled with the normal compliance,

the quasistatic version of Coulomb's law of dry friction, a regularizing condition of thermal contact. The weak variational formulation which consists of a system coupling a variational inequality for displacement field and a parabolic variational equality for the temperature. We establish the existence of unique solution using Banach fixed theorems and we prove the dependence of the solution on the contact boundary conditions and its convergence result.

The rest of the paper is structured as follows. In section 2, we describe the dynamic model of the frictional thermo-viscoelastic contact problem. In section 3, we list the assumptions of the problem data, we derive its variational formulation. In section 4, we state and prove our existence and uniqueness result. In the last section we study a contact problem for thermo-electro-viscoelastic with a small perturbation of the contact condition and we prove a convergence result.

2. Problem statement

We consider a body made of thermo-viscoelastic material which occupies in the reference configuration the domain $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, which will be supposed bounded with a smooth boundary $\partial\Omega = \Gamma$ that is divided into three disjoint measurable parts Γ_N, Γ_D and Γ_C such that $meas(\Gamma_D) > 0$. Let $T > 0$ and $[0, T]$ be the time interval of and $\rho : \Omega \mapsto \mathbb{R}^+$ mass density of the body. The body is clamped on $\Gamma_D \times (0, T)$ and so the displacement field vanish there; a volume force of density f_0 acts in $\Omega \times (0, T)$ and a volume thermal of density q_0. Surface traction of density f_1 act on $\Gamma_N \times (0, T)$. In $\Gamma_C \times (0, T)$ the body can arrive in frictional contact with the so called foundation which is thermally conductive. The variation of temperature is assumed to be zero on $\Gamma_D \cup \Gamma_N \times (0, T)$. We assume that the thermal potentials is maintained fix on θ_F. The normalized gap between $\Gamma_C \times (0, T)$ and the rigid foundation is denoted by g.
The space of second order symmetric tensors in the space $\mathbb{R}^d$ is denoted by $\mathbb{S}^d$ while "." and $\|.\|$ will represent the inner product and the Euclidean norm on $\mathbb{R}^d$ and $\mathbb{S}^d$, that is $\forall u, v \in \mathbb{R}^d$, $\forall \sigma, \tau \in \mathbb{S}^d$

$$u.v = u_i.v_i, \quad \|v\| = (v, v)^{\frac{1}{2}} \quad \text{and} \quad \sigma.\tau = \sigma_{ij}.\tau_{ij}, \quad \|\tau\| = (\tau, \tau)^{\frac{1}{2}}.$$

We use the notation u_ν and u_τ for the normal and tangential displacement that is $u_\nu = u.\nu$ and $u_\tau = u - u_\nu\nu$. We also denote by σ_τ the normal and tangential stress give by $\sigma_\nu = \sigma\nu.\nu$, $\sigma_\tau = \sigma\nu - \sigma_\nu\nu$.
The classical formulation of the thermo-viscoelastic contact problem is as follows:
• **Problem** (P) : Find a displacement field $u : \Omega \times]0, T[\longrightarrow \mathbb{R}^d$ and a temperature field $\theta : \Omega \times]0, T[\longrightarrow \mathbb{R}$ such that

$$\sigma(t) = \mathcal{A}\varepsilon(u(t)) + \mathcal{B}\varepsilon(\dot{u}(t)) - \theta(t)\mathcal{M} \qquad \text{in} \qquad \Omega \times (0, T), \qquad (2.1)$$

$$q(t) = -\mathcal{K}\nabla\theta(t) \qquad \text{in} \qquad \Omega \times (0, T), \qquad (2.2)$$

$$\rho\ddot{u}(t) = Div\,\sigma(t) + f_0(t) \qquad \text{in} \qquad \Omega \times (0, T), \qquad (2.3)$$

$$\dot{\theta}(t) + divq(t) - \mathcal{R}\varepsilon(u(t)) = q_0(t) \qquad \text{in} \qquad \Omega \times (0, T), \qquad (2.4)$$

$$u = 0 \qquad \text{on} \qquad \Gamma_D \times (0, T), \qquad (2.5)$$

$$\sigma(t)\nu = f_1(t) \qquad \text{on} \qquad \Gamma_N \times (0, T), \qquad (2.6)$$

$$\theta = 0 \qquad \text{on} \qquad (\Gamma_N \cup \Gamma_D) \times (0, T), \qquad (2.7)$$

$$u(0, x) = u_0, \ \dot{u}(0, x) = \dot{u}_0, \ \theta(0, x) = \theta_0 \qquad \text{in} \qquad \Omega, \qquad (2.8)$$

$$-\sigma_\nu(u(t) - g) = p_\nu(u_\nu(t) - g), \qquad \text{on} \qquad \Gamma_C \times (0, T), \qquad (2.9)$$

$$\left.\begin{array}{l} \|\sigma_\tau(t)\| \le p_\tau(u_\nu(t) - g), \\[2mm] \dot{u}_\tau(t) \ne 0 \Rightarrow \sigma_\tau(t) = -p_\tau(u_\nu(t) - g)\dfrac{\dot{u}_\tau(t)}{\|\dot{u}_\tau(t)\|}, \end{array}\right\} \quad \text{on } \ \Gamma_C \times (0, T) \qquad (2.10)$$

$$\frac{\partial q(t)}{\partial \nu} = k_c(u_\nu(t) - g)\phi_L(\theta(t) - \theta_F) \qquad \text{on} \quad \Gamma_C \times (0, T), \qquad (2.11)$$

where (2.1) and (2.2) are the thermo-viscoelastic constitutive law of the material, $\sigma = (\sigma_{ij}) : \Omega \times (0, T) \to \mathbb{S}^d$ the stress tensor, $q = (q_i) : \Omega \times (0, T) \to \mathbb{R}^d$ the heat flux vector, $\varepsilon(u) = (\varepsilon_{ij}(u)) = \frac{1}{2}(u_{i.j} + u_{j.i})$, $\mathcal{A} = (\mathcal{A}_{ijkl})$, $\mathcal{B} = (\mathcal{B}_{ijkl})$, $\mathcal{M} = (\mathcal{M}_{ij})$ and $\mathcal{K} = (\mathcal{K}_{ij})$ are respectively the linearized strain tensor, elastic tensor, (fourth-order) viscosity tensor, thermal expansion tensor and thermal conductivity tensor.

Equation (2.3) is the equation of motion. Equation (2.4) is the Fourier law of heat conduction where the function $\mathcal{R} = (\mathcal{R}_{ij})$ describe the influence of the displacement field. Here and below $Div\ \sigma = \sigma_{ij.j}$ and $div\ q = (q_{i,i})$ denote the divergence operator for tensor and vector valued functions. Conditions (2.5)-(2.7) are the displacement and thermal boundary condition. The initials conditions are represented in the equation (2.8). Relation (2.9) represents the normal compliance contact condition, where p_ν is a prescribed function. When it is positive, $u_\nu - g$ represents the penetration of the surface as parities into those of the foundation. The Coulomb's law of friction is considered in relation (2.10) where p_τ is a prescribed non-negative function, the so-called friction bound. Finally, the relation (2.11) represents a regularized thermal contact condition where $\dfrac{\partial q}{\partial \nu}$ is the normal derivative of q such that:

$$\phi_L(s) = \left\{\begin{array}{ccc} -L & \text{if} & s < -L, \\ s & \text{if} & -L \le s \le L, \\ L & \text{if} & s > L, \end{array}\right. \quad \left\{\begin{array}{lll} k_c(r) = 0 & \text{if} & r < 0, \\ k_c(r) > 0 & \text{if} & r \ge 0, \end{array}\right.$$

where L is a large positive constant, see [8].

3. VARIATIONAL FORMULATION

In order to obtain the variational formulation of **Problem** (P), let us denote by

$$H = \left[L^2(\Omega)\right]^d = \left\{u = (u_i) : \ u_i \in L^2(\Omega)\right\}, \qquad \mathcal{H} = \left\{\sigma = (\sigma_{ij}) : \ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\right\},$$

$$H_1 = \left\{u = (u_i) : \ \varepsilon(u) \in \mathcal{H}\right\} = \left[H^1(\Omega)\right]^d, \qquad \mathcal{H}_1 = \left\{\sigma \in \mathcal{H} : \ Div\ \sigma \in H\right\}.$$

These are real Hilbert spaces endowed with the inner products

$$(u, v)_H = \int_\Omega u_i v_i dx, \forall u, v \in H, \qquad (\sigma, \tau)_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij}dx, \forall \sigma, \tau \in \mathcal{H},$$

$$(u, v)_{H_1} = (u, v)_H + (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \qquad (\sigma, \tau)_{\mathcal{H}_1} = (\sigma, \tau)_{\mathcal{H}} + (Div\ \sigma, Div\ \tau)_H,$$

and the associated norms, $\|.\|_H$, $\|.\|_{H_1}$, $\|.\|_{\mathcal{H}}$, and $\|.\|_{\mathcal{H}_1}$.

Let X be a Banach space. For every $1 \le p \le \infty$, we will use the spaces $L^p(0, T; X)$, $C(0, T; X)$ and $W^{k,p}(0, T; X)$ with their standard norm.

Moreover, keeping in mind (2.5) and (2.7), we introduce the following space

$$V = \left\{v \in H_1(\Omega) : \ v = 0 \text{ on } \Gamma_D\right\}, \ \ V_{ad} = \left\{v \in V : \ u_\nu \le g \text{ on } \Gamma_C\right\},$$

$$Q = \left\{\eta \in H_1(\Omega) : \ \eta = 0 \text{ on } \Gamma_D \cup \Gamma_N\right\}.$$

endowed with the inner products and norms given by

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \qquad \|v\|_V = (v, v)_V^{\frac{1}{2}},$$

$$(\theta, \eta)_Q = (\nabla\theta, \nabla\eta)_H, \qquad \|\eta\|_Q = (\eta, \eta)_Q^{\frac{1}{2}}.$$

Since $meas(\Gamma_D) > 0$, the following Korn's inequality holds

$$\|\varepsilon(v)\|_{\mathcal{H}} \geq C_K \|v\|_{H_1}, \ \forall v \in V, \tag{3.12}$$

where $C_K > 0$ is a constant which depends only on Ω and Γ_D.
The following Frierichs-Poincaré inequality holds on Q at the bottom

$$\|\nabla \eta\|_H \geq C_F \|\eta\|_Q, \ \forall \eta \in Q. \tag{3.13}$$

Using Sobolev trace theorem, we have constants C_{S_1} and C_{S_2} depends only on Ω, Γ_D and Γ_C such that

$$\|v\|_{[L^2(\Gamma_C)]^d} \leq C_{S_1} \|v\|_V, \ \forall v \in V \text{ and } \|\eta\|_{[L^2(\Gamma_C)]} \leq C_{S_2} \|\eta\|_Q, \ \forall \eta \in Q. \tag{3.14}$$

Next, we define the elements $f(t) \in V$ and $q(t) \in Q$ respectively by

$$(f(t), v)_V \ := \ \int_\Omega f_0(t).v dx + \int_{\Gamma_N} f_1(t).v da, \ \forall v \in V. \tag{3.15}$$

$$(q_{th}(t), \eta)_Q \ := \ \int_\Omega q_0(t) \eta dx, \ \forall \eta \in Q. \tag{3.16}$$

We consider $j : V \times V \to \mathbb{R}$, $\chi : V \times Q \times Q \to \mathbb{R}$ as follows:

$$j(u(t), v) \ := \ \int_{\Gamma_C} p_\nu(u_\nu(t) - g) v_\nu da + \int_{\Gamma_C} p_\tau(u_\tau(t) - g) \|v_\tau\| da, \tag{3.17}$$

$$\chi(u(t), \theta(t), \eta) \ := \ \int_{\Gamma_C} k_c(u_\nu(t) - g) \phi_L(\theta(t) - \theta_F) \eta da, \tag{3.18}$$

and the following operators

$$
\begin{aligned}
a : V \times V \to \mathbb{R}, \quad & a(u, v) & := (\mathcal{A}\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \\
b : V \times V \to \mathbb{R}, \quad & b(u, v) & := (\mathcal{B}\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \\
d : Q \times Q \to \mathbb{R}, \quad & d(\theta, \eta) & := (\mathcal{K}\nabla\theta, \nabla\eta)_H, \\
m : Q \times V \to \mathbb{R}, \quad & m(\theta, v) & := (\mathcal{M}\theta, \varepsilon(v))_{\mathcal{H}}, \\
e : V \times Q \to \mathbb{R}, \quad & e(u, \eta) & := (\mathcal{R}\varepsilon(v), \eta)_{L^2(\Omega)}.
\end{aligned}
$$

Now, in order to study the problem (2.1)-(2.11), we need the following hypotheses on the data

(H1) The operators a, b, d, m and e satisfy the usual property of symmetry

$$\mathcal{A}_{ijkl} = \mathcal{A}_{jikl} = \mathcal{A}_{lkij} \in L^\infty(\Omega), \quad \mathcal{B}_{ijkl} = \mathcal{B}_{jikl} = \mathcal{B}_{lkij} \in L^\infty(\Omega),$$
$$\mathcal{K}_{ij} = \mathcal{K}_{ji} \in L^\infty(\Omega), \ \ \mathcal{M}_{ij} = \mathcal{M}_{ji} \in L^\infty(\Omega), \ \ \mathcal{R}_{ij} = \mathcal{R}_{ji} \in L^\infty(\Omega).$$

(H2) There exist positive constants m_a, m_b and m_d such that

$$a(v, v) \geq m_a \|v\|_V^2, \ \ b(v, v) \geq m_b \|v\|_V^2, \ \ d(\eta, \eta) \geq m_d \|\eta\|_Q^2.$$

(H3) The operators a, b, d, m and e satisfy the usual property of boundedness

$$|a(u, v)| \leq M_a \|u\|_V \|u\|_V, \quad |b(u, v)| \leq M_b \|u\|_V \|v\|_V,$$
$$|d(\theta, \eta)| \leq M_d \|\theta\|_Q \|\eta\|_Q, |m(\theta, v)| \leq M_m \|\theta\|_Q \|v\|_V, |e(u, \eta)| \leq M_e \|u\|_V \|\eta\|_Q,$$

(H4)   i) The forces, the traction and the thermal flux satisfy

$$f_0 \in W^{1,1}(0, T; L^2(\Omega)^d), \ \ f_1 \in W^{1,1}(0, T; L^2(\Gamma_N)^d) \text{ and } q_0 \in L^2(0, T; L^2(\Omega)).$$

ii) The gap function and the thermal potential satisfy

$$g \geq 0, \ g \in L^\infty(\Gamma_C), \text{ and } \theta_F \in L^2(0, T; L^2(\Gamma_C)).$$

iii) The mass density ρ satisfies $\rho \in L^\infty(\Omega)$, and there exists $\rho^* > 0$ such that $\rho(x) \geq \rho^*$ a.e. $x \in \mathbb{R}$.

iv) The functional j is proper, convex and lower semi-continuous on V.

(H5) The coefficient of heat exchange $k_c : \Gamma_C \times \mathbb{R} \to \mathbb{R}^+$ satisfy
  i) There exists $M_{k_c} > 0$ such that $|k_c(x, u)| < M_{k_c}$ for all $u \in \mathbb{R}, x \in \Gamma_C$
     $x \mapsto k_c(x, u)$ is measurable on Γ_C for all $x \in \mathbb{R}$, $k_c(x, u) = 0$ for all $x \in \Gamma_C$
     and $u \leq 0$.
  ii) There exists $L_{k_c} > 0$ such that $|k_c(x, u_1) - k_c(x, u_2)| \leq L_{k_c}|u_1 - u_2|$, for all
     $u_1, u_2 \in \mathbb{R}$.
(H6) The normal compliance function p_ν and the friction bound p_τ satisfy the following
     hypothesis for $r = \nu, \tau$
  i) $p_r : \Gamma_C \times \mathbb{R} \to \mathbb{R}_+$,
  ii) $x \to p_r(x, u)$ is measurable on Γ_C, for all $u \in \mathbb{R}$,
  iii) $x \to p_r(x, u) = 0$ for $u \leq 0$, a.e. $x \in \Gamma_C$,
  iv) There exists $L_r > 0$ such that $|p_r(., u) - p_r(., v)| \leq L_r|u - v|$, for all $u, v \in \mathbb{R}_+$.
(H7)   i) The initial data u_0 and $\dot{u}_0$ of **Problem** (P) satisfy

$$u_0 \in V_{ad}, \quad \dot{u}_0 \in D(\partial j),$$

     where ∂j denotes the subdifferential of j and $D(\partial j)$ represents its domain.
  ii) There exists $h \in L^2(\Omega)^d$ such that

$$b(\dot{u}_0, v - \dot{u}_0) + a(u_0, v - \dot{u}_0) + j(u_0, v) - j(u_0, \dot{u}_0) \geq (h, v - \dot{u}_0), \; \forall v \in V.$$

According to this notation and by using a standard procedure based on Green's formula, we can state the variational formulation of (2.1)-(2.11), in terms of displacement field and temperature.

Problem (PV): Find a displacement field $u : \Omega \times]0, T[\longrightarrow \mathbb{R}^d$ and a temperature $\theta : \Omega \times]0, T[\longrightarrow \mathbb{R}$ a.e. $t \in]0, T[$, $v \in V$ and $\eta \in Q$ such that

$$(\ddot{u}(t), v - \dot{u}(t)) + a(u(t), v - \dot{u}(t)) + b(\dot{u}(t), v - \dot{u}(t)) - m(\theta(t), v - \dot{u}(t)) \quad (3.19)$$
$$+ j(u(t), v) - j(u(t), \dot{u}(t)) \geq (f(t), v - \dot{u}(t))_V,$$
$$(\dot{\theta}(t), \eta) + d(\theta(t), \eta) - e(u(t), \eta) + \chi(u(t), \theta(t), \eta) = (q_{th}(t), \eta)_Q, \quad\quad (3.20)$$
$$u(0) = u_0, \; \dot{u}(0) = \dot{u}_0, \; \theta(0) = \theta_0. \quad\quad (3.21)$$

4. AN EXISTENCE AND UNIQUENESS RESULT

In this section, we present and we prove an existence and uniqueness result.

Theorem 4.1. *Assume that the assumptions (H1)-(H7) hold. Then, there exists a unique solution (u, θ) to problem* (3.19)-(3.21) *that satisfies*

$$u \in W^{1,\infty}(0, T; V) \cap W^{2,\infty}(0, T; H), \quad\quad (4.22)$$
$$\theta \in L^2(0, T; Q) \cap C([0, T]; L^2(\Omega)). \quad\quad (4.23)$$

The proof of this theorem is carried out in several steps and it is based on argument of second-order non linear evolution variational inequality and Banach fixed point theorem. Let $\alpha \in L^2(0, T; V)$ and $\beta \in L^2(0, T; L^2(\Omega))$ given by

$$(\alpha(t), v - \dot{u}_\alpha(t)) = m(\theta_\beta(t), v - \dot{u}_\alpha(t)), \quad\quad (4.24)$$
$$(\beta(t), \eta) = -e(u_\alpha(t), \eta) + \chi(u_\alpha(t), \theta_\beta(t), \eta). \quad\quad (4.25)$$

In the first, we consider the following variational problem of displacement field

Problem (PV_{df}): Find a displacement field $u_\alpha : \Omega \times]0, T[\longrightarrow \mathbb{R}^d$ a.e. $t \in]0, T[$ such that

$$(\ddot{u}_\alpha(t), v - \dot{u}_\alpha(t)) + a(u_\alpha(t), v - \dot{u}_\alpha(t)) + b(\dot{u}_\alpha(t), v - \dot{u}_\alpha(t)), \quad\quad (4.26)$$
$$-(\alpha(t), v - \dot{u}_\alpha(t)) + j(u_\alpha(t), v) - j(u_\alpha(t), \dot{u}_\alpha(t)) \geq (f(t), v - \dot{u}_\alpha(t))_V, \; \forall v \in V,$$
$$u_\alpha(0) = u_0, \; \dot{u}_\alpha(0) = \dot{u}_0. \quad\quad (4.27)$$

We have the following existence and uniqueness result.

Lemma 4.2. *For all $v \in V_{ad}$ and for a.e $t \in]0, T[$, the problem (4.26)-(4.27) has a unique solution u_α satisfies the regularity (4.22).*

Proof. By Riesz's representation theorem, we define the operator

$$(f_\alpha(t), v)_V = (f(t), v)_V - (\alpha(t), v)_V. \tag{4.28}$$

Then (4.26)-(4.27) can be written

$$(\ddot{u}_\alpha(t), v - \dot{u}_\alpha(t)) + b(\dot{u}_\alpha(t), v - \dot{u}_\alpha(t)) + a(u_\alpha(t), v - \dot{u}_\alpha(t)) \tag{4.29}$$
$$+ j(u_\alpha(t), v) - j(u_\alpha(t), \dot{u}_\alpha(t)) \geq (f(t), v - \dot{u}_\alpha(t))_V.$$

By assumptions (H1) and (H2) the operator a is continuous symmetric bilinear and coercive.

Next, we define the set-valued operator $B : V \mapsto V$ by $B = b + \partial j$.

From (H2), b is maximal monotone and by (H4)(iv), we observe that ∂j is maximal monotone. Consequently, [5, P. 39] the operator B is maximal monotone.

Moreover, by (H7), the initial data u_0 and $\dot{u}_0$ satisfy the condition $\{au_0 + b\dot{u}_0\} \cap L^2(\Omega)^d \neq \emptyset$.

Keeping in mind the regularity $f \in W^{1,1}(0, T; H)$ and $\alpha \in L^2(0, T; V)$, we deduce from (4.28) that $f_\alpha \in W^{1,1}(0, T; H)$.

Finally, using the Theorem presented in [5, P. 268], we obtain result. $\qquad \square$

In the second step, we consider the following variational problem of the temperature.
Problem (PV_{th}): Find a temperature $\theta_\beta : \Omega \times]0, T[\longrightarrow \mathbb{R}$ a.e. $t \in]0, T[$ and $\eta \in Q$ such that

$$(\dot{\theta}_\beta(t), \eta) + d(\theta_\beta(t), \eta) + (\beta(t), \eta) = (q_{th}(t), \eta)_Q, \tag{4.30}$$
$$\theta_\beta(0) = \theta_0. \tag{4.31}$$

The result of this problem presented in the following lemma.

Lemma 4.3. *There exists a unique solution for the problem (4.30)-(4.31) which satisfies (4.23).*

Proof. Using Riesz's representation theorem, there exists an operator q_β defined by

$$(q_\beta(t), \eta)_Q = (q_{th}(t), \eta)_Q - (\beta(t), \eta)_Q. \tag{4.32}$$

Then, the problem (4.30)-(4.31) can be written as follows

$$(\dot{\theta}_\beta(t), \eta) + d(\theta_\beta(t), \eta) = (q_\beta(t), \eta), \tag{4.33}$$
$$\theta_\beta(0) = \theta_0. \tag{4.34}$$

From (H1)-(H2) the operator d is a hemicontinuous and monotone.

By (4.32) and the regularity of q_{th}, we obtain that $q_\beta \in L^2(0, T; Q)$.

It follows now from Theorem presented [20, P. 48] that there exists a unique function θ_β which satisfies (4.23). $\qquad \square$

In the last step, for all $t \in [0, T]$ we define the operator

$$\Lambda(\alpha, \beta)(t) := (\Lambda_1(\alpha, \beta)(t), \Lambda_2(\alpha, \beta)(t)) \in V \times Q, \tag{4.35}$$

given by

$$(\Lambda_1(\alpha, \beta)(t), v) \quad := \quad m(\theta_\beta, v), \tag{4.36}$$
$$(\Lambda_2(\alpha, \beta)(t), \eta) \quad := \quad -e(u_\alpha(t), \eta) + \chi(u_\alpha(t), \theta_\beta(t), \eta), \tag{4.37}$$

and we have the following result

Lemma 4.4. *For $(\alpha, \beta) \in L^2(0, T; V) \times L^2(0, T; Q)$ the operator Λ is continuous. Moreover, there exists a unique $(\alpha^*, \beta^*) \in L^2(0, T; V \times Q)$ such that $\Lambda(\alpha^*, \beta^*) = (\alpha^*, \beta^*)$.*

Proof. Let $(\alpha, \beta) \in L^2(0, T; V \times Q)$ and let $t_1, t_2 \in [0, T]$.
From (4.36) and assumption (H3), it follows that

$$\|\Lambda_1(\alpha, \beta)(t_1) - \Lambda_1(\alpha, \beta)(t_2)\|_{V \times Q} \leq M_m \|\theta_\beta(t_1) - \theta_\beta(t_2)\|_Q. \tag{4.38}$$

Taking into account the regularity of θ_β, we found that $\Lambda_1(\alpha, \beta) \in C([0, T]; V)$.
The relation (4.37) combined with (H3), (H5) and (3.14), there exists a positive constant c depending on M_e, M_m, L, M_{kc}, C_{S_1} and C_{S_2} such that

$$\|\Lambda_2(\alpha, \beta)(t_1) - \Lambda_2(\alpha, \beta)(t_2)\|_{V \times Q} \tag{4.39}$$
$$\leq c \left(\|u_\alpha(t_1) - u_\alpha(t_2)\|_V + \|\theta_\beta(t_1) - \theta_\beta(t_2)\|_Q \right).$$

Then, $\Lambda_2(\alpha, \beta) \in C([0, T], Q)$. Consequently, we deduce that Λ is continuous.
Let $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in L^2(0, T; V \times Q)$ and $t \in [0, T]$. Similar to (4.38) and (4.39), we have that

$$\|\Lambda(\alpha_1, \beta_1)(t) - \Lambda(\alpha_2, \beta_2)(t)\|_{V \times Q}^2 \tag{4.40}$$
$$\leq c \left(\|u_{\alpha_1}(t) - u_{\alpha_2}(t)\|_V^2 + \|\theta_{\beta_1}(t) - \theta_{\beta_2}(t)\|_Q^2 \right).$$

Also from $u_{\alpha_i}(t) = \int_0^t \dot{u}_{\alpha_i}(s)ds + u_0$ for $i = 1, 2$, we found

$$\|u_{\alpha_1}(t) - u_{\alpha_2}(t)\|_V^2 \leq c \int_0^t \|\dot{u}_{\alpha_1}(s) - \dot{u}_{\alpha_2}(s)\|_V^2 ds. \tag{4.41}$$

Using inequality (4.26), we find

$$(\ddot{u}_{\alpha_1}(t) - \ddot{u}_{\alpha_2}(t), \dot{u}_{\alpha_1}(t) - \dot{u}_{\alpha_2}(t)) + b(\dot{u}_{\alpha_1}(t) - \dot{u}_{\alpha_2}(t), \dot{u}_{\alpha_1}(t) - \dot{u}_{\alpha_2}(t)) \tag{4.42}$$
$$a(u_{\alpha_1}(t) - u_{\alpha_2}(t), \dot{u}_{\alpha_1}(t) - \dot{u}_{\alpha_2}(t)) + (\alpha_1(t) - \alpha_2(t), \dot{u}_{\alpha_1}(t) - \dot{u}_{\alpha_2}(t))$$
$$j(u_{\alpha_1}(t), \dot{u}_{\alpha_1}(t)) - j(u_{\alpha_1}(t), \dot{u}_{\alpha_2}(t)) - j(u_{\alpha_2}(t), \dot{u}_{\alpha_1}(t)) + j(u_{\alpha_2}(t), \dot{u}_{\alpha_2}(t)) \leq 0.$$

Moreover, from (3.14), (3.17) and (H5), we have

$$|j(u_{\alpha_1}(t), \dot{u}_{\alpha_1}(t)) - j(u_{\alpha_1}(t), \dot{u}_{\alpha_2}(t)) - j(u_{\alpha_2}(t), \dot{u}_{\alpha_1}(t)) + j(u_{\alpha_2}(t), \dot{u}_{\alpha_2}(t))| \tag{4.43}$$
$$\leq C_{S_1}^2(L_\nu + L_\tau) \|u_{\alpha_1}(t) - u_{\alpha_2}(t)\|_V \|\dot{u}_{\alpha_1}(t) - \dot{u}_{\alpha_2}(t)\|_V.$$

We integrate the relation (4.42) to 0 at t and use (4.43), (H2) and the initial condition $\dot{u}_{\alpha_1}(0) = \dot{u}_{\alpha_2}(0) = \dot{u}_0$ to obtain

$$m_b \int_0^t \|\dot{u}_{\alpha_1}(s) - \dot{u}_{\alpha_2}(s)\|_V^2 ds + \frac{1}{2} \|\dot{u}_{\alpha_1}(t) - \dot{u}_{\alpha_2}(t)\|_V^2 \tag{4.44}$$
$$\leq - \int_0^t (\alpha_1(s) - \alpha_2(s), \dot{u}_{\alpha_1}(t) - \dot{u}_{\alpha_2}(t)) ds$$
$$+ (M_a + C_{S_1}^2(L_\nu + L_\tau)) \int_0^t \|u_{\alpha_1}(s) - u_{\alpha_2}(s)\|_V \cdot \|\dot{u}_{\alpha_1}(s) - \dot{u}_{\alpha_2}(s)\|_V ds.$$

Then, using the inequality $xy \leq \kappa x^2 + \frac{1}{4\kappa} y^2$, $(\kappa > 0)$ and Gornwall inequality, we get that

$$\|u_{\alpha_1}(t) - u_{\alpha_2}(t)\|_V^2 \leq c \int_0^t \|\alpha_1(s) - \alpha_2(s)\|_V^2 ds. \tag{4.45}$$

In order words from (4.26), it follows that

$$\left(\dot{\theta}_{\beta_1}(t) - \dot{\theta}_{\beta_2}(t), \theta_{\beta_1}(t) - \theta_{\beta_2}(t) \right) + d(\theta_{\beta_1}(t) - \theta_{\beta_2}(t), \theta_{\beta_1}(t) - \theta_{\beta_2}(t)) \tag{4.46}$$
$$+ (\beta_1(t) - \beta_2(t), \theta_{\beta_1}(t) - \theta_{\beta_2}(t)) = 0.$$

Similar to (4.45), we have

$$\|\theta_{\beta_1}(t) - \theta_{\beta_2}(t)\|_Q^2 \le c \int_0^t \|\beta_1(s) - \beta_2(s)\|_Q^2 \, ds. \tag{4.47}$$

Now, we combine (4.40), (4.45) and (4.47), we obtain

$$\|\Lambda(\alpha_1, \beta_1)(t) - \Lambda(\alpha_2, \beta_2)(t)\|_{V \times Q}^2 \le c \int_0^t \|(\alpha_1, \beta_1)(s) - (\alpha_2, \beta_2)(s)\|_{V \times Q}^2 \, ds. \tag{4.48}$$

Reiterating this inequality n times result in

$$\|\Lambda^n(\alpha_1, \beta_1) - \Lambda^n(\alpha_2, \beta_2)\|_{L^2(0,T;V \times Q)}^2 \tag{4.49}$$

$$\le \frac{(CT)^n}{n!} \|(\alpha_1, \beta_1) - (\alpha_2, \beta_2)\|_{L^2(0,T;V \times Q)}^2.$$

Which implies that for n sufficiently large, Λ^n is a contraction operator in the Banach space $L^2(0,T;V \times Q)$.

Therefore, Λ has a unique fixed point. $\qquad\qquad\square$

Now we have everything that is required to prove Theorem (4.1)

Proof of Theorem (4.1).
Existence: Let $(\alpha^*, \beta^*) \in L^2\left(0,T;V \times L^2(\Omega)\right)$ be the fixed point of the operator Γ and denote $u_{\alpha^*}^*, \theta_{\beta^*}^*$ be the solution of (4.26)-(4.27) and (4.30)-(4.31) respectively, for $(\alpha, \beta) = (\alpha^*, \beta^*)$, the definition of Λ we find that the pair $\left(u_{\alpha^*}^*, \theta_{\beta^*}^*\right)$ is a solution of **Problem** (PV).
Uniqueness: The uniqueness of the solution is a consequence of the fixed point of operator Λ given by (4.35). $\qquad\qquad\square$

5. Continuous dependence of problem disturbed

In this section we consider p_ν^δ and p_τ^δ for any $\delta > 0$ the perturbation of p_ν and p_τ which satisfies (H6) and we study the dependence of the solution to (3.19)-(3.21).

We define the functional j^δ from j by replacing p_ν and p_τ with p_ν^δ and p_τ^δ. We introduce the following variational problem

Problem (PV^δ): For $\delta > 0$ find a displacement field $u^\delta : \Omega \times]0,T[\longrightarrow \mathbb{R}^d$ and a temperature $\theta^\delta : \Omega \times]0,T[\longrightarrow \mathbb{R}$ a.e. $t \in]0,T[$, $v \in V$ and $\eta \in Q$ such that

$$\left(\ddot{u}^\delta(t), v - \dot{u}^\delta(t)\right) + a\left(u^\delta(t), v - \dot{u}^\delta(t)\right) + b\left(\dot{u}^\delta(t), v - \dot{u}^\delta(t)\right) \tag{5.50}$$

$$-m\left(\theta^\delta(t), v - \dot{u}^\delta(t)\right) + j^\delta\left(u^\delta(t), v\right) - j^\delta(u^\delta(t), \dot{u}^\delta(t)) \ge \left(f(t), v - \dot{u}^\delta(t)\right)_V,$$

$$\left(\dot{\theta}^\delta(t), \eta\right) + d\left(\theta^\delta(t), \eta\right) - e\left(u^\delta(t), \eta\right) + \chi\left(u^\delta(t), \theta^\delta(t), \eta\right) = (q_{th}(t), \eta)_Q, \tag{5.51}$$

$$u^\delta(0) = u_0, \ \dot{u}^\delta(0) = \dot{u}_0, \ \theta^\delta(0) = \theta_0. \tag{5.52}$$

This problem has a unique solution, and the proof is similar to that used in Section 4. We present now, the assumptions of the contact function p_ν^δ and p_τ^δ

$$\begin{cases} \text{There exists } \omega_\tau \in \mathbb{R} \text{ and } \varphi_\tau : \mathbb{R}_+ \to \mathbb{R}_+ \text{ such that:} \\ (i) \left|p_\tau^\delta(x,r) - p_\tau(x,r)\right| \le \varphi_\tau(\delta)(|r| + \omega_\tau), \ \ \forall r \in \mathbb{R}^N, a.e. \ x \in \Gamma_C, \\ (ii) \lim_{\delta \to 0} \varphi_\tau(\delta) = 0, \end{cases} \tag{5.53}$$

and

$$\begin{cases} \text{There exists } \omega_\nu \in \mathbb{R} \text{ and } \varphi_\nu : \mathbb{R}_+ \to \mathbb{R}_+ \text{ such that:} \\ (i) \left|p_\nu^\delta(x,r) - p_\nu(x,r)\right| \le \varphi_\nu(\delta)(|r| + \omega_\nu), \ \ \forall r \in \mathbb{R}^N, a.e. \ x \in \Gamma_C, \\ (ii) \lim_{\delta \to 0} \varphi_\nu(\delta) = 0. \end{cases} \tag{5.54}$$

Next, we present the following convergence result

Theorem 5.1. *The solution* $(u^\delta, \theta^\delta)$ *of* (5.50)-(5.52) *converge uniformly to a solution of* (3.19)-(3.21) *i.e.,*

$$u^\delta \longrightarrow u, \ \dot{u}^\delta \longrightarrow \dot{u}, \ and \ \theta^\delta \longrightarrow \theta \ as \ \delta \longrightarrow 0.$$

Proof. Replacing v by $\dot{u}^\delta$ in (3.19) and by $\dot{u}$ in (5.50), we have

$$\left(\ddot{u}(t) - \ddot{u}^\delta(t), \dot{u}(t) - \dot{u}^\delta(t) \right) + a \left(u(t) - u^\delta(t), \dot{u}(t) - \dot{u}^\delta(t) \right) \tag{5.55}$$
$$+ b \left(\dot{u}(t) - \dot{u}^\delta(t), \dot{u}(t) - \dot{u}^\delta(t) \right) - m \left(\theta(t) - \theta^\delta(t) \right)$$
$$j \left(u(t), \dot{u}(t) \right) - j \left(u(t), \dot{u}^\delta(t) \right) + j^\delta \left(u^\delta(t), \dot{u}^\delta(t) \right) - j^\delta \left(u^\delta(t), \dot{u}(t) \right) \le 0.$$

Using (H2) and (H3) we found the inequality

$$m_b \left\| \dot{u}(t) - \dot{u}^\delta(t) \right\|_V^2 + \frac{1}{2} \frac{d}{dt} \left\| \dot{u}(t) - \dot{u}^\delta(t) \right\|_V^2 \tag{5.56}$$
$$\le (M_a + M_m) \left\| \dot{u}(t) - \dot{u}^\delta(t) \right\|_V \left(\left\| u(t) - u^\delta(t) \right\|_V + \left\| \theta(t) - \theta^\delta(t) \right\|_Q \right) + \left| R_{j,j^\delta} \right|.$$

The assumption (5.53)-(5.54) and (3.17) allows us to obtain

$$\left| R_{j,j^\delta} \right| \ = \ \left| \int_{\Gamma_C} \left(p_\nu(u_\nu(t) - g) - p_\nu^\delta(u_\nu^\delta(t) - g) \right) \left(\dot{u}_\nu^\delta(t) - \dot{u}_\nu(t) \right) da \right. \tag{5.57}$$
$$+ \ \left. \int_{\Gamma_C} \left(p_\nu(u_\nu(t) - g) - p_\nu^\delta(u_\nu^\delta(t) - g) \right) \left(\| \dot{u}_\nu^\delta(t) \| - \| \dot{u}_\nu(t) \| \right) da \right|$$
$$\le \ c \left(\varphi_\nu(\delta) + \varphi_\tau(\delta) \right) \left\| \dot{u}^\delta(t) - \dot{u}(t) \right\|_V.$$

Combining (5.56), (5.57) and using Gronwall inequality, it follows that

$$\left\| u(t) - u^\delta(t) \right\|_V^2 + \left\| \dot{u}(t) - \dot{u}^\delta(t) \right\|_V^2 \tag{5.58}$$
$$\le c \left(\int_0^t \left\| \theta(s) - \theta^\delta(s) \right\|_Q^2 ds + \left(\varphi_\nu(\delta) + \varphi_\tau(\delta) \right) \right).$$

Now, we replace η by $\theta - \theta^\delta$ in (3.20) and in (5.51), we have the following relation

$$\left(\dot{\theta}(t) - \dot{\theta}^\delta(t), \theta(t) - \theta^\delta(t) \right) + d \left(\theta(t) - \theta^\delta(t), \theta(t) - \theta^\delta(t) \right) \tag{5.59}$$
$$- e \left(u(t) - u^\delta(t), \theta(t) - \theta^\delta(t) \right) + \chi \left(u(t), \theta(t), \theta(t) - \theta^\delta(t) \right)$$
$$- \chi \left(u^\delta(t), \theta^\delta(t), \theta(t) - \theta^\delta(t) \right) = 0.$$

Using the property of operator d and e, we deduce that

$$m_d \left\| \theta(t) - \theta^\delta(t) \right\|_Q^2 + \frac{1}{2} \frac{d}{dt} \left\| \theta(t) - \theta^\delta(t) \right\|_Q^2 \le \tag{5.60}$$
$$M_e \left\| u(t) - u^\delta(t) \right\|_V \left\| \theta(t) - \theta^\delta(t) \right\|_Q + \left| R_\chi \right|.$$

From (H4) and (3.14), we have the following estimate of R_χ

$$\left| R_\chi \right| \le M_{k_c}.L.C_{S_2}^2 \left\| \theta(t) - \theta^\delta(t) \right\|_Q^2. \tag{5.61}$$

Similar to (5.59), we conclude that

$$\left\| \theta(t) - \theta^\delta(t) \right\|_Q^2 \le c \int_0^t \left\| u(s) - u^\delta(s) \right\|_V^2 ds. \tag{5.62}$$

Moreover, from (5.59) and (5.62) combined with Gronwall inequality we find

$$\left\| u(t) - u^\delta(t) \right\|_V^2 + \left\| \dot{u}(t) - \dot{u}^\delta(t) \right\|_V^2 + \left\| \theta(t) - \theta^\delta(t) \right\|_Q^2 \le c \left(\varphi_\nu(\delta) + \varphi_\tau(\delta) \right). \tag{5.63}$$

Hence, since $\delta \longrightarrow 0$, we obtain the result of Theorem (5.1). $\qquad\square$

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