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STRONG BANACH-SAKS OPERATORS

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Abstract. In this paper, we introduce a new class of operators, called strong Banach-Saks operators, related to the Banach-Saks and L-weakly compact operators. We first prove that every strong Banach-Saks operator from a Banach space Z into a Banach lattice F is Banach-Saks. Then we show that if F is order continuous, the notions of strong Banach-Saks and Banach-Saks operators coincide. Finally, we close this paper by a new characterization of order continuous Banach lattices.

Вводиться новий клас операторів, так звані сильні оператори Банаха-Сакса, пов'язані з операторами Банаха-Сакса і L-слабко компактними операторами. Доведено, що кожен сильний оператор Банаха-Сакса з банахового простору Z у банахову решітку F є оператором Банаха-Сакса. Далі, якщо F є порядково неперервним, то властивості оператора Банаха-Сакса і сильного оператора Банаха-Сакса співпадають. Нарешті, в статті дано нову характеризацію порядково неперервних банахових решіток.

1. Introduction

In [3], S. Banach and S. Saks showed that for $1 < p < \infty$, every bounded sequence in $L_p[0, 1]$ has a subsequence (y_n) whose arithmetic means converge in norm. That is

$$\frac{1}{n}\sum_{k=1}^{n} y_k \xrightarrow{\|\cdot\|_p} x.$$

This prompted A. Brunel and L. Sucheston [6] to qualify every Banach space with this property as a Banach-Saks space. Every Banach-Saks space is reflexive, see [11, Proposition 2.3]. The converse statement is not true in general. That is, there are reflexive spaces without the Banach-Saks property [2]. Inspired by the preceding papers, B. Beauzamy introduced in [4] the notion of a Banach-Saks operator. We say that an operator $T : X \longrightarrow Y$ between two Banach spaces is a Banach-Saks operator if T maps the closed unit ball B_X of X onto a Banach-Saks subset of Y. A bounded subset A of X is said to be Banach-Saks if each sequence (x_n) in A has a subsequence (y_n) whose arithmetic means converge in norm. Observe that a compact operator must be a Banach-Saks operator. The two notions coincide when Y has the Schur property. Every Banach-Saks operator is weakly compact. If Y has the positive Schur property, then weakly compact and Banach-Saks operators coincide.

The class of L-weakly compact operators was introduced by Meyer-Nieberg[8]. Recall that a bounded subset A of a Banach lattice E is said to be L-weakly compact, if $\|x_n\| \longrightarrow 0$ for every disjoint sequence $(x_n)_n$ in the solid hull of A. A linear operator T from a Banach space X into a Banach lattice F is said to be L-weakly compact if so is $T(B_X)$. Note that (by Proposition 3.6.5 in [9]) every L-weakly compact operator is weakly compact.

In this paper, we introduce a new class of operators, called strong Banach-Saks operators, related to the Banach-Saks and L-weakly compact operators. We first prove that every strong Banach-Saks operator from a Banach space Z into a Banach lattice

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F is Banach-Saks. We also show that if F is order continuous, the notion of strong Banach-Saks and Banach-Saks operators coincide. Finally, we close this paper by a new characterization of order continuous Banach lattices.
Our terminology and notations are standard, and we refer to [1] and [9] for unexplained definitions and properties about Banach lattices and operators on them.

2. Strong Banach-Saks Operators

We start by the following definition.

Definition 2.1. A linear operator T from a Banach space X into a Banach lattice F is said to be strong Banach-Saks if for every bounded sequence $(x_n)_n$ in X, the sequence of images $(Tx_n)_n$ has a subsequence which is Cesàro L-weakly compact in F (i.e, there exists a subsequence $(Ty_k)_k$ such that $\{\frac{1}{n}\sum_{l=1}^{n} Ty_l,\ \ n \in \mathbb{N}\}$ is an L-weakly compact subset of F).

To continue our discussion, we need the next Lemma.

Lemma 2.2. [5, Lemma 2.4] *For every nonempty bounded subset $A \subset E$, the following assertions are equivalent.*
(1) *A is L-weakly compact.*
(2) *$f_n(x_n) \to 0$ for every sequence (x_n) of A and every disjoint sequence (f_n) of $B_{E'}$.*

Note that every L-weakly compact operator $T : X \to F$ is strong Banach-Saks. Indeed, let $(x_n)_n$ be a bounded sequence of X. Since T is L-weakly compact, it follows from Lemma 2.2 that $f_n(Tx_n) \to 0$ for every disjoint sequence (f_n) of $B_{F'}$. Thus, $\frac{1}{n}\sum_{k=1}^{n} f_k(Tx_k) \to 0$. This shows that T is strong Banach-Saks. Every strong Banach-Saks operator is a weakly compact operator. The details follow.

Theorem 2.3. *Every strong Banach-Saks operator T from a Banach space Z into a Banach lattice F is weakly compact.*

Proof. Let (x_n) be a bounded sequence in Z. Since T is strong Banach-Saks, it follows that there exists a subsequence $(y_n)_n$ of $(x_n)_n$ such that $\{\frac{1}{n}\sum_{k=1}^{n} Ty_k;\ \ n \in \mathbb{N}\}$ is L-weakly compact subset of F. On the other hand, note that the sequence $(e_n)_n$ is not Banach-Saks, where (e_n) is the standard basis of l_1. Now, an easy application of Theorem 4.32 in [1] shows that $\{\frac{1}{n}\sum_{k=1}^{n} e_k;\ \ n \in \mathbb{N}\}$ is not relatively weakly compact, in particular, $\{\frac{1}{n}\sum_{k=1}^{n} e_k;\ \ n \in \mathbb{N}\}$ is not L-weakly compact (see Proposition 3.6.5 in [9]). Hence, by the Rosenthal's l_1 Theorem, there exists a subsequence (z_n) of (y_n) such that (Tz_n) is weak Cauchy. According to Theorem 9.3.1 in [7], there exists some $z'' \in F''$ so that $Tz_n \xrightarrow{\sigma(F'',F')} z''$. Now, since $\{\frac{1}{n}\sum_{k=1}^{n} Tz_k;\ \ n \in \mathbb{N}\}$ is L-weakly compact, it follows from Proposition 3.6.5 in [9] that there is a subsequence $(t_n)_n$ of $(z_n)_n$ such that $\frac{1}{n}\sum_{k=1}^{n} Tt_k \xrightarrow{\sigma(F,F')} z \in F$. So $z = z''$, and consequently $(T(t_n))_n$ converges weakly to $z \in F$. □

Recall that a Banach lattice E is said to be order continuous if $\lim_\alpha \|x_\alpha\| = 0$ for every decreasing net $(x_\alpha)_\alpha$ in E such that $\wedge_\alpha x_\alpha = 0$. Let E be a Banach lattice. An element $e \in E$ is called weak unit if for $h \in E$, $e \wedge h = 0$ implies $h = 0$.

If E is an order continuous Banach lattice with weak unit, then there exists a probability space (Ω, Σ, μ), an order ideal I of $L_1(\Omega, \Sigma, \mu)$, a lattice norm $\| \, . \, \|_I$ on I, and an order isometry j from E onto $(I, \| \, . \, \|_I \,)$ such that the canonical inclusion from I into $L_1(\Omega, \Sigma, \mu)$ is continuous with $\|f\|_1 \leq \|f\|_I$(see Theorem 1.b.14 in [14]). This implies that j is continuous as an operator from E into $L_1(\Omega, \Sigma, \mu)$. Note that a separable subspace X of an order continuous Banach lattice E is included in some closed order ideal Y of E with weak unit (see Proposition 1.a.9 in [14]). Thus, $\overline{F_X}$ (F_X the ideal generated by X) has a weak unit. In terms of order continuous Banach lattices, the convergence of a bounded sequence is characterized as follows.

Lemma 2.4. *Let E be a Banach lattice with order continuous norm, and $(g_n)_n$ be a bounded sequence in E($\overline{E_{[g_n]}}$ representable as an order ideal in $L_1(\Omega, \Sigma, \mu)$ for some probability measure μ). Then:*
$(g_n)_n$ is convergent in E if and only if $(g_n)_n$ is L-weakly compact and $\| \, . \, \|_1-$convergent.

Proof. Since $[g_n]$ is a separable subspace of E, it follows from Proposition 1.a.9 in [14] that $E_1 = \overline{E_{[g_n]}}$ ($E_{[g_n]}$ the ideal generated by $(g_n)_n$) is a Banach lattice with weak unit. Thus, from Lemma 1.4.2 in [12] it easily follows that $(g_n)_n$ is convergent in E_1 if and only if (g_n) is L-weakly compact and $\| \, . \, \|_1-$convergent, which finishes the proof. □

A Banach space E has the weak Banach-Saks property (or it is weakly Banach-Saks) if every weakly convergent sequence $(x_n)_n$ in E has a subsequence which is Cesàro convergent.

Theorem 2.5. (*Szlenk* [16]) *Let (Ω, Σ, μ) be a probability space. Then $L_1(\Omega, \Sigma, \mu)$ is weakly Banach-Saks.*

Our next goal is to establish that a strong Banach-Saks operator is a Banach-Saks operator. To do this, we need the following Lemma. That is the general version of Theorem 5.66 in [1].

Lemma 2.6. *Let A be an L-weakly compact subset of E. If E_A is the ideal generated by A, then $\overline{E_A}$ is a Banach lattice with order continuous norm.*

Proof. Let E_A be the ideal generated by A in E. By Theorems 4.13 and 4.11 in [1], it suffices to show that every order bounded disjoint sequence in E_A is norm convergent to zero. Let $(y_n)_n$ be a disjoint sequence with $0 \le y_n \le y$ for all n and some $y \in E_A$. Then there exist $x_1, ..., x_{n_0} \in A_+$ and $\lambda > 0$ such that

$$y \le \lambda \sum_{i=1}^{n_0} x_i.$$

From the Riesz decomposition property(see Theorem 1.13 in [1]), there exist $y_1^n, ..., y_{n_0}^n$ in E_+ with

$$y_n = y_1^n + ... + y_{n_0}^n, \text{ and } y_i^n \le \lambda x_i$$

for all $n \in \mathbb{N}$ and $i \in \{1, ..., n_0\}$. Clearly, for each i the sequence $(y_i^n)_n$ is disjoint and included in Sol$A$. The L-weak compactness of A guarantees that $y_n \to 0$ in norm. So $\overline{E_A}$ is order continuous. □

Let X be a Banach space. A sequence (x_n) in X is said to be Cesàro convergent if its Cesàro means converge in norm. An operator T from a Banach space X to a Banach space Y is called a Banach-Saks operator if for any norm bounded sequence (x_n) in X, $(Tx_n)_n$ has a Cesàro convergent subsequence.

Theorem 2.7. *Every strong Banach-Saks operator T from a Banach space Z into a Banach lattice F is Banach-Saks.*

Proof. Let $(x_n)_n$ be a bounded sequence in Z. Since T is strong Banach-Saks, it follows that there exists a subsequence $(y_n)_n$ of $(x_n)_n$ such that $\{\frac{1}{n}\sum_{k=1}^n Ty_k; \quad n \in \mathbb{N}\}$ is L-weakly compact subset in F. Let $A = \{\frac{1}{n}\sum_{k=1}^n Ty_k; \qquad n \in \mathbb{N}\}$. Then from Lemma 2.6 we see that $\overline{F_A}$(F_A is the ideal generated by A)is order continuous.
Now, since $X := [Ty_n]$ is a separable subspace of F_A, it follows from Proposition 1.a.9 in [14] that $\overline{F_X}$ is an order ideal with a weak order unit and so can be represented as

a dense order ideal of $L_1(\Omega, \Sigma, \mu)$ for some probability measure μ, such that the formal inclusion

$$j : \overline{F_X} \hookrightarrow L_1(\Omega, \Sigma, \mu)$$

is continuous ([14], Theorem 1.b.14). By Theorem 2.3 the sequence (y_n) has a subsequence (z_n) such that (Tz_n) converges weakly to some $z \in \overline{F_X}$. Thus, $j(Tz_n)$ converges weakly in $L_1(\Omega, \Sigma, \mu)$. The fact that $L_1(\Omega, \Sigma, \mu)$ is weakly Banach-Saks(by Theorem 2.5), and hence without loss of generality we can assume that there is some $f \in L_1(\Omega, \Sigma, \mu)$ such that:

$$\|j(\frac{1}{n}\sum_{k=1}^{n} Tz_k) - f\|_1 \longrightarrow 0.$$

Since $\{\frac{1}{n}\sum_{k=1}^{n} Tz_k;\ n \in \mathbb{N}\}$ is L-weakly compact subset in F_X, it follows from Lemma 2.4 that $\frac{1}{n}\sum_{k=1}^{n} Tz_k$ converges in F. □

It is interesting to know that the converse of the preceding theorem is in general false, as shown in the following.

Example 2.8. Consider the linear operator $T : l_1 \to l_\infty$ defined by

$$T(x_1, x_2, ...) = (\sum_{i=1}^{\infty} x_i, \sum_{i=1}^{\infty} x_i, ...) = \sum_{i=1}^{\infty} x_i(1, 1, ...).$$

Evidently, T is Banach-Saks (it has rank one). Let (e_n) be the sequence of standard unit vectors in l_1. Then (e_n) is a disjoint sequence in the solid hull of $\{\frac{1}{n}\sum_{k=1}^{n} Te_k,\ n \in \mathbb{N}\}$ and $\|e_n\|_\infty = 1$. Consequently the operator T fails to be strong Banach-Saks.

However, it holds for order continuous Banach lattices, as follows from the next theorem:

Theorem 2.9. *A bounded linear operator from a Banach space Z into an order continuous Banach lattice F is Banach-Saks if and only if it is strong Banach-Saks.*

Proof. From Theorem 2.7, it is enough to show that every Banach-Saks operator from Z into F is strong Banach-Saks. To this end, let (x_n) be a bounded sequence in Z. Since T is Banach-Saks, it follows that there is a subsequence (z_n) of (x_n) such that for every subsequence (y_n) of (z_n), we have that $T(y_n)$ is Cesàro convergent to some $y \in F$, that is:

$$\|\frac{1}{n}\sum_{k=1}^{n} Ty_k - y\| \to 0. \tag{2.1}$$

Let $(w_k)_k$ be a disjoint sequence in the solid hull of $\{\frac{1}{n}\sum_{k=1}^{n} Ty_k;\ n \in \mathbb{N}\}$. Without loss of generality we can assume that $|w_n| \le |\frac{1}{n}\sum_{k=1}^{n} Ty_k|$ holds for all $n \in \mathbb{N}$. Then

$$|w_n| \le |\frac{1}{n}\sum_{k=1}^{n} Ty_k - y| + |y|.$$

From the Riesz decomposition property (see [1, Theorem 1.13]), it follows that for each n there exist $w_n^1, w_n^2 \in F_+$ such that $|w_n| = w_n^1 + w_n^2$ with:

$$w_n^1 \le |\frac{1}{n}\sum_{k=1}^{n} Ty_k - y| \quad \text{and} \quad w_n^2 \le |y|.$$

By 2.1 the sequence $(w_n^1)_n$ converges to 0 in F. On the other hand, since $(w_n^2)_n$ is order bounded and disjoint $(0 \le w_n^2 \le |w_n|)$, it follows from Theorem 4.14 in [1] that $(w_n^2)_n$ converges to 0 in F. Thus $\lim_n \|w_n\| = 0$, and so T is strong Banach-Saks. □

Recall that a subset A of a Banach space X is called Banach-Saks if every sequence in A has a Cesàro convergent subsequence. A Banach space X is said to have the Banach-Saks property when its unit ball is a Banach-Saks set.

Corollary 2.10. *For a Banach lattice E the following statements are equivalent.*
(1) *E has the Banach-Saks property.*
(2) *The identity operator $I : E \to E$ is strong Banach-Saks.*

Proof. First, assume that E has the Banach-Saks property. From Corollary of Theorem 1 in [10] we know that E is reflexive, in particular it is order continuous (see Theorem 4.70 in [1]). Since the identity operator $I : E \to E$ is Banach-Saks, it follows from Theorem 2.9 that $I : E \to E$ is strong Banach-Saks.

For the converse assume that $I : E \to E$ is strong Banach-Saks, it follows from Theorem 2.7 that E is Banach-Saks. □

A Banach lattice E has the positive Schur property if weakly null sequences with positive terms are norm null.

Corollary 2.11. *For a bounded linear operator T from a Banach space Z into a Banach lattice F with positive Schur property the following statements are equivalent.*
(1) *T is L-weakly compact.*
(2) *T is strong Banach-Saks.*
(3) *T is Banach-Saks.*
(4) *T is weakly compact.*

Proof. (1) $\Longrightarrow$ (2). Let $(x_n)_n$ be a bounded sequence of Z. By Theorem 5.61 in [1] the sequence $T(x_n)$ has a weakly convergent subsequence in F (which we shall denote by $T(x_n)$ again). Therefore, the sequence $(\frac{1}{n}\sum_{k=1}^{n} Tx_k)_n$ also has a weakly convergent subsequence in F. Next, let (w_n) be a disjoint sequence in the solid hull of $(\frac{1}{n}\sum_{k=1}^{n} Tx_k)_n$, then $(|w_n|)$ is also in the solid hull of $(\frac{1}{n}\sum_{k=1}^{n} Tx_k)_n$. Now an easy application of Theorem 4.34 shows that $(|w_n|)$ converges weakly to zero. Since F has the positive Schur property, it follows that $\lim_n \| |w_n| \| = \lim_n \|w_n\| = 0$. Consequently, $(\frac{1}{n}\sum_{k=1}^{n} Tx_k)_n$ is L-weakly compact subset of F, and so T is strong Banach-Saks.
(2) $\Longrightarrow$ (3). It is a direct application of Theorem 2.7.
(3) $\Longrightarrow$ (4). It follows from Proposition 2.3 in [11].
(4) $\Longrightarrow$ (1). Since F has the positive Schur property and $T(B_Z)$ is relatively weakly compact subset of F, it follows from Theorem 4.34 in [1] that T is L-weakly compact. □

The domination property for strong Banach-Saks operators is the following.

Theorem 2.12. *Let E and F be Banach lattices. If $0 \le S \le T : E \longrightarrow F$ with T is strong Banach-Saks, then S is also strong Banach-Saks.*

Proof. Suppose that T is strong Banach-Saks and let (x_n) be a sequence in E. Then there exists a subsequence $\{y_n\}$ of $\{x_n\}$ such that $\{\frac{1}{n}\sum_{k=1}^{n} T|y_k|; \quad n \in \mathbb{N}\}$ is L-weakly compact subset. Since

$$|\frac{1}{n}\sum_{k=1}^{n} Sy_k| \le \frac{1}{n}\sum_{k=1}^{n} T|y_k|,$$

it follows that $\{\frac{1}{n}\sum_{k=1}^{n} Sy_k; \quad n \in \mathbb{N}\}$ is L-weakly compact. □

In what follows:

$L(X, F)$ will denote the space of all operators from X into F,

$SBS(X, F)$ will denote the space of all strong Banach-Saks operators from X into F.

Theorem 2.13. *The set of all strong Banach-Saks operators from a Banach space X to a Banach lattice F is a closed vector subspace of $L(X,F)$.*

Proof. Clearly, the collection $SBS(X,F)$ of all strong Banach-Saks operators from X to F is a vector subspace of $L(X,F)$.

Now, let $(T_n)_n$ be a sequence of strong Banach-Saks operators such that $\|T_n - T\| \to 0$ in $L(X,F)$. Let $\epsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that $n \geq N_0$, implies $\|T_n - T\| < \frac{\epsilon}{2}$. Let (x_n) be a bounded sequence in X, by passing to a subsequence, we can assume without loss of generality that $\{\frac{1}{n}\sum_{k=1}^{n} T_{N_0}x_k; \quad n \in \mathbb{N}\}$ is L-weakly compact. The inequality

$$|T(\frac{1}{n}\sum_{k=1}^{n} x_k)| \leq |(T - T_{N_0})(\frac{1}{n}\sum_{k=1}^{n} x_k)| + |T_{N_0}(\frac{1}{n}\sum_{k=1}^{n} x_k)|,$$

guarantees that $\{\frac{1}{n}\sum_{k=1}^{n} Tx_k; \quad n \in \mathbb{N}\}$ is L-weakly compact. □

Proposition 2.14. *Let $E = \prod_{i=1}^{n} E_i$ be the direct sum of Banach lattices, and let $A_{ij} : E_j \longrightarrow E_i$ be a strong Banach-Saks operator for all $1 \leq i,j \leq n$. Then the matrix operator $T : E \longrightarrow E$ defined by*

$$T = \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,n} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,1} & A_{n,2} & \cdots & A_{n,n} \end{pmatrix}$$

is strong Banach-Saks.

Proof. Let $\{X_k = (x_k^1, x_k^2, ..., x_k^n), k \in \mathbb{N}\}$ be a norm bounded sequence in E. Then $(x_k^i)_k$ is bounded in E_i for each $1 \leq i \leq n$. Since A_{ij} is strong Banach-Saks, then, by passing to a subsequence, we can suppose that $\{\sum_{j=1}^{n} \frac{1}{k}\sum_{l=1}^{k} A_{1j}x_l^j, \quad n \in \mathbb{N}\}$ is L-weakly compact subset of E_i for all $1 \leq i \leq n$. Then

$$\frac{1}{k}\sum_{l=1}^{k} TX_l = \begin{pmatrix} \sum_{j=1}^{n} \frac{1}{k}\sum_{l=1}^{k} A_{1j}x_l^j \\ \sum_{j=1}^{n} \frac{1}{k}\sum_{l=1}^{k} A_{2j}x_l^j \\ \vdots \\ \sum_{j=1}^{n} \frac{1}{k}\sum_{l=1}^{k} A_{nj}x_l^j \end{pmatrix}$$

is L-weakly compact subset of E. □

3. Order continuous Banach lattices

The next Theorem gives a characterization of Banach lattices with order continuous norms.

Theorem 3.1. *For a Banach lattice E the following statements are equivalent.*
(1) *E is order continuous.*
(2) *E is Dedekind $\sigma-$complete and $x_n \downarrow 0$ implies $\frac{1}{n}\sum_{l=1}^{n} x_l \to 0$.*
(3) *For every $x \in E_+$, the order interval $[-x,x]$ is Banach-Saks.*

Proof. (1) $\Longrightarrow$ (2) If $x_n \downarrow 0$, then $\|x_n\| \longrightarrow 0$. Thus, $\frac{1}{n}\sum_{l=1}^{n} x_l \to 0$.
(2) $\Longrightarrow$ (1) Let $x_n \downarrow 0$, then $\frac{1}{n}\sum_{l=1}^{n} x_l \to 0$. In particular, from Lemma 2.6 it follows that $\overline{E_A}$ is order continuous, where $A = \{\frac{1}{n}\sum_{l=1}^{n} x_l, \quad n \in \mathbb{N}\}$. Since $x_n \in E_A$ for all $n \in \mathbb{N}$, then from Theorem 4.9 in [1] it should be clear that $\|x_n\| \longrightarrow 0$.
(1) $\Longrightarrow$ (3) Let $(x_n)_n$ be a sequence in E satisfying $0 \leq x_n \leq x$ for all n and some $x \in E_+$. Since E is order continuous, it follows from Theorem 4.9 in [1] that $[0,x]$ is weakly compact. By passing to a subsequence, we can assume that $x_n \xrightarrow{\sigma(E,E')} y$

for some $y \in E$. Then $x_n \xrightarrow{\sigma(L_1(\mu),L_\infty(\mu))} y$ for some probability measure μ. Thus, by Theorem 2.5 there exist a subsequence (y_n) of (x_n) such that $\frac{1}{n}\sum_{k=1}^{n} y_k \xrightarrow{\|\cdot\|_1} y$. On the other hand, since E is order continuous and $0 \leq \frac{1}{n}\sum_{k=1}^{n} y_k \leq x$ for all n, we see that $A = \{\frac{1}{n}\sum_{l=1}^{n} y_l, \quad n \in \mathbb{N}\}$ is L-weakly compact subset of E (see Theorem 4.14 in [1]). According to Lemma 2.4, we have $\frac{1}{n}\sum_{k=1}^{n} y_k$ converges to y in E.
(3) $\Longrightarrow$ (1) The implication follows from Proposition 2.3 in [11].

□

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