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A GLIMPSE ON BIRKHOFF-JAMES ORTHOGONALITY IN BANACH SPACES

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Abstract. This paper is an overview of various results on Birkhoff-James orthogonality of operators in Hilbert space and Banach spaces. We mainly focus on Birkhoff orthogonality of linear(bounded and compact) operators in terms of matrices, projection angles, Hilbert C^*-modules as well as on Banach modules. The article concludes with some open problems regarding possible correlation between Birkhoff-James orthogonality and Carlsson orthogonality, particularly in the case of Pythagorean orthogonality.

Дано огляд різноманітних результатів щодо ортогональності в сенсі Біркгофа-Джеймса операторів у гільбертових і банахових просторах. Переважно розглядається ортогональність за Біркгофом лінійних (обмежених і компактних) операторів у термінах матриць, кутів, гільбертових C*-модулів, а також банахових модулів. Наведені деякі відкриті питання стосовно співвідношень ортогональністю Біркгофа-Джеймса та ортогональністю Карлссона, зокрема для випадку піфагорової ортогональності.

1. Introduction

The concept of Birkhoff orthogonality began in 1935 [1]. In the literature of orthogonality this is known with some other names such as; Birkhoff- James orthogonality and Blaschke Birkhoff-James orthogonality (see [2]). In this paper [1, 3], an orthogonality which satisfies homogeneity but neither symmetric nor additive is defined by $x \perp y$ if and only if $\|x + \lambda y\| \geq \|x\|$ for all λ, is known as Birkhoff orthogonality or Birkhoff-James orthogonality. The geometrical meaning of Birkhoff orthogonality is that if x is an unit vector of a Banach space X and $y \in X$, then x is Birkhoff orthogonal to y means that the straight line $\{x + \lambda y : \lambda \in K\}$ is tangent to the unit ball of X at x. This concept is similar to the statement: suppose two lines l_1 and l_2 intersect at the point m, then $l_1 \perp l_2$ if and only if the distance from a point of l_2 to a given point n of l_1 is never less than the distance from m and n. [3] For any hyper-plane $H \subset X$, x is said to be orthogonal to H if $\forall x \in H, x \perp h$.

Bhatia and Semrl in [4] generalize the definition of Birkhoff orthogonality in terms of matrices. For any matrices A and B they denote the symbol $\|A\|$ for operator norm of A and A is orthogonal to B in the sense of Birkhoff-James iff for any complex number z, $\|A + zB\| \geq \|A\|$. A matrix A is orthogonal to B iff there exist a unit vector $x \in H$ such that $\|Ax\| = \|A\|$ and $\langle Ax, Bx \rangle = 0$ [4]. They also introduced Birkhoff- James orthogonality in [4] as $A \perp B$ if and only if $\|A + zB\|_p \geq \|A\|_p$, where $\|A\|_p$ denotes Schatten p-norm of A defined by $\|A\|_p = [\sum_{j=1}^{n} S_j(A)^p]^{\frac{1}{p}}$ for $1 \leq p < \infty$ and $S_1(A) \geq$ ......$S_n(A)$ are singular values of A. Taking the special case for $p = 2$, Bhatia and Semrl in [4] also proved that the given orthogonality is equivalent to usual Hilbert space condition $\langle A, B \rangle = 0$, which defines an inner-product on the space of matrices as $\langle A, B \rangle = tr(A^*B)$. The norm associated to this inner product is $\|.\|_2$. In an infinite dimensional case [4], for

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any bounded operators in a Hilbert space H, $A \perp B$ if and only if there exist a sequence $\{x_n\}$ of unit vectors such in H that $\|Ax\| \to \|A\|$, and $\langle Ax_n, Bx_n \rangle \to 0$.

Benitz et al. [5] proved that X is an inner-product space if and only if for any linear operators A and C in a finite dimensional normed space X, $A \perp C \Leftrightarrow \exists u \in S_X : \|Au\| = \|u\|, Au \perp Cu$, where $S_X = \{x \in X : \|x\| = 1\}$ and "$\perp''$" denotes the Birkhoff-James orthogonality.

Theorem 1.1. [5] *If S_X is not an ellipse(X is not an inner-product space), then there exists linear operators A and C in X such that $A \perp C$, but there does not exists $u \in S_X$ such that $\|A\| = \|AU\|$ and $Au \perp Cu$.*

Theorem 1.2 ([5]). *A real finite dimensional normed space X is an inner-product space if and only if , for $A, C \in L(X)$, $A \perp C \Leftrightarrow \exists x \in S_X : \|A\| = \|Ax\|$, $Ax \perp Cx$.*

where, $P_{xy} = D_{xy} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $D_{xy} = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$.

Theorem 1.3. [6] *The q-angle has the following properties:*

(i) Part of parallelism property: $A_q(x, y) = 0$ iff x and y are linearly dependent.
(ii) Part of homogeneity property: $A_q(Ax, By) = A_q(x, y)$ for every $x, y \in X$ and $A, B \in \mathbb{R} - \{0\}$.

In [6] Chen Zhi-Zhi et al. have given slightly different definition of Birkhoff orthogonality in such a way that; x is Birkhoff orthogonal to y iff $A_q(x, y) = \frac{\pi}{2}$ by using projections of the angles between two vectors x and y in a real two dimensional normed space X.

Definition 1.4. [6] The g-angle between two vectors x and y is given by $g(x, y) = \cos^{-1} \frac{g(x,y)}{\|x\| \ \|y\|}$, where $g(x, y) = \frac{1}{2}\|x\|[\tau_+(x, y) + \tau_-(x, y)]$ and $\tau_\pm(x, y) = \lim_{t \to \pm 0} \frac{\|x+ty\| - \|x\|}{t}$. In that case $x \perp_g y$ if $g(x, y) = 0$ or $A_g(x, y) = \frac{\pi}{2}$.

For any $x = (x_1, x_2)^T$ and $y = (y_1, y_2)^T$ in a two dimensional real normed space X,

$$q(x, y) = \begin{cases} 0 & \text{if x and y are linearly dependent} \\ \|P_{xy}\|^{-1}, & \text{if x and y are linearly independent.} \end{cases}$$

Continuity property: If $x_n \to x$ and $y_n \to y$, then $A_q(x_n, y_n) \to A_q(x, y)$, where $A_q(x, y)$ is q-angle between x and y defined by $A_q(x, y) = \sin^{-1}[q(x, y)]$.

Lemma 1.5. [6] *If x is Birkhoff orthogonal to y. Then for any $m, n \in \mathbb{R}$, $\|mx + ny\| \geq \|mx\|$.*

Proof. If $m = 0$, the conclusion is obviously true. If $m \neq 0$ and if x is Birkhoff orthogonal to y,

$$\|mx + ny\| = |m|\|x + \frac{m}{n}y\| \geq |m|\|x\| = \|mx\|. \qquad \square$$

Theorem 1.6. [7] *Let $x = (x_1, x_2)^T$ and $y = (y_1, y_2)^T$ be two vectors in a two dimensional real normed space X with basis $\{e_1, e_2\}$. Then x is Birkhoff-orthogonal to to y iff $A_q(x, y) = \frac{\pi}{2}$ i.e. $\|P_{xy}\| = 1$.*

2. Orthogonality on C^*-module

[8] Let A be a C^*-algebra and H be a (left) $\mathscr{A}$ module. Suppose that the linear structure given on $\mathscr{A}$ and H are compatible, that is, $\lambda(ax) = a(\lambda x)$ for every $\lambda \in \mathbb{C}$ and $a \in H$. Then there exists a mapping $\langle .,. \rangle : H \times H \to \mathscr{A}$ with the following properties:

(i) $\langle x, x \rangle \geq 0$ for every $x \in H$,
(ii) $\langle x, x \rangle = 0$ iff $x = 0$,
(iii) $\langle x, y \rangle = \langle y, x \rangle^*$ for every $x, y \in H$,

(iv) $\langle ax, y\rangle = a\langle x, y\rangle$ of every $a \in \mathscr{A}$ and $x, y \in H$,
(v) $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ for every $x, y, z \in H$

The pair $\{H, \langle ., .\rangle\}$ is called a (left) pre-Hilbert $\mathscr{A}$ module. The map $\langle ., .\rangle$ is called an $\mathscr{A}$-valued inner-product. If the pre-Hilbert $\mathscr{A}$-module $\{H, \langle ., .\rangle\}$ is complete with respect to the norm $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$, then it is called $\mathscr{A}$ Hilbert C^*-module over $\mathscr{A}$. Rajic et al., in [7, 8] introduced a new concept of Birkhoff-James orthogonality in a Hilbert C^*-modules over a C^*-algebra $\mathscr{A}$ and proved that such orthogonality with respect to $\mathscr{A}$-valued inner product coincide if and only if $\mathscr{A}$ is isomorphic to $\mathbb{C}$.

[8] A mapping $T : V \to W$ between $\mathscr{A}$-modules V and W is called adjointable if there exists mapping $T^* : W \to V$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $v \in V, y \in W$. Such a mapping T is bounded, linear and satisfies $T(xa) = T(x)a$ for all $x \in V$ and $a \in \mathscr{A}$. The space of all adjointable mapping from V into W is denoted by $B(V, W)$. Let $\theta_{x,y}(z) = x(y, z)$, where $\theta_{x,y} \in B(V, W)$ and $K(B, V)$ denotes the closed linear subspace of $B(V, W)$ spanned by $\{Q_{xy} : x \in W, y \in V\}$ is called space of compact operators.

Proposition 2.1. [8] *Let $A, B \in B(H)$. Then $min_{\lambda\in\mathbb{C}}\|A + \lambda B\|^2 = sup_{\|xi\|=1}M_{A,B}(\xi)$, where*

$$M_{A,B}(\xi) = \begin{cases} \|A\xi\|^2 - \frac{|\langle A\xi, B\xi\rangle|^2}{\|B\xi\|^2} & if, \quad B\xi \neq 0 \\ \|A\xi\|^2 & if, \quad B\xi = 0 \end{cases}$$

Proposition 2.2. [8] *let $\mathscr{A}$ be a C^*-algebra, and $a, b \in \mathscr{A}$. Then $min_{\lambda\in\mathbb{C}}\|a + \lambda b\|^2 = max_{\varphi\in S(A)}M_{A,B}(\varphi)$, where*

$$M_{a,b}(\varphi) = \begin{cases} \varphi(a^*a) - \frac{|\varphi(a^*b)^2}{\varphi(b^*b)} & if, \quad \varphi(b^*b) \neq 0 \\ \varphi(a^*a) & if, \quad \varphi(b^*b) = 0 \end{cases}$$

Theorem 2.3. [8] *Le V be a Hilbert C^*-module over a C^*-algebra $\mathscr{A}$ and $x, y \in V$. Then $min_{\varphi\in\mathbb{C}}\|x + \varphi y\|^2 = max_{\varphi\in S(A)}M_{x,y}(\varphi)$, where $M_{x,y}(\varphi) \in \mathscr{A}$ is defined by*

$$M_{x,y}(\varphi) = \begin{cases} \varphi(\langle x, x\rangle) - \frac{|\varphi(\langle x,y\rangle)^2}{\varphi(\langle y,y\rangle)} & if, \quad \varphi(\langle y, y\rangle) \neq 0 \\ \varphi(\langle x, x\rangle) & if, \quad \varphi(\langle y, y\rangle) = 0 \end{cases}$$

Theorem 2.4. [8] *Let V be a Hilbert C^*-module over a C^*-algebra $\mathscr{A}$. Let $x, y \in V$. Then $x\perp_B y \Leftrightarrow \exists\varphi \in S(\mathscr{A}) : \varphi(\langle x, x\rangle) = \|x\|^2$ and $\varphi(\langle x, y\rangle) = 0$.*

Theorem 2.5. [8] *Let V be a Hilbert C^*-module over a C^*-algebra $\mathscr{A}$ and $x, y \in V$. Then*

(i) $x\perp_B y \Leftrightarrow \langle x, x\rangle \perp \langle x, y\rangle \Leftrightarrow \langle x, x\rangle \perp_B \langle y, x\rangle$.
(ii) $x\perp_B y \Rightarrow x\perp_B x\langle x, y\rangle$ and $x\perp_B x\langle y, x\rangle$.

Arambasic and Rajic (see in[8]) characterized Hilbert C^*-modules where the Birkhoff orthogonality coincides with the usual orthogonality with respect to inner-product space. By using the Gelfand-Mazur theorem, it can be proved that $\mathscr{A}$ is isomorphic to $\mathbb{C}$ and using this concept, $\mathbb{C}$ is only the unital C^*-algebra in which Birkhoff orthogonality $x\perp_B y$ coincides with $x^*y = 0$ for all elements $x, y \in \mathscr{A}$.

Theorem 2.6. *let $V \neq \{0\}$ be a full Hilbert $\mathscr{A}$-module. then the following statements are equivalent:*

(i) For all $x, y \in V$ the condition $(x\perp_B y \Leftrightarrow \langle x, y\rangle = 0)$ is always true.
(ii) $\mathscr{A}$ is isomorphic to $\mathbb{C}$.

3. Generalization of Bhatia-Semrl Property

In 2013, Sain and Paul [9] linked the Bhatia-Semrl property with norm attaining operators in a finite dimensional normed spaces which attain its norm on connected closed subset of S_X and proved that the linear operator T satisfies the condition; $T\perp_B A \Rightarrow \exists x \in D : Tx\perp_B Ax$, where A is a linear operator on $L(X)$ and D is connected closed subset of S_X. For the normed linear space X of dimension 2, their next research in 2015 (see [10]) explore the converse of previous result as obtained in [9]. They proved that if a linear operator T satisfies Bhatia-Semrl property, then the set of unit vectors S_X, on which T attains norm, is connected in the projective space $RP^{'} = S_X \setminus \{x \sim -x\}$ and conversely. For a strictly convex normed space X, the set of operators in $L(X)$ satisfying the Bhatia-Semrl property is dense in $L(X)$. [10] Let T be a linear operator on a normed space X. Then the set of unit vectors in S_X at which T attains norm is given by $M_T = \{x \in S_X : \|Tx\| = \|T\|\}$. Such a T satisfies Bhatia-Semrl property if for any operator $A \in L(X), T\perp_B A \Rightarrow \exists x \in M_T : Tx\perp_B Ax$.. Sain et al. proved a slight different concept depending on the nature of M_T described in [9] by stating that ; if $M_T \neq D \cup (-D)$ and the condition on the form of M_T implying that T may not satisfies the Bhatia-Semrl property.

Theorem 3.1. [10] *Let T be a linear operator on a finite dimensional real normed space X and $M_T = \{x \in S_X : \|Tx\| = \|T\|\}$. If M_T can be partitioned into tow non-empty sets which are contained in complementary subset of X, then there is a linear operator A on X such that $T\perp_B A$ but $Tx \not\perp_B Ax$.*

Theorem 3.2. [10] *Let T be a linear operator on a finite dimensional real smooth normed space X. If $M_T = \{x \in S_X : \|Tx\| = \|T\|\}$ is a countable set with more than 2 points. Then for any $x \in M_T$ there is a linear operator A on X such that $T\perp_B A$ but $Tx \not\perp_B Ax$*

Theorem 3.3. *Let T be a linear operator on a two dimensional real normed space X, and let $M_T = \{x \in S_X : \|Tx\| = \|T\|\}$. If M_T has more than two components, then for any $x \in M_T$ there is a linear operator A on X such that $T\perp_B A$ but $Tx \not\perp_B Ax$.*

4. Strong Birkhoff-James orthogonality

Paul et al. in the paper [11] proved that a normed linear space X is strictly convex if and only if for all $x \in S_X$ there is bounded linear operator A which attain its norm only at the points of the form λx with $\lambda \in S_k$. To prove this, they have introduced a concept of strong Birkhoff-James orthogonality. [11] For any normed linear space X, x is said to be strongly orthogonal to y in the sense of Birkhoff-James iff $\|x\| < \|x + \lambda y\|$ for all $\lambda \neq 0$. The notation $x\perp SBy$ was used to indicate the strongly Birkhoff-James orthogonality and proved that the strongly Birkhoff-James orthogonality implies Birkhoff orthogonality, but the converse may not be true. To illustrate this concept, two elements $(1, 0)$ and $(0, 1)$ are taken in $l_\infty(\mathbb{R}^2)$, showing that $(1, 0)$ and $(0, 1)$ are orthogonal in the sense of Birkhoff-James but not strongly orthogonal to each other.

Definition 4.1. (Strongly orthogonal set)[11]: A finite set of elements $\{x_1, ...x_k\}$ is said to be strongly orthogonal set in the sense of Birkhoff-James iff for each $m \in \{1, 2, ...k\}$ $\quad \|x_m\| < \|x_m + \sum_{m=1, m\neq n}^{k} \lambda_n x_n\|$, whenever $\lambda_n \neq 0$.

In case of an infinite set, if every finite subset of the set is strongly orthogonal in the sense of Birkhoff-James, then the infinite set is said to strongly orthogonal and conversely.

Theorem 4.2. [11] *Let X be a normed linear space and $x_0 \in S_x$. If there exists a Hamel basis of X containing x_0 which is strongly orthogonal relative to x_0 in the sense of Birkhoff-James, then x_0 is an extreme point of B_X.*

Theorem 4.3. [11] *Let X be a normed linear space and $x_0 \in S_X$ be an exposed point of B_X. Then there exists a Hamel basis of X containing x_0 which is strongly orthogonal relative to x_0 in the sense of Birkhoff-James.*

Theorem 4.4. [11] *Let X be a normed linear space and $x_0 \in S_X$. If there exist a Hamel basis of X containing x_0 which is strongly orthogonal relative to x_0 in the sense of Birkhoff-James, then there exists a bounded invertivle linear operator A on X such that $\|A\| = \|A_0\| > \|Ay\|$ for all $y \in S_X$ with $y \neq \lambda x_0, \lambda \in S_k$.*

Theorem 4.5. [11] *For a normed space X, and a point $x \in span(X)$, the following are equivalent:*

(i) x is an exposed point of B_X.
(ii) There is a Hamel basis of X containing x which is strongly orthonormal relative to x in the sense of Birkhoff-James.
(iii) There exists a bounded linear operator A on X which attains only at the points of the form λx with $\lambda \in S_k$.

Theorem 4.6. [11] *For a normed linear space X, the following are equivalent.*

(i) X is strictly convex.
(ii) For each $x \in S_X$, there exist a Hamel basis of X containing x which is strongly orthonormal relative to x in the sense of Birkhoff-James.

5. Orthogonality of operators in complex Banach Spaces

To study the difference of orthogonality in the complex case in comparison to the real case, Paul et al. in 2018 [12] came with a new concept of Birkhoff-James orthogonality by introducing new definitions on a complex reflexive Banach spaces and introduced more than one equivalent characterization of Birkhoff-James orthogonality of compact linear operators in the complex case. [12] For any bounded linear operator $T, A \in L(X)$, T is said to be Birkhoff-James orthogonal to A if $\|T + \lambda A\| \geq \|T\| \quad$ for all $\lambda \in \mathbb{C}$ and $M_T = \{x \in S_X : \|Tx\| = \|T\|\}$. In the real Banach space X, Sain introduced two sets x^+ and x^- in his paper [13] by

(i) $x^+ = \{y \in X : \|x + \lambda y\| \geq \|x\| \quad \text{for all} \quad \lambda \geq 0\}$ and
(ii) $x^+ = \{y \in X : \|x + \lambda y\| \geq \|x\| \quad \text{for all} \quad \lambda \leq 0\}$

For the complex Banach space, Paul et al. in 2018 introduced the following notations [12] depending on Sain's concept : For any $\gamma \in V$,

(i) $x_\gamma^+ = \{y \in X : \|x + \lambda y\| \geq \|x\| \quad \text{for all} \quad \lambda = tr, t \geq 0\}$
(ii) $x_\gamma^- = \{y \in X : \|x + \lambda y\| \geq \|x| \quad \text{for all} \quad \lambda = tr, t \leq 0\}$
(iii) $x^{\frac{1}{\gamma}} = \{y \in X : \|x + \lambda y\| \geq \|x\| \quad \text{for all} \quad \lambda = tr, t \in \mathbb{R}\}$
(iv) where $V = \{\gamma \in \mathbb{C} : |\gamma| = 1, arg(\gamma) \in [0, 2\pi]\}$.
(v) If $\mu = e^{i\pi}\gamma$, then $x_\mu^+ = x_\gamma^-, x_\mu^- = x_\gamma^+$ and $x^{\frac{1}{\mu}} = x^{\frac{1}{\gamma}}$. In the complex Banach space,
(vi) $x^+ = \cap\left\{x_\gamma^+ : \gamma \in V\right\}, x^- = \cap\left\{x_\gamma^- : \gamma \in V\right\}$ and $x^\perp = \cap\left\{x^{\frac{1}{\gamma}} : \gamma \in V\right\}$

Proposition 5.1. [13] *Let $x, y \in X$, where X is an complex Banach space and $\gamma \in V$. Then following statements are true*

(i) Either $y \in x_\gamma^+$ or $y \in x_\gamma^-$.
(ii) $x \perp \gamma y \Leftrightarrow y \in x_\gamma^+$ or $y \in x_\gamma^-$.
(iii) $y \in x_\gamma^+ \Rightarrow \eta y \in (\xi x)_\gamma^+$ for all $\eta, \xi > 0$.
(iv) $y \in x_\gamma^+ \Rightarrow -y \in x_\gamma^-$ and $y \in (-x)_\gamma^-$.
(v) $y \in x_\gamma^- \Rightarrow \eta y \in (\xi x)_\gamma^-$ for all $\eta, \xi > 0$.
(vi) $y \in x_\gamma^- \Rightarrow -y \in x_\gamma^+$ and $y \in (-x)_\gamma^+$.
(vii) $y \in x_\gamma^+ \Rightarrow \mu y \in (\mu x)_\gamma^+$ for all $\mu \in \mathbb{C}$.

(viii) $y \in x_\gamma^- \Rightarrow \mu y \in (\mu x)_\gamma^-$ *for all* $\mu \in \mathbb{C}$.

Proposition 5.2. [13] *Let* $x, y \in X$*, where* X *is a complex Banach space. Then the following are true*

(i) $x \perp_B y \Leftrightarrow y \in x^+$ *and* $y \in x^-$.
(ii) $y \in x^+ \Rightarrow \eta y (\xi x)^+$ *for all* $\eta, \xi > 0$.
(iii) $y \in x^+ \Rightarrow -y \in x^-$ *and* $y \in (-x)^-$.
(iv) $y \in x^- \Rightarrow -y \in x^+$ *and* $y \in (-x)^+$.
(v) $y \in x^- \Rightarrow \eta y \in (\xi x)^-$ *for all* $\eta, \xi > 0$.

Theorem 5.3. [13] *Let* X *be a reflexive complex Banach space, and* Y *be any complex Banach space. Let* $T, A \in K(x, y)$*. Then* $T \perp_B A \Leftrightarrow \forall \gamma \in V, \quad \exists \quad x = x(\gamma), y = y(\gamma) \in M_T : Ax \in (Tx)_\gamma^+$ *and* $Ty \in (Ty)_\gamma^-$.

Theorem 5.4. [13] *Let* X *be a complex Banach Space. Let* $x, y \in X$ *and* $r = e^{i\theta}$*, where* $\theta \in [0, 2\pi]$*. If* $y \in x_\gamma^+$*, then either* $y \in x_\mu^+$ *for all* μ *with* $\arg \mu \in [0, \theta]$ *or* $y \in x_\mu^+$ *for all* μ *with* $\arg \mu \in [0, \pi]$.

Theorem 5.5. [13] *Let be a linear operator on a finite dimensional complex Banach space X, such that* M_T *is a closed connected subset of* S_X*. Then for* $A \in L(X)$*,* $T \perp_B A \Leftrightarrow \forall \gamma \in V \quad \exists \quad x = x(\gamma) \in M_T : Tx \perp_\aleph Ax$.

Theorem 5.6. [13] *Let* T *be a linear operator in a finite dimensional complex Banach space* X *such that* M_T *is a closed connected subset of the unit sphere of* X*. Then for* $A \in L(X)$*,* $T \perp_B A \Leftrightarrow \exists \quad \theta \in [0, \pi]$ *and* $x, y \in M_T : Ax \in (Tx)_\gamma^+$ *for all* γ *with* $\arg \gamma \in [\theta - \pi, \theta]$ *and* $Ay \in (Ty)_\gamma^+$ *for all* γ *with* $\arg \gamma \in [\theta, \theta + \pi]$.

6. Geometric Properties

Definition 6.1. [14] Let $x, y \in X$ and $T = \{\mu \in K : |\mu| = 1\}$. Then x is said to be norm parallel to y if $\|x + \mu y\| = \|x\| + \|y\|$ for all $\mu \in T$.

Norm parallelism is symmetric as well as homogeneous; whereas, Birkhoff-James orthogonality is homogeneous but not symmetric in a Banach space. [14] In the case of Hilbert space, two elements are linearly dependent iff they are norm- parallel; however, in normed spaces two linearly dependent vectors are norm-parallel, but the converse may not be true. For instance, $(1, 1)$ and $(1, 0)$ are norm parallel but not linearly dependent. Depending on the concept of Birkhoff-James orthogonality and strong Birkhoff-James orthogonality Paul et al.[14] introduce a new geometric notion of semi-rotund point. For any normed linear space X, $\beta \neq x \in X$ is said the semi-rotund point of X if $\exists y \in X : x \perp_{SB} y$. If for every $x \neq 0 \in X$, x is a semi-rotund point, the normed space X is said to be semi-rotund space. Dragomir introduced the concept of approximate Birkhoff-James orthogonality [15] as follows: x is said to be approximate Birkhoff-James orthogonal to y if $\|x + \mu y\| \geq (1 - \epsilon)\|x\|$ for all $\mu \in K$ and $\epsilon \in [0, 1]$; however, Chmielinski [14, 16] defined approximate Birkhoff-James orthogonality as ; $x \perp_D^\epsilon \Leftrightarrow \|x + \mu y\| \geq \sqrt{1 - \epsilon^2}\|x\|$ for all $\mu \in K$. The concept of approximate parallelism was developed by Zamani and Moslehian [17] by stating that x is approximately parallel to y if $\inf \{\|x + \lambda y\| : \lambda \in K\} \leq \epsilon \|x\|$ for all $\epsilon \in [0, 1]$.

Proposition 6.2. [14] *let* X *be a bounded linear operator form a normed space* X *to normed space* Y *and* $x \in M_T$*. Then for any* $\epsilon \in [0, 1]$ *and* $y \in X$*, we have* $x||_y^\epsilon \Rightarrow Tx||^\epsilon Ty$.

Theorem 6.3. [14] *Let* T *and* A *are compact linear operators form a reflexive Banach space* X *to any normed space* Y*. Then* $T||A \Leftrightarrow \exists \quad x \in M_T \cap M_A : Tx||Ax$.

Theorem 6.4. [14] *If T and A are bounded linear operators form a normed space X to Y. Then $T||A \Leftrightarrow \quad \exists \{x_n\} \in S_X : \lim_{n\to\infty} \|Tx_n\| = \|T\|, \lim_{n\to\infty} \|Ax_n\| = \|A\|$ and $\lim_{n\to\infty} \|Tx_n + \mu Ax_n\| = \|T\| + \|A\|$, for some $\mu \in K$.*

Proposition 6.5. [14] *Let T be a bounded linear operator form a normed space X into normed space Y and let $x \in M_T$. Then $Tx \perp_D^{\epsilon} Ty \Rightarrow x \perp_D^{\epsilon} y$ for any $\epsilon \in [0,1]$ and $y \in X$.*

Theorem 6.6. [14] *let T and A are bounded linear operators from finite dimensional Banach spaces X to Y. Then $T \perp_{SB} A \Leftrightarrow \forall \epsilon > 0, \exists \mu_\epsilon > 0 : \forall |\mu| < \mu_\epsilon, \exists y_\mu \in (\cup_{x \in M_T} B(x,\epsilon)) \cap S_x : \|Ty_\mu + \mu Ay_\mu\| > \|T\|$.*

Theorem 6.7. [14] *Let T and A are compact linear operators fron a reflexive Banach space X to any normed space Y be such that $T \perp_B A$ but $T \not\perp_{SB} A$. Then there exists $x \in M_T$ such that $Tx \perp_B Ax$.*

Theorem 6.8. [14] *Let T and A are bounded linear operators from a normed space X to Y. If $T \perp_B A$ but $T \not\perp_{SB} A$, then there exists a sequence $\{x_n\}$ in S_X such that $\|Tx_n\| \to \|T\|, Ax_n \to 0$ or there exist a sequence $\{x_n\}$ in S_X and sequence $\{\epsilon_n\}$ in $\mathbb{R}^+$ such that $\|Tx_n\| \to \|T\|, \epsilon_n \to 0$, and $Tx_n \perp_D^{\epsilon_n} Ax_n$.*

7. Relation between Birkhoff-Jame, Robert, and isosceles orthogonality in terms of bounded linear operators

Recently, Bottazzi et al. in [18] has introduced a new generalization of earlier results on orthogonality of bonded linear operators. They discussed about Birkhoff-James,Isosceles, and Robert orthogonality in Banach spaces in terms of bounded linear operators. For better description of Birkhoff-James orthogonality, they introduced the sets,
$\mathscr{O} = \{x \in S_X : Tx \perp_B Ax\}$ for any $T, A \in B(X)$ and $M_T = \{x \in S_X : \|Tx\| = \|T\|\}$.
For any bounded linear operator A on the Hilbert space H; $A^*, R(A)$, and $N(A)$ denotes the adjoint, range and kernal of A respectively. The bounded linear operators A and B in a real or complex Hilbert space H have a disjoint support if $AB^* = BA^* = 0$.

Theorem 7.1. [18] *Let X be reflexive Banach space and Y be Banach spaces, either both real, or both complex. Let T and A are compact linear operators from X to Y be such that for any $x_0 \in S_X$,*

$$M_T = \begin{cases} \pm x_0 & \text{in the real case} \\ e^{i\theta}x_0 : \theta \in [0, 2\pi] & \text{in the complex case} \end{cases}$$

Then $T \perp_B A \Leftrightarrow \mathscr{O}_{T,A} \cap M_T \neq \phi$.

Theorem 7.2. [18] *Let T and A are compact linear operators from a reflexive Banach space X to any real Banach space Y. If T is Birkhoff-James orthogonal to A, then the set $\mathscr{O}_{T,A}$ is non-empty.*

Theorem 7.3. [18] *Let X, Y be two Banach spaces, either both real,or, both complex. let and T and A are bounded linear operators from X to Y. Then, $\mathscr{O}_{T,A} = S_X \Rightarrow T \perp_B A$.*

Theorem 7.4. [18] *A real or complex Hilbert space H is of finite dimensional if and only if for any bounded linear operators in H, $T \perp_B A \Rightarrow \mathscr{O}_{T,A} \neq \phi$.*

Proposition 7.5. [18] *For any bounded linear operators A and T in a real or complex Hilbert space H satisfying $T^*A = 0$, then the following statements holds:*

(i) *$A \perp_B T$ and $T \perp_B A$,*
(ii) *$A \perp_R T$, and in particular, $A \perp_I T$*

Proposition 7.6. [18] *Let X be real or complex normed space. Let $x, y \in X$ and assume that $x + y \perp_B y$ and $x - y \perp_B y$. Then $x \perp_I y$*

Remark 7.7. In order to illustrate the concept regarding to the converse part of the above proposition Sain et al in [18] introduced strongly Isosceles orthogonality in the real Banach space by stating that: An element $x \in X$ is said to strongly orthogonal to $y \in X$ (written as $x \perp_{SI} y$) if the following conditions are satisfied;

(i) $x \perp_I y$,
(ii) there exists a real sequence $\{\lambda_n\}_{n\in\mathbb{N}}$, with $\lambda_n > 0$ such that $\lim_{n\to\infty} \lambda_n = 0$ and $x \perp_I \lambda_n y$ for all $n \in \mathbb{N}$.

Theorem 7.8. [18] *Let $x, y \in X$. Then $x \perp_{SI} y \Rightarrow x \perp_B^r y$ and in particular and is X is real normed space then $x \perp_{SI} y \Rightarrow x \perp_B y$.*

8. Birkhoff-James orthogonality by applying semi-inner product

The concepts of Birkhoff-James orthogonality has been widely used by various researchers since 1935. The latest research on this topic by Sain, Mal, and Paul [19]have studied Birkhoff-James orthogonality of compact linear operators between Hilbert space and Banach spaces by applying the notion of semi-inner product in normed linear spaces.

Definition 8.1. [19] For any normed linear space x, A scalar valued function $(.,.) : X \times X \to K$ is a semi-inner product if for any $\xi, \eta \in K$ and for any $x, y, z \in X$, it satisfies the following conditions:

(i) $(\xi x + \eta y, z) = \xi(x, z) + \eta(x, z)$,
(ii) $(x, x) > 0$, whenever $x \neq 0$.
(iii) $| (x, y) |^2 \leq (x, x)(y, y)$,
(iv) $(x, \xi y) = \bar{\xi}(x, y)$.

Every semi-inner product space is a normed space with the norm $\|x\|^2 = (x, x)$ and the norm of any normed space can be generated through a semi-inner product in infinitely many ways. Sain et al. in [19] characterized the Birkhoff-James orthogonality set of any compact linear operators between a reflexive Banach space any Banach spaces. They also proved that there is an relationship between the concept of semi-inner product spaces and the sets $x^+ = \{y \in X : \|x + \gamma y\| \geq \|x\| \text{for} \gamma \geq 0\}$ and $x^+ = \{y \in X : \|x + \gamma y\| \geq \|x\| \text{for} \gamma \leq 0\}$

Theorem 8.2. [19] *Let T and A be compact linear operators from a reflexive Banach space X to any Banach space Y. If any one of the following conditions holds;*

(i) M_T is a connected subset of S_X.
(ii) M_T is not connected but $M_T = D \cup (-D)$, where D is a non-empty subset of S_X.

Then $T \perp_B A \Leftrightarrow \exists x \in M_T : Tx \perp_B Ax$.

Theorem 8.3. [19] *For a finite-dimensional Banach space X, the following statements are are equivalent.*

(i) For any linear operator T on X, M_T is the unit sphere of some subspace of X.
(ii) For any linear operator T on X, $M_T = D_T \cup (-D_T)$, where D_T is connected subset of X.
(iii) X is an Euclidean space.

As an correlation between the semi-inner product space and geometric concepts of the sets Sain et al. proved the following theorem.

Theorem 8.4. [19] *Let $x, y \in X$, where X is a normed linear space. Then the following are true.*

(i) $y \in x^+$ iff there exists a semi-inner product $(.,.)$ on X with $(y, x) \geq 0$.
(ii) $y \in x^-$ iff there exists a semi-inner product $(.,.)$ on X with $(y, x) \leq 0$.

Theorem 8.5. *Let T and A be compact linear operators from a reflexive Banach space X to any Banach space Y be such that $T \perp_B A$. let $\mathscr{O}_Y$ denotes the collection of all semi-inner product on Y. Then*

$$\|T\| = \begin{cases} \sup\{(Tx, y) : x \in S_X, y \in S_Y, (.,.) \in \mathscr{O}_Y, (Ax, y) \geq 0\} \\ \sup\{(Tx, y) : x \in S_X, y \in S_Y, (.,.) \in \mathscr{O}_Y, (Ax, y) \leq 0\} \end{cases}$$

Theorem 8.6. [19] *Let T and A be bounded linear operators form a normed space X to Y be such that $T \perp_B A$. Ley $\mathscr{O}_Y$ denotes the collection of semi-inner product space on Y. Let $\epsilon > 0$ be arbitrary but fixed after A choice. Then*

(i) $\|T\| = \max\{l_1(\epsilon), l_2(\epsilon)\} = \max\{l_1(\epsilon), l_3(\epsilon)\}$*, where*
(ii) $l_1(\epsilon) = \sup\{(Tx, y) : x \in S_X, y \in S_Y, (.,.) \in \mathscr{O}_Y, |\ (Ax, y)\ | < \epsilon\}$
(iii) $l_2(\epsilon) = \sup\{(Tx, y) : x \in S_X, y \in S_Y, (.,.) \in \mathscr{O}_Y, Ax \in (y)^{+\epsilon}\}$
(iv) $l_3(\epsilon) = \sup\{(Tx, y) : x \in S_X, y \in S_Y, (.,.) \in \mathscr{O}_Y, Ax \in (y)^{-\epsilon}\}$

Theorem 8.7. [19] *Let X be normed linear space such that X^* is strictly convex. Let $f, g \in X^*$ be such that $f \perp_B g$. then*

$$\|f\| = \begin{cases} \sup\{f(x) : x \in S_x, g(x) \geq 0\} \\ \sup\{f(x) : x \in S_x, g(x) \leq 0\}. \end{cases}$$

Theorem 8.8. [19] *Let T and A are compact linear operators from a reflexive Banach space X to any Banach space Y be such that for each $\lambda \in \mathbb{R}, M_{T+\lambda A} = D_\lambda \cup (-D_\lambda)$, where D_λ is a non-empty connected subset of S_X. Let $\mathscr{O}_Y$ denotes the collection of all semi-inner product space on Y. Then*

$$dist(T, span\{A\}) = \sup\{(Tx, y) : x \in S_X, y \in S_Y, (.,.) \in \mathscr{O}_Y, (Ax, y) = 0\}.$$

Theorem 8.9. [19] *Let X be a reflexive Banach space and Y be any Banach space. Let $\mathcal{Z}$ be a finite dimensional subspace of $K(X, Y)$. Let $T \in K(X, Y) \setminus \mathscr{Z}$. Let us further assume that for any $\lambda \in \mathbb{R}$ and for any $A \in \mathscr{Z}, M_{T+\lambda A} = D_{\lambda,A} \cup (-D_{\lambda,A})$, where $D_{\lambda,A}$ is non-empty connected subset of S_X. Then there exist $A_0 \in \mathscr{Z}$ such that*

$$dis(T, \mathscr{Z}) = \sup\{(Tx, y) : x \in S_X, y \in S_Y, (A_0x, y) = 0\}.$$

Moreover, A_0 is the best approximation of T in $\mathscr{Z}$.

9. Modular Birkhoff orthgonality in Banach modules

We have already mentioned that Rajic et al. in[8] studied Birkhoff-James orthogonality in a Hilbert C^*-modules over a C^*-algebra. The most current research as generalization of Birkhoff-James orthogonality from Hilbert space to Banach spaces in [20], Sain and Tanaka studied the stronger version of modular Birkhoff-James orthogonality in the set of bounded and compact linear operators. In order to prove their study they introduced the following notions: $X^\perp = \{y \in X : x \perp_B y\}$ and $M_A = \{x \in S_X : \|Ax\| = \|A\|\}$. An element $x \neq 0 \in X$ is said to be smooth point in X if $\mathscr{T}(x) = \{f \in S_X^* : f(x) = \|x\|\}$ is a singleton set. For any Banach space X, an element $x \in X$ is said to be left symmetric in X if for any $y \in X$, $x \perp_B y \Rightarrow y \perp_B x$. Similarly x is said to be right symmetric in X if for any $y \in X$, $y \perp_B x \Rightarrow x \perp_B y$. If x is both left as well as right symmetric, then x is said to be a symmetric point.

Definition 9.1. [20] A Banach space X is called a right $\mathscr{A}$-module (where $\mathscr{A}$ is a Banach algebra) if there exists a mapping of $X \times \mathscr{A}$ into X such that for each $a, b \in \mathscr{A}$ and $x \in X$, $x(ab) = (xa)b$ and $\|ax\| \leq \|x\|\|a\|$.

An element $x \in X$ is said to be right-modular Birkhoff-James orthogonal to $y \in X$, if $x \perp_B ya$ for all $a \in \mathscr{A}$ and left-modular Birkhoff-James orthogonal to y if $x \perp_B ay$ for all $a \in \mathscr{A}$.

Theorem 9.2. [20] *Let T and A be compact linear operators form a reflexive real Banach space X to any real Banach space Y such that $M_A = \{\pm x_0\}$ for some $x_0 \in S_X$. Then $A^{\perp}_{B(X)} \Leftrightarrow T(X) \subset (Ax_0)^{\perp}$.*

Definition 9.3. [20] A Banach space X is said to be Kadets-Klee if whenever $\{x_n\}$ is a sequence in X and $x \in X$ is such that $\{x_n\}$ converges weekly to x and $\lim\limits_{n\to\infty} \|x_n\| = \|x\|$, then $\lim_{n\to\infty} \|x_n - x\| = 0$.

Theorem 9.4. [20] *Let A be a compact linear operator from a reflexive Kadets-Klee real Banach space to any real Banach space be such that $M_T = \{\pm x_0\}$ for some $x_0 \in S_X$. Then given any bounded linear operator $T \in B(X,Y), A \perp_{B(X)} T \Leftrightarrow T(X) \subset (Ax_0)^{\perp}$.*

Theorem 9.5. [20] *Let X, Y be real Banach spaces. Let $A \in B(X,Y)$ be a smooth point in $B(X,Y)$ such that $M_A \neq 0$. Then given any $T \in B(X,y)$, $A \perp_{B(X)} T \Leftrightarrow T(X) \subset (Ax_0)^{\perp}$, where $M_A = \{\pm x_0\}$.*

Theorem 9.6. *Let T and A are compact linear operators from a reflexive complex Banach space X to any complex Banach space Y be such that $M_A = \left\{e^{i\theta}x_0 : \theta \in [0,2\pi]\right\}$ for some $x_0 \in S_X$. Then given any compact linear operator T, $A \perp_{B(X)} T \Leftrightarrow T(X) \subset (Ax_0)^{\perp}$.*

Theorem 9.7. [20] *Let T and A are compact linear operators from a reflexive real Banach Space X to any real Banach space Y be such that $M_A = \{\pm x_0\}$ for some $x_0 \in S_X$. Then given any compact linear operator T, $A \perp^*_{B(Y)} T = \Leftrightarrow Tx_0 = 0$. Moreover,if X is Kadets-Klee, then same is true for any $T \in B(X,y)$.*

Theorem 9.8. [20] *Let T and A are compact linear operators from a reflexive complex Banach space to any complex Banach space Y be such that $M_A = \left\{e^{i\theta}x_0 : \theta \in [0,2\pi]\right\}$ for some $x_0 \in S_X$. Then given any $T \in K(X,Y), A \perp^*_{B(Y)} T \Leftrightarrow Tx_0 = 0$.*

If A is a bounded linear operator from a normed spaces X to Y, then its adjoint $A^* \in B(Y^*, X^*)$ is defined by $(A^*y^*) = y^*Ax$ for each $x \in X$, $y^* \in Y^*$ and $\|A^*\| = \|x\|$. For any subsets R and S of a Banach space X, $R \perp_B S$ if $x \perp_B y$ for all $x \in R$ and $y \in S$.

Proposition 9.9. [20] *Let T and A are bounded linear operators from a Banach space X to Y. If $A(x) \perp_B T(X)$, then $A \perp_B T$.*

Theorem 9.10. [20] *Let X and Y be finite dimensional Banach spaces with $\dim(X) \geq \dim(Y) > 0$, and let $A \in B(X,y)$ and suppose that $A(X) = Y$. Then A is right symmetric for $\perp_{B(X)}$ in $B(X,Y)$.*

10. Open problems

Definition 10.1. [21] In a normed linear space X,

$$x \perp y \Leftrightarrow \sum_{k=1}^{m} a_k \|b_k x + c_k y\|^2 = 0,$$

where $m \geq 2$ and a_k, b_k, c_k are real numbers such that

$$\sum_{k=1}^{m} a_k b_k c_k = 1, \quad \sum_{k=1}^{m} a_k b_k^2 = \sum_{k=1}^{m} a_k c_k^2 = 0$$

.

Problem 10.2. Birkhoff-James, Robert, and isosceles orthogonality has been studied in terms of linear operators in Hilbert space and general Banach spaces.This fact raises a question- can Carlsson orthogonality(in particular Pythagorean orthogonality) be characterized in terms of operators in Hilbert C^* as well as Banach modules?

Problem 10.3. According to proposition-7.8 in [20] if two bounded linear operators in a real or complex Hilbert space satisfy $T^*A = 0$, then these operators are Birkhoff-James, Robert and isosceles orthogonal. This fact leaves behind a question if we can prove the condition of Pythagorean orthogonality by introducing some different nature of operators A and T in the same space or not.

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