Methods of Functional Analysis and Topology
Vol. 27 (2021), no. 1, pp. 1–9
https://doi.org/10.31392/MFAT-npu26_1.2021.01

MFAT METHODS OF FUNCTIONAL ANALYSIS AND TOPOLOGY

DIFFEOMORPHISMS OF FOLIATED MANIFOLDS

G.M. ABDISHUKUROVA AND A.YA. NARMANOV

Abstract. The set $Diff(M)$ of all diffeomorphisms of a manifold M onto itself is the group related to composition and inverse mapping. The group of diffeomorphisms of smooth manifolds is of great importance in differential geometry and in analysis. It is known that the group $Diff(M)$ is a topological group in compact open topology. In this paper we investigate the group $Diff_F(M)$ of diffeomorphisms foliated manifold (M, F) with foliated compact open topology. It is proven that foliated compact open topology of the group $Diff_F(M)$ has a countable base. It is also proven that the group $Diff_F(M)$ is a topological group with foliated compact open topology. Also some one-parameter subgroups of the group $Diff_F(M)$ are found and studied for the foliations generated by special submersions.

Множина $Diff(M)$ всіх дифеоморфізмів многовиду M є групою відносно композиції та взяття оберненого і топологічною групою в компактно-відкритій топології. Групи дифеоморфізмів гладких многовидів мають велике значення в диференціальній геометрії та аналізі. У цій роботі досліджується група дифеоморфізмів шаруватого многовиду з розшарованою компактно-відкритою топологією. Показано, що ця топологія має зліченну базу. Знайдені деякі однопараметричні підгрупи групи $Diff(M)$ і досліджені для шарувань, породжених спеціальними субмерсіями.

1. Introduction

In this paper the group of diffeomorphisms of a foliated manifold with foliated compact open topology are studied. The foliated compact open topology was introduced in the paper [7] and studied in [9].

Let M be a smooth connected manifold of dimension n. Smoothness in this paper means the class C^∞-smoothness.

Let us recall the definition of a foliation.

Definition 1.1. A foliation F on M of dimension k (codimension $n-k$) is a partition of M into arcwise connected subsets L_α with the following properties:
1. $M = \bigcup L_\alpha$,
2. $L_\alpha \bigcap L_\beta = \emptyset$ if $\alpha \neq \beta$,
3. For every point $p \in M$ there is an open neighborhood U of p and a chart $x = (x_1, x_2, \cdots, x_k, y_1, y_2, \cdots, y_{n-k})$ such that for each leaf L_α the connected components of $L_\alpha \cap U$ are defined by the equations $y_1 = const, y_2 = const, \cdots, y_{n-k} = const$. Such a chart is a distinguished chart.

The connected components of the sets $y_1 = const, y_2 = const, \cdots, y_{n-k} = const$ in a distinguished chart are called plaques (plates) of F. Fixing $y_1 = const, y_2 = const, \cdots, y_{n-k} = const$, the map $x \to (x, y))$ is a smooth embedding, therefore the plaques are connected k-dimensional submanifolds of M. This shows that each leaf L_α is a union of plaques and there exists a differential structure σ_α on L_α such that $(L_\alpha, \sigma_\alpha)$ is a k-dimensional connected manifold. Note that the canonical injection $i : (L_\alpha, \sigma_\alpha) \to M$ is an immersion, but it is not necessarily an embedding [3].

An example of a foliation is given by a smooth submersions. Let us recall definition of a submersion.

2020 *Mathematics Subject Classification.* 57R30, 57M07, 57S05.
Keywords. foliated manifold, submersion, foliated compact open topology, diffeomorphisms of foliated manifold, Lie group.

Definition 1.2. Differentiable mapping of maximal rank $\pi : M \to B$, where M and B are manifolds of dimensions n, m respectively, is called a submersion if $n > m$.

By the rank theorem of a differentiable function the full inverse image $L_p = \pi^{-1}(p)$ of every point $p \in B$ is a submanifold of dimension $k = n-m$. So the connected components of the inverse images of points $p \in B$ define a $k = n - m$-dimensional foliation of M.

For a point $q \in L_p$ we denote by T_qF the tangent space of the leaf L_p at the point q, by $H(q)$ the orthogonal complement of the tangent space T_qF of the leaf L_p, i.e. $T_qM = T_qF \oplus H(q)$. We have two distributions $TF : q \to T_qF$, $H : q \to H(q)$. The distribution $TF : q \to T_qF$ is a completely integrable distribution whose maximum integral submanifolds are leaves of the foliation F, the distribution of $H : q \to H(q)$, which is the orthogonal complement of TF, is not necessarily completely integrable.

Each vector field X can be represented as $X = X^v + X^h$, where X^v, X^h are the orthogonal projections of X onto P, H respectively. Here for convenience P, H are considered as subbundles of the tangent bundle TM. If $X^h = 0$, then X is called a vertical field (it is tangent to the foliation), and if $X^v = 0$ the vector field X is called a horizontal field.

Definition 1.3. A submersion of $\pi : M \to B$ is called Riemannian if its differential $d\pi$ preserves the length of horizontal vectors.

Many studies deal with geometry of Riemannian submersions, i.g., [12], [4],[5],[11]. In particular, in [10], fundamental equations of Riemannian submersion were obtained.

Now we construct an example of a Riemannian submersion using Killing vector fields.

Denote by $V(M)$ the set of all smooth vector fields defined on the manifold M, by $[X, Y]$ the Lie bracket vector field $X, Y \in V(M)$. With respect to the Lee bracket, the set $V(M)$ is a Lie algebra.

For a point $x \in M$ by $t \to X^t(x)$ we denote the integral curve of the vector field X passing through the point x at $t = 0$. The map $t \to X^t(x)$ is defined in some region $I(x) \subset R$ which generally depends on the field X and the starting point x. In what follows, everywhere in formulas of the form $X^t(x)$ we will assume that $t \in I(x)$.

Recall that the vector field X on M is called a Killing vector field if the one-parameter group of local transformations $x \to X^t(x)$ generated by the field X consists of isometries [8], [1], [6].

Consider the following vector fields on the plane $R^2(x_1, x_2)$ with the Cartesian coordinates (x_1, x_2):

$$X_1 = \frac{\partial}{\partial x_1}, \qquad X_2 = \frac{\partial}{\partial x_2}, \qquad X_3 = -x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2}. \tag{1.1}$$

We can define a submersion

$$\pi : R^3 \to R^2 \tag{1.2}$$

using the vector fields X_1, X_2, X_3 by putting

$$\pi(t_1, t_2, t_3) = X_3^{t_3}(X_2^{t_2}((X_1^{t_1}(O)...))),$$

where O is the origin of the coordinate system (x_1, x_2). As follows from [8], for every point $(x_1, x_2) \in R^2$ there exist points $(t_1, t_2, t_3) \in R^3$ such that $\pi(t_1, t_2, t_3) = (x_1, x_2)$. The submersion π in the coordinate system has following form:

$$\pi(t_1, t_2, t_3) = \{\pi_1(t_1, t_2, t_3), \pi_2(t_1, t_2, t_3)\},$$

where

$$\pi_1(t_1, t_2, t_3) = t_1 \cos t_3 - t_2 \sin t_3,$$
$$\pi_2(t_1, t_2, t_3) = t_1 \sin t_3 + t_2 \cos t_3.$$

Jacobi matrix of the map π is following matrix

$$J(\pi) = \begin{pmatrix} \cos t_3 & -\sin t_3 & -t_1 \sin t_3 - t_2 \cos t_3 \\ \sin t_3 & \cos t_3 & t_1 \cos t_3 - t_2 \sin t_3 \end{pmatrix}.$$

It is easy to check that the rank of the Jacobi matrix of the map π equal to 2, i.e., π is a submersion. This submersion generates a one-dimensional foliation leaves of that are one-dimensional manifolds. For every point $p \in R^2$ the leaf $\pi^{-1}(p)$ is given by the equations

$$t_1 = x_1 \cos u + x_2 \sin u, t_2 = -x_1 \sin u + x_2 \cos u, t_3 = u \tag{1.3}$$

The leaf is a curve of constant curvature and constant torsion, which is verified by a direct calculation,

$$k = \frac{\sqrt{(x_1^2 + x_2^2)(1 + x_1^2 + x_2^2)}}{(2 + x_1^2 + x_2^2)^{\frac{3}{2}}},$$
$$\sigma = \frac{1}{1 + x_1^2 + x_2^2}.$$

The tangent vector of the curve (1.3), i.e., the vertical vector field has following form:

$$V = t_2 \frac{\partial}{\partial t_1} - t_1 \frac{\partial}{\partial t_2} + \frac{\partial}{\partial t_3}.$$

The distribution orthogonal to this one-dimensional foliation is generated by the following horizontal vector fields:

$$H_1 = \frac{\partial}{\partial t_1} - t_2 \frac{\partial}{\partial t_3}$$
$$H_2 = \frac{\partial}{\partial t_2} + t_1 \frac{\partial}{\partial t_3}.$$

This distribution is not completely integrable, since it is not involutive due to the fact that the Lie bracket $[H_1, H_2] = 2\frac{\partial}{\partial t_3}$ is not expressed linearly in terms of the vector fields H_1, H_2.

Now we introduce a Riemannian metric g on R^2, with respect to which the submersion (1.2) is a Riemannian submersion.

Let $p \in R^2$, $u, v \in R_p^2$ be tangent vectors at p, and X, Y be such vector fields that $X(p) = u, Y(p) = v$. Since the Euclidean space R^3 is a complete manifold, there exist horizontal vector fields $\widetilde{X}, \widetilde{Y}$ such that $d\pi(\widetilde{X}) = X, d\pi(\widetilde{Y}) = Y$ (They are called horizontal lifts of the vector fields X, Y) [11].

The vertical vector field V is a Killing field. Therefore, it satisfies the equality [8]

$$V\langle Z_1, Z_2\rangle = \langle [V, Z_1], Z_2\rangle + \langle Z_1, [V, Z_2]\rangle, \tag{1.4}$$

where Z_1, Z_2 are arbitrary vector fields, $[\cdot, \cdot]$ is the Lie bracket, $< \cdot, \cdot >$ is the inner product. From the equality (1.4) we have $V\langle \widetilde{X}, \widetilde{Y}\rangle = 0$. This means that the scalar product $\langle \widetilde{X}, \widetilde{Y}\rangle$ is constant at the points q of the leaf $\pi^{-1}(p)$. Therefore, by putting $g(u, v)_q = \langle \widetilde{X}, \widetilde{Y}\rangle_q$ we define a new Riemannian metric g on R^2. With respect this Riemannian metric, the submersion

$$\pi : R^3 \to (R^2, g)$$

is Riemannian.

2. Diffeomorphisms of foliated manifolds

Denote by (M, F) a manifold M with a foliation F of dimension k and call it a foliated manifold.

Definition 2.1. A diffeomorphism $\varphi : M \to M$ is called a diffeomorphism of the foliated manifold (M, F), if the image $\varphi(L_\alpha)$ of each leaf L_α is a leaf of the foliation F.

The diffeomorphism $\varphi : M \to M$ of the foliated manifold (M, F), is denoted by $\varphi : (M, F) \to (M, F)$. The set of all diffeomorphisms of a foliated manifold is denoted by $Diff_F(M)$. The set $Diff_F(M)$ is a group with respect to the superposition of mappings and is a subgroup of the group $Diff(M)$ of diffeomorphisms of the manifold M.

The group $Diff_F(M)$ was studied in the papers [7], [9], and, in particular, in [9] it was proved that this group is a closed subgroup of the group $Diff(M)$ with respect to the compact open topology.

Let $Iso_F(M)$ denote the subset of the set $Diff_F(M)$ consisting of isometries of the Riemannian manifold (M, g). This set is a subgroup of the group $Diff_F(M)$. In [9] it was proved that the group $Iso_F(M)$ is a Lie group with respect to the compact open topology.

Example 2.2. Let $M = R^2(x_1, x_2)$ be a Euclidean plane with the Cartesian coordinates (x_1, x_2). The foliation F is given by the submersion $f(x_1, x_2) = x_2 - x_1^2$. Diffeomorphism of the plane $\varphi_\lambda \colon R^2 \to R^2$ determined by the formula

$$\varphi(x, y) = (x_1, x_2 + \lambda f(x_1, x_2))$$

is a diffeomorphism of the foliated plane (R^2, F) for every $\lambda \in R$, $\lambda \neq -1$. Diffeomorphisms $(x_1, x_2) \to (x_1, x_2 + h)$ and $(x_1, x_2) \to (-x_1, x_2 + h)$ are elements of the group $Iso_F(M)$ for $h \in R$.

We will consider the group $Diff_F(M)$ with foliated compact open topology that depends on the foliation F and coincides with the compact open topology when F is an n-dimensional foliation. Foliated compact open topology was introduced in [7].

We recall the notion of the foliated compact open topology. Let $\{K_\lambda\}$ be a family of all compact sets where each K_λ is a subset of some leaf L_λ of the foliation F, and let $\{U_\beta\}$ be the family of all open sets in M. We consider, for each pair K_λ and U_β, the set of all mappings $f \in Diff_F(M)$ for which $f(K_\lambda) \subset U_\beta$. This set of mappings is denoted by $[K_\lambda, U_\beta] = \{f : M \to M | f(K_\lambda) \subset U_\beta\}$.

It is not difficult to show that every possible finite intersections of sets of the form $[K_\lambda, U_\beta]$ forms a base for some topology. This topology will be called a foliated compact open topology or in brief an F-compact open topology. The space $Diff_F(M)$ with the F-compact open topology is a Hausdorff topological space [7]. Since K runs only over all compact subsets of leaves, the F-compact open topology on $Diff_F(M)$ is weaker than the usual compact open topology induced from $Diff(M)$. It can be proved as follows.

Lemma 2.3. *The space $Diff_F(M)$ with an F-compact open topology is a topological space with countable base.*

Proof. Since M is a smooth manifold there exists a countable base $O_1, O_2, \cdots$, for the topology of M [14]. Since the smooth manifold M is locally compact we can assume that the closure $\overline{O_i}$ of every O_i is a compact set. Let $f \in [K, U] = \{f : M \to M | f(K) \subset U\}$, where K is a subset of some leaf L, and U is an open subset of M. For every point $x \in K$ there exist O_i and O_j such that $x \in O_i$ and $f(O_i) \subset O_J$. Since K, $f(K)$ are compact sets we can find finite coverings $O_{i_1}, O_{i_2}, \cdots, O_{i_m}$ of K and $O_{j_1}, O_{j_2}, \cdots, O_{j_m}$ of $f(K)$ such that $f(\overline{O_{i_l}}) \subset O_{j_l}$ for $l = 1, 2, ..., m$. Hence,

$$f \in \bigcap_{l=1}^{m} [K_l, O_{j_l}] \subset [K, U],$$

where $K_l = K \bigcap \overline{O}_{i_l}$. It follows that the set O_F all finite intersections of the sets $\bigcap_{l=1}^{m} [K_l, O_{j_l}]$ forms a base for the F-compact open topology. □

Theorem 2.4. *Let (M, F) be a smooth foliated manifold.Then the group $Diff_F(M)$ is a topological group with respect to the F-compact open topology.*

Proof. Let g be complete Riemannian metric on M. It is known that smooth manifold M possesses a complete Riemannian metric [2, p. 186], [13].

We will show that the mapping $\chi : f \to f^{-1}$ is continuous. Since by Lemma 2.3 the space $Diff_F(M)$ with the F-compact open topology is a space with a countable base, we can use consequences.

Assume that $f_i \to g$ as $i \to \infty$ in the F-compact open topology. We have to show that $f_i^{-1} \to g^{-1}$ as $i \to \infty$ in the F-compact open topology.

Let A be a neighborhood of g^{-1} in the F-compact open topology. Actually, it is sufficient to show this fact if A is an element of a prebase, i. e., $A = [K, V] = \{f \in Diff_F(M) : f(K) \subset V\}$, where K is a compact subset of a leaf L, and V is an open subset of M.

Let U be a neighborhood of $g^{-1}(K)$ in M with compact closure $\overline{U}$ such that $\overline{U} \subset V$.

We put

$$U_\varepsilon(g) = \{f \in Diff_F(M) : d(g(x), f(x)) < \frac{\varepsilon}{2},\ \forall x \in \overline{U}\},$$

where $\varepsilon = d(K, M\backslash g(U)) = \inf\{d(x, y) : x \in K, y \in (M\backslash g(U))\}$.

Let us show that, if $h \in U_\varepsilon(g)$, then $h^{-1}(K) \subset V$, i.e., $h^{-1} \in A$. We will show that $h^{-1}(K) \subset U$.

Let us assume that this is not true. Let for the some $h \in U_\varepsilon(g)$ there exist a point $y \in K$ such that $h^{-1}(y) \in M\backslash U,$, i.e., $y \in M\backslash h(U)$. Then, since $g^{-1}(y) \in U$, we have $d(y = g(g^{-1}(y)), h(g^{-1}(y))) < \frac{\varepsilon}{2}$.

Let γ be a shortest geodesics in M (in virtue of completeness of (M, g) there exists a shortest geodesics between any two points) going from the point y to the point $h(g^{-1}(y))$. If $z \in \gamma \cap \partial(h(U))$ then $h^{-1}(z) \in \overline{U}$, and besides $d(g(h^{-1}(z)), h(h^{-1}(z)))) < \frac{\varepsilon}{2}$. In addition, since the length of the geodesics γ is less than $\frac{\varepsilon}{2}$, we have $d(y, z) < \frac{\varepsilon}{2}$. It follows then that $d(y, g(h^{-1}(z))) \leq d(y, z) + d(z, g(h^{-1}(z))) < \varepsilon$.

But on the other hand, since $z \notin h(U)$, we have $g(h^{-1}(z)) \in M\backslash g(U)$. Since $y \in K \subset g(U)$, $g(h^{-1}(z)) \notin g(U)$ and $\varepsilon = d(K, M\backslash g(U))$ we have $d(y, g(h^{-1}(z))) \geq \varepsilon$. This contradiction shows that $h^{-1}(K) \subset U$. Hence, $h^{-1} \in A$.

Since $f_i \to g$ as $i \to \infty$ in the F-compact open topology, for a point $x \in \overline{U}$ there exists an integer n_x such that $d(g(x), f_i(x)) < \frac{\varepsilon}{2}$ for $i \geq n_x$. Since $d(g(x), f_i(x))$ is a continuos function there exists a neighborhood U_x of x in M such that $d(g(y), f_i(y)) < \frac{\varepsilon}{2}$ for $y \in U_x$ and $i \geq n_x$.

We can find a finite covering $U_{x_1}, U_{x_2}, \cdots, U_{x_m}$ of $\overline{U}$, such that $d(g(y), f_i(y)) < \frac{\varepsilon}{2}$ for $y \in U_{x_i}$ as $i \geq n_{x_i}$, where $i = 1, 2, ..., m$. It follows that $d(g(y), f_i(y)) < \frac{\varepsilon}{2}$ for $y \in \overline{U}$ if $i \geq p$ where $p = \max\{n_{x_1}, n_{x_2}, \cdots, n_{x_m}\}$.

This implies that $f_i^{-1}(K) \in U$ for $i \geq p$. Thus $f_i^{-1} \to g^{-1}$ as $i \to \infty$ in the F-compact open topology. Hence the map $\chi : f \to f^{-1}$ is continuous.

Let us now show that the mapping $(g, h) \to g \circ h$ is continuous for $g, h \in Diff_F(M)$ with respect to the F-compact topology, where $g \circ h(x) = g(h(x))$. Since the space $Diff_F(M)$ has a countable base in the F-compact open topology, we use consequences. Assume that $h_i \to h$, $g_i \to g$ as $i \to \infty$ with respect to the F-compact open topology. We have to show that $g_i \circ h_i \to g \circ h$ as $i \to \infty$ in the F-compact open topology.

Let $g \circ h \in [K, G]$, where K is a compact subset of some leaf L of the foliation F, and G be a open subset of M.

Then $h(K) \subset g^{-1}(G)$ and, since $h(K)$ is a compact set, there exists a neighborhood U of $h(K)$ in M with compact closure $\overline{U}$ such that $g(\overline{U}) \subset G$. Since $h_i \to h$ as $i \to \infty$ in the F-compact open topology and $h(K) \subset U$, there is an integer n_1 such that $h_i(K) \subset U$ for $i \geq n_1$. Also, since $g_i \to g$ as $i \to \infty$ in the F-compact open topology and $g(h(K)) \subset G$ there is an integer n_2 such that $g_i(h(K)) \subset G$ for $i \geq n_2$. It follows that $g_i(h_i(K)) \subset G$

for $i \geq m$, where $m = \max\{n_1, n_2\}$. Hence $g_i \circ h_i \in [K, G]$ for $i \geq m$. Therefore, the map $(g, h) \to g \circ h$ is continuous. □

3. Some subgroups of the group of diffeomorphisms of foliated manifolds

In this section we study some diffeomorphisms of foliated manifolds when foliations generated by special submersions.

Let us consider a submersion $\pi : R^{n+1} \to R^1$,

$$\pi(x_1, x_2, \cdots, x_n, x_{n+1}) = x_{n+1} - f(x_1, x_2, \cdots, x_n), \tag{3.5}$$

where $f(x_1, x_2, \cdots, x_n)$ is a differentiable function.

The submersion (3.5) generates a foliation F of R^{n+1} as pointed out in Introduction. Leaves of the foliation F of R^{n+1} defined by the submersion (3.5) are level surfaces of this submersion.

In this section we will study some subgroups of the group $Diff_F(R^{n+1})$ of diffeomorphisms of the foliated manifold (R^{n+1}, F).

Definition 3.1. A diffeomorphism $\varphi : (M, F) \to (M, F)$ is called an isometry of the foliated manifold (M, F) if the restriction of the mapping φ to each leaf of the foliation F is an isometry, that is, for each leaf L_α the map $\varphi : L_\alpha \to f(L_\alpha)$ is an isometry between the manifolds L_α and $\varphi(L_\alpha)$.

Theorem 3.2. *A diffeomorphism $\varphi : R^{n+1} \to R^{n+1}$ defined by the formula*

$$\varphi_\lambda(x_1, x_2, \cdots x_n, x_{n+1}) = (x_1, x_2, \cdots, x_n, x_{n+1} + \lambda\pi) \tag{3.6}$$

is an isometry of the foliated manifold (R^{n+1}, F) for $\lambda \neq -1$.

Proof. First of all we note that each leaf $L_c = \pi^{-1}(c), c \in R^1$ of the foliation F generated by submersion (3.5) is a graph of the function

$$x_{n+1} = f(x, x, \cdots, x_n) + c. \tag{3.7}$$

The diffeomorphism $\varphi : R^{n+1} \to R^{n+1}$ maps each leaf L_c to the leaf $L_{(1-\lambda)c}$ of this foliation. Indeed, if a point $x = (x_1, x_2, \cdots, x_n, x_{n+1})$ belongs to the leaf L_c, then its coordinates satisfy equation (3.11). Then it is easy to verify that the coordinates of the point $\varphi(x_1, x_2, \cdots x_n, x_{n+1})$ satisfy the equation

$$x_{n+1} = f(x, x, \cdots, x_n) + (1 + \lambda)c. \tag{3.8}$$

We show that the differential $d\varphi_q$ of the map φ at the point q preserves the inner product on the tangent space T_qF of the leaf L_c at the point q.

Let $A = (\frac{\partial \varphi_i}{\partial x_j})$ be the Jacobi matrix of the map φ at the point q, $i, j = 1, 2, ..., n + 1$. Then the matrix A has the following form:

$$A(t) = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & & 1 & 0 \\ \lambda\frac{\partial\pi}{\partial x_1} & \lambda\frac{\partial\pi}{\partial x_2} & \cdots & \lambda\frac{\partial\pi}{\partial x_n} & 1+\lambda \end{pmatrix}.$$

The vector fields

$$r_i = \frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_{n+1}}, \qquad i = 1, 2, \cdots, n, \tag{3.9}$$

form a basis of the tangent space T_qF of the leaf L_c at the point q. Here the ith component of the vector r_i is 1, the last component is $-\frac{\partial f}{\partial x_i}$, where $i = 1, 2, \cdots, n$.

The tangent vector fields r_i are orthogonal to the gradient $grad\pi$ of the function $\pi : R^{n+1} \to R^1$, i.e, $< r_i, grad\pi >= 0$ for every i, where $< \cdot, \cdot >$ is the inner product.

Using these equalities we obtain that

$$< Ar_i, Ar_j >=< r_i, r_j > \tag{3.10}$$

for all i, j. In particular, we have

$$|Ar_i|^2 = |r_i|^2. \tag{3.11}$$

It follows that the differential $d\varphi_q$ of the map φ at the point q preserves the scalar product on the tangent space T_qF of the fiber L_c at the point q. Therefore, a map of the form (3.10) is an isometry of the foliated manifold (R^{n+1}, F) for $\lambda \neq -1$. □

Theorem 3.3. *Let F be a foliation of R^{n+1} defined by the submersion (3.5). Then the set of diffeomorphisms*

$$G_\Lambda = \{\varphi_\lambda : \lambda \in R^1, \lambda \neq -1\}, \tag{3.12}$$

is a subgroup of the group $Diff_F(R^{n+1})$.

Proof. The multiplication of the diffeomorphisms φ_{λ_1}, φ_{λ_2} in the group $G_F(M)$ is their composition,

$$\varphi_{\lambda_1} \cdot \varphi_{\lambda_2}(x) = \varphi_{\lambda_1}(\varphi_{\lambda_2}(x)). \tag{3.13}$$

It is easy to verify that

$$\varphi_{\lambda_1} \cdot \varphi_{\lambda_2} = \varphi_\mu, \tag{3.14}$$

where $\mu = \lambda_1 + \lambda_2 + \lambda_1\lambda_2$. It is easily seen that $\mu \neq -1$. Indeed, if $\mu = -1$, then $1 + \lambda_1 + \lambda_2 + \lambda_1\lambda_2 = 0$. This is equivalent to the identity $(1 + \lambda_1)(1 + \lambda_2) = 0$, which is impossible due to the fact that $\lambda_i \neq -1$. Thus $\varphi_\mu \in G_\Lambda$.

An inverse element to φ_λ is the element φ_μ, where $\mu = -\frac{\lambda}{1+\lambda}$. The number $-\frac{\lambda}{1+\lambda}$ is also not equal to -1. Therefore $\varphi_\mu \in G_\Lambda$ for $\mu = -\frac{\lambda}{1+\lambda}$. The single element is the diffeomorphism φ_λ for $\lambda = 0$. Thus the set G_Λ is a subgroup of the group $G_F(R^{n+1})$. □

The following interesting example of a Lie group follows from the proof of Theorem 3.3.

Lemma 3.4. *The set G_Λ is a one-dimensional Lie group.*

Proof. Using the mapping $\varphi_\lambda \to \lambda$ we identify the set G_Λ with the set $R^1 \setminus \{-1\}$ of real numbers other than -1.

On the set $R^1 \setminus \{-1\}$ we define a multiplication as follows:

$$\lambda_1 \cdot \lambda_2 = \lambda_1 + \lambda_2 + \lambda_1\lambda_2, \tag{3.15}$$

The inverse element is determined by the formula

$$\lambda^{-1} = -\frac{\lambda}{1+\lambda} \tag{3.16}$$

and it is obvious that the maps

$$(\lambda_1, \lambda_2) \to \lambda_1 \cdot \lambda_2, \qquad \lambda \to \lambda^{-1}$$

are differentiable maps. Therefore with these group operations the one-dimensional manifold $R^1 \setminus \{-1\}$ is a Lee group. □

Remark 3.5. Although the subgroup G_Λ is a one-parameter group, due to the fact that $\varphi_{\lambda_1} \cdot \varphi_{\lambda_2} \neq \varphi_{\lambda_1+\lambda_2}$, it is not a flow of a vector field. In the following example, we show a subgroup of the group of diffeomorhisms of the foliated manifold which is generated by a flow of a vector field.

Example 3.6. Consider the submersion $\pi : R^3 \to R^1$,

$$\pi(x_1, x_2, x_3) = x_3 - f(x_1, x_2), \tag{3.17}$$

where $f(x_1, x_2) = x_1^2 + x_2^2$. This submersion generates a two-dimensional foliation F. The gradient of this function has the form $grad\pi = \{2x_1, 2x_2, -1\}$. The following vector fields

$$V_1 = \frac{\partial}{\partial x_1} + 2x_1\frac{\partial}{\partial x_3}, V_2 = \frac{\partial}{\partial x_2} + 2x_2\frac{\partial}{\partial x_3}$$

are vertical vector fields.

Recall that a vector field X is called foliated if for each vertical vector field Y the Lie bracket $[X, Y]$ is also vertical. It is known that the flow of a foliated vector field consists of diffeomorphisms of a foliated manifold (M, F) [3].

The vector field

$$X = -x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}$$

is a foliated vector field for the foliation F, as shown by the following identities:

$$[V_1, X] = V_2, \qquad [V_2, X] = -V_1.$$

The vector field X is a Killing vector field. Therefore, the flow of the vector field X consists of isometries of a foliated manifold. Indeed, the flow of the vector field X consists of the diffeomorphisms

$$x \to A(t)x + bt,$$

where $t \in R$, $b = \{0, 0, 1\}^T$, $x = (x_1, x_2, x_3)^T$,

$$A(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which are isometries of the foliated manifold (R^3, F).

Theorem 3.7. *Suppose for a vector field*

$$V = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial x_i}$$

we have that $V(f) = 0$. Then the flow of the vector field

$$X = V + \frac{\partial}{\partial x_{n+1}}$$

consists of diffeomorphisms of the foliated manifold (F, R^{n+1}) generated by submersion (3.5). If the field V is a Killing field then the flow of the vector field X consists of isometries of the foliated manifold (R^{n+1}, F).

Proof. Let a point $x = (x_1, x_2, \cdots, x_n, x_{n+1})$ belong to the leaf L_c. Then its coordinates satisfy equation (3.11).

Let $t \to X^t(x)$ be an integral curve of the vector field X passing through the point x at $t = 0$ and

$$X^t(x) = (X_1^t(x), X_2^t(x), \cdots, X_n^t(x), X_{n+1}^t(x)).$$

An integral curve of the vector field X satisfies the following system of differential identities:

$$\frac{dX_i^t(x)}{dt} = \xi_i(X^t(x)), \qquad \frac{dX_{n+1}^t(x)}{dt} = 1, \tag{3.18}$$

where $i = 1, 2, \cdots, n$.

Due to the fact that $V(f) = 0$, the value of the function f remains constant along the trajectory of the vector field V.

Therefore we have

$$f(X_1^t(x), X_2^t(x), \cdots, X_n^t(x)) = f(x_1, x_2, \cdots, x_n).$$

It follows from last the equation in system (3.18) we have the equality $X_{n+1}^t(x) = x_{n+1} + t$. It means that the flow of the vector field $\frac{\partial}{\partial x_{n+1}}$ consists of parallel translations.

Therefore, the coordinates of the point $X^t(x)$ satisfy the equation

$$x_{n+1} = f(x_1, x_2, \cdots, x_n) + c - t. \tag{3.19}$$

Hence, the flow of the vector field X consists of diffeomorphisms of the foliated manifold (R^{n+1}, F) that maps each leaf L_c to the leaf L_{c-t} of this foliation.

Suppose that the vector field V is a Killing field. The vector field $\frac{\partial}{\partial x_{n+1}}$ is also a Killing vector field, since its flow consists of parallel translations.

Note that a linear combination of Killing fields over a field of real numbers is also a Killing field [8]. Then the vector field X as linear combination of Killing fields V and $\frac{\partial}{\partial x_{n+1}}$ is also a Killing field.

Therefore the flow

$$t \to (X_1^t(x), X_2^t(x), \cdots, X_n^t(x), X_{n+1}^t(x))$$

consists of isometries of the foliated manifold (R^{n+1}, F). □

References

[1] A. Narmanov and E. Rajabov, *On the geometry of orbits of conformal vector fields*, Journal of Geometry and Symmetry in Physics **51** (2019), 29–39.

[2] D. Gromoll, W. Klingenberg, and W. Meyer, *Riemannian geometry in the large*, Mir, Moscow, 1971.

[3] P. Molino, *Riemannian foliations*, translated from the French by Grant Cairns, Progress in Mathematics, 73, Birkhäuser Boston, Inc., Boston, MA, 1988.

[4] A. Narmanov and G. Abdushukurova, *On the geometry of riemannian submersions*, Uzbek Mathematical journal **2** (2016), 3–8.

[5] A. Narmanov and Sh. Norjigitov, *On the geometry of manifolds of nonnegative curvature*, Uzbek Mathematical journal **3** (2014), 83–89.

[6] A. Narmanov and O. Qasimov, *On the geometry of the set of orbits of killing vector fields on euclidean space*, Journal of Geometry and Symmetry in Physics **55** (2020), 39–49.

[7] A. Narmanov and A. Sharipov, *On the group of foliation isometries*, Methods Funct. Anal. Topology **15** (2009), 195–200.

[8] A. Narmanov and S. Saitova, *On the geometry of orbits of killing vector fields*, Differential Equations **50** (2014), 1582–1589.

[9] A. Narmanov and A. Zoyidov, *On the group of diffeomorphisms of foliated manifolds*, Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Kompyuternye Nauki **30** (2020), 49–58.

[10] B. O'Neil, *The fundamental equations of submersions*, Michigan Mathematical Journal **13** (1966), 459–469.

[11] B. L. Reinhart, *Foliated manifolds with bundle-like metrics*, Annals of Mathematics, Second Series **69** (1959), 119–132.

[12] R. Hermann, *A sufficient condition that a mapping of riemannian manifolds to be a fiber bundle*, Proc. Amer. Math. Soc. **11** (1960), 236–242.

[13] S. Helgason, *Differential geometry, lie groups and symmetric spaces*, Academic Press, Toronto, 1978.

[14] M. W. Hirsch, *Differential topology*, Springer, Verlag, New York, 1976.

G.M. Abdishukurova: -
Faculty of Mathematics, National University of Uzbekistan, Tashkent, 100174, Tashkent, Uzbekistan

A.Ya. Narmanov: narmanov@yandex.ru
Faculty of Mathematics, National University of Uzbekistan, Tashkent, 100174, Tashkent, Uzbekistan

Received 17/08/2020; Revised 02/09/2020