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ON THE RITT CONDITION ON LOCALLY CONVEX VECTOR SPACES

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Abstract. In this paper, we show that the Ritt condition in the case of locally convex spaces can be related to the power boundedness of a universally bounded operator. We will characterize this condition by two geometric properties of the powers and we prove that the Ritt condition will be shown to be equivalent to the Tadmor condition. We study the Ritt condition for a quasinilpotent operator acting on locally convex spaces. Also, an upper bound for the norm of the powers of operators acting on locally convex spaces under Ritt condition was given.

Показано, що у випадку локально опуклих просторів умова Рітта пов'язана з обмеженістю степенів універсально обмеженого оператора. Ця умова характеризується в термінах геометричних властивостей степенів. Доведено, що умова Рітта еквівалентна умові Тедмора. Досліджена умова Рітта для вирадку квазінільпотентних операторів у локально опуклих просторах. Знайдена також верхня оцінка норм степенів операторів, які задовольняють умову Рітта.

1. Introduction

During the last decades, there has been huge interest in the study of power boundedness under various resolvent conditions of operators acting on Banach spaces at first and lately, more generally, on locally convex spaces. In the present paper, we study the Ritt condition of universally bounded operators acting on locally convex spaces, for more information on this class we refer to [3, 4]. This condition extends the Banach spaces one [14].

The principal difficulty is that there are many non-equivalent definitions of bounded operators on locally convex spaces. The concept of bounded element of a locally convex algebra was introduced by Allan [1].

In [10], it was shown that if Ritt resolvent condition holds for an operator T acting on a Banach space, then $\|T^n\| =| O(\log n)$ as $n \longrightarrow \infty$, and $\left\|T^n - T^{n+1}\right\| \to 0$ as $n \longrightarrow \infty$. This result was generalized by Pater for operators acting on locally convex spaces [13, Theorem 3]. The first aim of the present article is to improve this result by giving a characterization of the Ritt resolvent condition by two geometric properties of the powers. In particular, the geometric characterization in terms of the behavior of the powers gives easily that the product of two commuting Tadmor operators is Tadmor operator [16]. Next, We prove that the Ritt condition will be shown to be equivalent to the Tadmor condition. We study the Ritt condition for a quasinilpotent operator on the real line.

It was proved independently by Yu. Lyubich [6], B. Nagy and J. Zemánek [10], and O. Nevanlinna [11] that if Ritt condition holds, outside the unit ball, for an operator T then it is power bounded. An upper bound was given by N. Borovykh, D. Drissi and M. N. Spijker, see [2]. Recently the authors have broadened the Known Banach setting to the locally convex spaces one, bringing upfront similar yet more general results and adapted proofs (see, e. g., [5]). In connecting with this, we will conclude this work by given an upper bound of the powers of universally bounded operators satisfying Ritt's condition.

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2. Preliminaries

Let $\mathcal{X}$ be a Hausdorff locally convex vector space over the complex field $\mathbb{C}$. A calibration for $\mathcal{X}$ is a family $\mathcal{P}$ of seminorms generating its topology (in the sense that the topology of $\mathcal{X}$ is the coarsest with respect to which all seminorms of $\mathcal{P}$ are continuous). Such family of seminorms was used in [9]. We denote by $(\mathcal{X}, \mathcal{P})$ a locally convex space $\mathcal{X}$ with a calibration $\mathcal{P}$.

Recall that a linear operator T on a locally convex space $\mathcal{X}$ is quotient-bounded with respect to a calibration $\mathcal{P}$ if for every seminorm $p \in \mathcal{P}$ there exists some $c_p > 0$ such that

$$p(Tx) \leq c_p p(x), \quad \forall x \in \mathcal{X}.$$

The class of quotient-bounded operators with respect to a calibration $\mathcal{P}$, which was introduced in [4, 8, 9], will be denoted by $\mathcal{Q}_{\mathcal{P}}(\mathcal{X})$. For each $p \in \mathcal{P}$ and $T \in \mathcal{Q}_{\mathcal{P}}(\mathcal{X})$ we define

$$\hat{p}(T) = \inf\{r > 0 : \ p(Tx) \leq rp(x), \ \forall x \in \mathcal{X}\}.$$

For each $p \in \mathcal{P}$, $\hat{p}$ is then a sub-multiplicative seminorm on $\mathcal{Q}_{\mathcal{P}}(\mathcal{X})$ satisfying $\hat{p}(I) = 1$. The space $\mathcal{Q}_{\mathcal{P}}(\mathcal{X})$ will be endowed with a topology $\tau_{\hat{\mathcal{P}}}$ generated by $\hat{\mathcal{P}} = \{\hat{p} : p \in \mathcal{P}\}$. We note that $\mathcal{Q}_{\mathcal{P}}(\mathcal{X})$ becomes a Hausdorff local multiplicative convex (l.m.c.) algebra with respect to the topology determined by $\hat{\mathcal{P}}$, for more information see [5].

In [15], S. M. Stoian proved the following lemma.

Lemma 2.1. [15] *If $\mathcal{X}$ is a sequentially complete convex space, then $\mathcal{Q}_{\mathcal{P}}(\mathcal{X})$ is a sequentially complete m-convex algebra for all calibration $\mathcal{P}$.*

An operator $T \in Q_{\mathcal{P}}(X)$ is a bounded element of the algebra $Q_{\mathcal{P}}(X)$ if it is a bounded element in the sense of G. R. Allan [1], i.e some scalar multiple of it generates a bounded semigroup. By $(\mathcal{Q}_{\mathcal{P}}(\mathcal{X}))_0$ we denote the algebra of all bounded elements in $\mathcal{Q}_{\mathcal{P}}(\mathcal{X})$. One can show (by [1], see also [5]) that

$$(\mathcal{Q}_{\mathcal{P}}(\mathcal{X}))_0 = \{T \in \mathcal{Q}_{\mathcal{P}}(\mathcal{X}) \ : \ r_{\mathcal{P}}(T) < \infty\}.$$

Let $T \in \mathcal{Q}_{\mathcal{P}}(\mathcal{X})$, the $\mathcal{P}$-spectral radius of T, denoted by $r_{\mathcal{P}}(T)$, is defined as the boundedness radius in the sense of Allan [1]

$$r_{\mathcal{P}}(T) = \inf\{\lambda > 0 : \ \text{the sequence } \left(\left(\lambda^{-1}T\right)^n\right)_{n\in\mathbb{N}} \text{ is bounded in } \mathcal{Q}_{\mathcal{P}}(\mathcal{X})\}.$$

If $T \in (\mathcal{Q}_{\mathcal{P}}(\mathcal{X}))_0$, we said that $\lambda \in \mathbb{C}$ is in the Waelbroeck resolvent set if there exists $(\lambda I - T)^{-1} \in (\mathcal{Q}_{\mathcal{P}}(\mathcal{X}))_0$. We denote the Waelbroeck resolvent set of $T \in (\mathcal{Q}_{\mathcal{P}}(\mathcal{X}))_0$ by $\rho_W(T)$, the resolvent function of T by $R(T, \lambda) := (\lambda I - T)^{-1} \in (\mathcal{Q}_{\mathcal{P}}(\mathcal{X}))_0$, $\lambda \in \rho_W(T)$ and the Waelbroeck spectrum of T will be denoted by $\sigma_W(T)$ (see, [18, 19]). It is well-known that for $|\lambda| > r_{\mathcal{P}}(T)$, we have

$$R(T, \lambda) = \sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}.$$

Let $\mathcal{P}$ be a calibration on $\mathcal{X}$. Recall [9], that a linear operator $T : \mathcal{X} \to \mathcal{X}$ is universally bounded on $(\mathcal{X}, \mathcal{P})$ if there exists $r > 0$ such that

$$p(Tx) \leq rp(x), \ \text{ for all } x \in \mathcal{X} \ \text{ and } p \in \mathcal{P}.$$

We will denote by $\mathcal{B}_{\mathcal{P}}(\mathcal{X})$ the collection of all universally bounded operators on $(\mathcal{X}, \mathcal{P})$. First, we have $\mathcal{B}_{\mathcal{P}}(\mathcal{X}) \subset \mathcal{Q}_{\mathcal{P}}(\mathcal{X})$ (see [4]).

Next, $\mathcal{B}_{\mathcal{P}}(\mathcal{X})$ is an unital normed algebra with respect to the norm

$$\|T\|_{\mathcal{P}} = \inf\{r > 0 \ : \ p(Tx) \leq rp(x) \ \text{ for all } p \in \mathcal{P} \text{ and all } x \in \mathcal{X}\}.$$

Furthermore

$$\|T\|_{\mathcal{P}} = \sup\{\hat{p}(T) \ , \ p \in \mathcal{P}\}, \ \text{ for all } T \in \mathcal{B}_{\mathcal{P}}(\mathcal{X}).$$

Recall [4, Definition 1.2], that two families $\mathcal{P}$ and $\mathcal{P}'$ of seminorms on a linear space are called $\mathcal{B}$-equivalent (denoted $\mathcal{P} \simeq \mathcal{P}'$) provided each seminorm in each is a positive number multiple of a seminorm in the other.

Proposition 2.2. [4] *Let $(\mathcal{X}, \mathcal{P})$ be a locally convex space. Then*

1. *For each calibration $\mathcal{P}'$ with the property $\mathcal{P} \simeq \mathcal{P}'$, we have $\mathcal{B}_{\mathcal{P}}(\mathcal{X}) = \mathcal{B}_{\mathcal{P}'}(\mathcal{X})$ and $\|T\|_{\mathcal{P}} = \|T\|_{\mathcal{P}'}$.*
2. *An operator $T \in \mathcal{Q}_{\mathcal{P}}(\mathcal{X})$ is bounded in the algebra $\mathcal{Q}_{\mathcal{P}}(\mathcal{X})$ if and only if there exists some calibration $\mathcal{P}'$ such that $\mathcal{P} \simeq \mathcal{P}'$ and $T \in \mathcal{B}_{\mathcal{P}'}(\mathcal{X})$.*

In view of the previous Proposition, for each bounded operator T of $\mathcal{Q}_{\mathcal{P}'}(\mathcal{X})$, we are able to choose an equivalent family of seminorms $\mathcal{P}$ such that $T \in \mathcal{B}_{\mathcal{P}}(\mathcal{X})$. In the following, we consider a family of seminorms $\mathcal{P}$ such that $T \in \mathcal{B}_{\mathcal{P}}(\mathcal{X})$.

An operator $T \in (\mathcal{Q}_{\mathcal{P}}(\mathcal{X}))_0$ is called power bounded, if

$$\sup_{\hat{p} \in \hat{\mathcal{P}}} \hat{p}\left(T^n\right) \leq C, \quad \text{for all } n \in \mathbb{N}. \tag{2.1}$$

In the present article, we prove that the Ritt condition of a bounded operator $T \in \mathcal{B}_{\mathcal{P}}(\mathcal{X})$ on a locally convex space

$$\begin{aligned} &\sigma_W(T) \subset \mathbb{D} \cup \{1\}, \text{ and} \\ &\sup_{\hat{p} \in \hat{\mathcal{P}}} \hat{p}\left[R(T, \lambda)\right] \leq \frac{M}{|\lambda - 1|} \text{ for all } |\lambda| \geq 1,\ 0 < |\lambda - 1| \leq \eta, \end{aligned} \tag{2.2}$$

is equivalent to the Tadmor condition [16]

$$\begin{aligned} &\sigma_W(T) \subset \{z \in \mathbb{C} \ :\ |z| \leq 1\}, \text{ and} \\ &\sup_{\hat{p} \in \hat{\mathcal{P}}} \hat{p}\left[R(T, \lambda)\right] \leq \frac{L}{|\lambda - 1|} \text{ for all } |\lambda| > 1. \end{aligned} \tag{2.3}$$

This characterization in the case of bounded linear operators acting on Banach spaces was presented in [2]. In [13], F. Pater has noted that if (2.3) hold, then

$$\sup_{\hat{p} \in \hat{\mathcal{P}}} \hat{p}\left(T^n - T^{n+1}\right) \longrightarrow 0, \qquad n \longrightarrow \infty. \tag{2.4}$$

On the other hand, the obviously necessary condition $\sigma_W(T) \subset \mathbb{D} \cup \{1\}$ is not sufficient for (2.4).

We need to formulate a simple lemma, which we shall use to prove that (2.3) imply (2.2), (see [1, Theorem 3.8, ii.]).

Lemma 2.3. [1] *Let $T \in \mathcal{B}_{\mathcal{P}}(\mathcal{X})$. If $\mu \in \rho_W(T)$ and $\alpha = |\lambda - \mu| \|R(T, \mu)\|_{\mathcal{P}} < 1$, then also $\lambda \in \rho_W(T)$ and*

$$\|R(T, \lambda)\|_{\mathcal{P}} \leq \frac{\|R(T, \mu)\|_{\mathcal{P}}}{1 - \alpha}.$$

3. Main results

Consider the family of semi-norms $\mathcal{P}$ such that $T \in \mathcal{B}_{\mathcal{P}}(\mathcal{X})$. Let $\delta > 0$ and consider the set

$$K_\delta = \left\{\lambda = 1 + re^{i\theta},\ r > 0,\ |\theta| < \frac{\pi}{2} + \delta\right\}.$$

If T satisfies (2.3), then the resolvent condition

$$\|R(T, \lambda)\|_{\mathcal{P}} \leq \frac{M}{|\lambda - 1|}, \quad \lambda \in K_\delta, \tag{3.5}$$

holds for some $\delta > 0$ and $M > 0$, see [12, Lemma 1].

It was shown, in [13, Theorem 3], that condition (2.3) implies $\lim_n \|T^n\|_{\mathcal{P}} = O(\log n)$ as well as $\lim_n \|T^n - T^{n+1}\|_{\mathcal{P}} \longrightarrow 0$. In the following, we get strengthening of this result.

Theorem 3.1. *Let $T \in \mathcal{B}_{\mathcal{P}}(\mathcal{X})$. The operator T satisfy (2.3) if and only if*

$$\sup_{n\in\mathbb{N}} \|T^n\|_{\mathcal{P}} < \infty, \tag{3.6}$$

and

$$\sup_{n\in\mathbb{N}} n\|T^n - T^{n+1}\|_{\mathcal{P}} < \infty. \tag{3.7}$$

Proof. Assume that (2.3) holds, then by the above discussion we get (3.5). Thus, in the second part of [12, Theorem 2], it suffices to assume Ritt's condition outside the unit disc only. Then, we obtain (3.6) as well as (3.7). The converse follows from [13, Theorem 4]. □

It was shown in [17, Proposition 1] that in the case of Banach spaces, the product of two commuting Ritt operators is a Ritt operator. In the following, we extend this result to the case of locally convex spaces.

Proposition 3.2. *Let the operators $T, S \in \mathcal{B}_{\mathcal{P}}(\mathcal{X})$ be two commuting operators on a locally convex space satisfying (2.3). Then their product TS satisfies also the condition (2.3).*

Proof. Indeed, it is easy to see that the product of two commuting power bounded operators is a power pounded operator. On the other hand, one can show that

$$n\|(TS)^n - (TS)^{n+1}\|_{\mathcal{P}} \leq \|T^n\|_{\mathcal{P}}(n\|S^n - S^{n+1}\|_{\mathcal{P}})\|S^n\|_{\mathcal{P}}\|I-T\|_{\mathcal{P}}, \quad \text{for all } n \in \mathbb{N}.$$

Since T and S are power bounded and $n\|S^n - S^{n+1}\|_{\mathcal{P}} < \infty$. Hence, $n\|(TS)^n - (TS)^{n+1}\|_{\mathcal{P}} < \infty$. Then the result follows from Theorem 3.1. □

In the following theorem, we prove that condition (2.2) and condition (2.3) can be regarded to be equivalent.

Theorem 3.3. *Let $T \in \mathcal{B}_{\mathcal{P}}(\mathcal{X})$. There exist constants M and $\eta > 0$ such that (2.2) holds, if and only if there is constant L such that (2.3) is valid.*

Proof. Suppose that T satisfies condition (2.2). Since the function $F(\lambda) = |\lambda-1|\|R(T,\lambda)\|_{\mathcal{P}}$ is continuous on $\rho_W(T)$, and $F(\lambda) \longrightarrow 1$ for all $|\lambda| \longrightarrow \infty$, there is a finite constant C such that

$$F(\lambda) \leq C \quad \text{for all} \quad \lambda \in \mathbb{C} \quad \text{with} \quad |\lambda| \geq 1, \; |\lambda - 1| \geq \eta.$$

Now by applying (2.2), we arrive at (2.3) with $L = \max\{C, M\}$.

Conversely, Suppose that T satisfies (2.3). Let μ be a complex number such that $|\mu| = 1, \; \mu \neq 1$. By choosing $|\lambda| > 1$, sufficiently close to μ we obtain by using the notation of Lemma 2.3

$$\alpha = |\lambda - \mu|\|R(T,\lambda)\|_{\mathcal{P}} \leq |\mu - \lambda|.\frac{L}{|\lambda - 1|} < 1.$$

From the lemma, we conclude that $\lambda \in \rho_W(T)$ with

$$|\lambda - 1|(1-\alpha)\|R(T,\mu)\|_{\mathcal{P}} \leq L.$$

By letting $\lambda \longrightarrow \mu$, it follows that $|\mu - 1|\|R(T,\mu)\|_{\mathcal{P}} \leq L$. Thus, condition (2.2) holds with $M = L$ and any $\eta > 0$. □

Next, our goal is to obtain a similar result of [17, Propositions 2]. A simple computation gives:

Proposition 3.4. *Let the operators $T, S \in \mathcal{B}_{\mathcal{P}}(\mathcal{X})$ be two commuting power bounded operators on a locally convex space, $0 \leq t \leq 1$. Then the convex combination $tT + (1-t)S$ is a power bounded operator.*

Proposition 3.5. *Let $\mathcal{X}$ be a locally convex space and $Q \in \mathcal{B}_{\mathcal{P}}(\mathcal{X})$ such that $\sigma(Q) = \{0\}$. If $I - Q$ satisfies condition (2.3), then so does $I - tQ$ for $t \geq 0$. Consequently, $(1-t)I + t(I-Q)^2$ satisfies also the condition (2.3) for $t \geq 0$.*

Proof. Suppose that $I - Q$ satisfies (2.3). In view of Theorem 3.3, it suffices to show the result for any $\lambda \in \mathbb{C}$ such that $|\lambda| \geq 1$ and $|\lambda - 1| \leq \eta$ with $\eta = \frac{1}{t}$.

Case 1: If $0 \leq t \leq 1$, then

$$\left|\frac{\lambda}{t} - \frac{1}{t} + 1\right| \geq \left|\frac{\lambda}{t}\right| - \left|1 - \frac{1}{t}\right| = \frac{|\lambda|}{t} - \frac{1}{t} + 1 \geq 1.$$

Since

$$\|R(I - tQ, \lambda)\|_{\mathcal{P}} = \frac{1}{t}\left\|\left(\left(\frac{\lambda}{t} - \frac{1}{t} + 1\right) - I + Q\right)^{-1}\right\|_{\mathcal{P}}.$$

Hence, by (2.2), we have

$$\|R(I - tQ, \lambda)\|_{\mathcal{P}} \leq \frac{1}{t}\frac{M}{|\lambda - 1|}, \quad \text{for all } 0 \leq t \leq 1, \ |\lambda| \geq 1 \text{ and } |\lambda - 1| \leq \eta. \tag{3.8}$$

Case 2: If $t > 1$, the resolvent $(\lambda I - I + tQ)^{-1}$ can be expressed by the integral

$$(\lambda I - I + tQ)^{-1} = \frac{1}{2\pi i}\int_{\Gamma}(\lambda - z)^{-1}(zI - I + tQ)^{-1}dz \tag{3.9}$$

$$= \frac{1}{2\pi t i}\int_{\Gamma}(\lambda - z)^{-1}\left(\left(\frac{z-1}{t} + 1\right)I - I + tQ\right)^{-1}dz, \tag{3.10}$$

where Γ is any contour enclosing the spectrum of $I - tQ$. By choosing $\Gamma = \left\{z \in \mathbb{C} \ : \ |z| = 2 + \frac{1}{t}\right\}$ and by (3.10), we have

$$\left\|(\lambda I - I + tQ)^{-1}\right\|_{\mathcal{P}} \leq \frac{1}{2\pi t}\int_{\Gamma}\frac{1}{|\lambda - z|}\left\|\left(\left(\frac{z-1}{t} + 1\right)I - I + tQ\right)^{-1}\right\|_{\mathcal{P}}dz,$$

using (2.2), we have the estimate

$$\left\|(\lambda I - I + tQ)^{-1}\right\|_{\mathcal{P}} \leq \frac{1}{2\pi}\int_{\Gamma}\frac{1}{|\lambda - z|}.\frac{L}{|z-1|}dz. \tag{3.11}$$

Since $|z - 1| \geq |z| - 1 = 2 + \frac{1}{t} - 1 = 1 + \frac{1}{t}$, and $|\lambda - 1| \leq \frac{1}{t}$, thus $|z - 1| \geq |\lambda - 1|$, hence

$$\frac{1}{|z-1|} \leq \frac{1}{|\lambda - 1|}.$$

On the other hand

$$|\lambda - z| \geq |z| - |\lambda| = 2 + \frac{1}{t} - 1 - \frac{1}{t} = 1.$$

By (3.11) and the last two estimates, we get

$$\left\|(\lambda I - I + tQ)^{-1}\right\|_{\mathcal{P}} \leq \frac{L}{|\lambda - 1|}, \quad \text{for all } t > 1, \ |\lambda| \geq 1 \text{ and } |\lambda - 1| \leq \eta. \tag{3.12}$$

To conclude that $I - tQ$ satisfies condition (2.3), it remains to combine (3.8), (3.12) and using Theorem 3.3. For the consequence, it suffices to use Propositions 3.2, 3.4 and Theorem 3.1. □

In the following theorem, our goal is to get an upper bound for $\|T\|_{\mathcal{P}}$ under condition (2.3) in the case of universally bounded operators acting on a locally convex space X. Related to this, F. Pater [13] showed that if condition (2.3) holds, then $\|T^n\|_{\mathcal{P}} = O(n)$.

Theorem 3.6. *Let $T \in \mathcal{B}_{\mathcal{P}}(\mathcal{X})$ satisfy (2.3). Then*

$$\|T^n\|_{\mathcal{P}} \leq \frac{e}{2}L^2, \quad n = 1, 2, \ldots. \tag{3.13}$$

Proof. Suppose that (2.3) is true. By choosing the integration path $\Gamma = \{\lambda \in \mathbb{C} \;:\; |\lambda| = r,\; r > 1\}$, with the aid of the functional calculus from the algebra $\mathcal{B}_{\mathcal{P}}(\mathcal{X})$, we have

$$T^n = \frac{1}{2\pi i}\int_\Gamma \lambda^n(\lambda I - T)^{-1}d\lambda.$$

By partially integrating, we have

$$T^n = \frac{1}{2\pi i(n+1)}\int_\Gamma \lambda^{n+1}(\lambda I - T)^{-2}d\lambda.$$

Appliqueing (2.3), we get

$$\|T^n\|_{\mathcal{P}} \leq \frac{r^{n+1}L^2}{2\pi(n+1)}.J, \tag{3.14}$$

where

$$J = \int_{-\pi}^{\pi} \frac{r}{|re^{it}-1|^2}dt.$$

Since

$$\left|\frac{1}{re^{it}-1} - \frac{1}{r^2-1}\right| = \frac{r}{r^2-1},$$

then J is the length of the curve $z = (re^{it}-1)^{-1},\; -\pi \leq t \leq \pi$. We see that this curve is the circle with center $\frac{1}{r^2-1}$ and radius $\frac{r}{r^2-1}$. Hence

$$J = \frac{2\pi r}{r^2-1}. \tag{3.15}$$

Using (3.14) and (3.15), we have

$$\|T^n\|_{\mathcal{P}} \leq L^2.F(n,r), \tag{3.16}$$

where

$$F(x,r) = \frac{r^{x+2}}{(x+1)(r^2-1)}, \quad x \geq 1,\; r > 1.$$

Then a simple computation gives that

$$\min_{r>1} F(x,r) = F\left(x, \sqrt{1+\frac{2}{x}}\right) \quad \text{and} \quad \sup_{x\geq 1} F\left(x, \sqrt{1+\frac{2}{x}}\right) = \frac{e}{2}.$$

Consequently, if we choose $r = \sqrt{1+\frac{2}{n}}$, then by (3.16), we obtain (3.13). □

Example 3.7. Let $\mathcal{X} = \ell^2(\mathbb{N})$. Let $\mathcal{P}$ be the calibration $\{p_0, p_1, \cdots\}$ defined by $p_i(x) = |x_i|$ for all $i \in \mathbb{N}$ and all $x = (x_0, x_1, x_2, \cdots) \in \mathcal{X}$. The family $\mathcal{Q}_{\mathcal{P}}(\mathcal{X})$ will be endowed with the family of seminorms $\hat{\mathcal{P}} = \{\hat{p}_i :\; i \in \mathbb{N}\}$, where

$$\hat{p}_i(T) = \sup\{p_i(Tx) :\; p_i(x) \leq 1,\; x \in \mathcal{X}\},\; i \in \mathbb{N},\; T \in \mathcal{Q}_{\mathcal{P}}(\mathcal{X}).$$

$\mathcal{B}_{\mathcal{P}}(\mathcal{X})$ is unital normed algebra with respect to the norm

$$\|T\|_{\mathcal{P}} = \sup\{\hat{p}_i(T)\;,\; i \in \mathbb{N}\}, \text{ for all } T \in \mathcal{B}_{\mathcal{P}}(\mathcal{X}).$$

Let $\alpha = (\alpha_k)_{k\in\mathbb{N}}$ be a sequence in $(\alpha_k)_{k\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$. The sequence $\alpha = (\alpha_k)_{k\in\mathbb{N}}$ will be suitably chosen later. Define the operator T_α on $\mathcal{X}$ by

$$\begin{array}{rcl} T_\alpha : \mathcal{X} & \longrightarrow & \mathcal{X} \\ (x_k)_{k\in\mathbb{N}} & \longmapsto & (\alpha_0 x_0, \alpha_1 x_1, \alpha_2 x_2, \ldots). \end{array}$$

First, we prove that T_α is universally bounded.

$$p_i(T_\alpha x) = |\alpha_i x_i| \leq d.|x_i| = d.p_i(x),$$

for all $x = (x_0, x_1, x_2, \cdots) \in \mathcal{X}$ and $i \in \mathbb{N}$, with $d = \|x\|_\infty$.

Now, we choose $\alpha = (\alpha_k)_{k\in\mathbb{N}}$ to be a sequence in $(0,1)$ such that $\alpha_k \nearrow 1,$ as $k \to \infty$. Then

$$\sigma(T_\alpha) = \{\alpha_k\} \cup \{1\}.$$

It is easy to show that

$$T_\alpha^n x = (\alpha_0^n x_0, \alpha_1^n x_1, \alpha_2^n x_2, \ldots).$$

and

$$T_\alpha^n (I-T)x = (\alpha_0^n(1-\alpha_0)x_0, \alpha_1^n(1-\alpha_1)x_1, \alpha_2^n(1-\alpha_2)x_2, \ldots).$$

Thus, by [10, Formula (3)], we get

$$(\lambda-1)(T_\alpha - \lambda I)^{-1}x =$$
$$-x + \left(\sum_{n=1}^{\infty} \alpha_0^{n-1}(1-\alpha_0)\lambda^{-n}x_0, \sum_{n=1}^{\infty} \alpha_1^{n-1}(1-\alpha_1)\lambda^{-n}x_1, \sum_{n=1}^{\infty} \alpha_2^{n-1}(1-\alpha_2)\lambda^{-n}x_2, \cdots\right),$$

for all $\lambda \in \mathbb{C}$ such that $|\lambda| > 1$. Hence,

$$(\lambda-1)(T_\alpha - \lambda I)^{-1}x = -x + \left(\frac{1-\alpha_0}{\lambda-\alpha_0}x_0, \frac{1-\alpha_1}{\lambda-\alpha_1}x_1, \frac{1-\alpha_2}{\lambda-\alpha_2}x_2, \cdots\right),$$

for all $\lambda \in \mathbb{C}$ such that $|\lambda| > 1$.

Since $|1-\alpha_i| < |1-\lambda|$ for all $i \in \mathbb{N}$ and $|\lambda| > 1$. Thus, for all $i \in \mathbb{N}$ and $x = (x_0, x_1, x_2, \cdots) \in \mathcal{X}$ such that $p_i(x) \le 1$, we obtain

$$p_i\left((\lambda-1)(T_\alpha - \lambda I)^{-1}x\right) \le 1 + \left|\frac{1-\alpha_i}{\lambda-\alpha_i}x_i\right| \le 2.$$

Hence,

$$\|(\lambda-1)(T_\alpha - \lambda I)\|_{\mathcal{P}} \le 2.$$

This means that T_α satisfies condition (2.3). Therefore, by Theorem 3.6,

$$\|T_\alpha^n\|_{\mathcal{P}} \le 2e, \quad n = 1, 2, \ldots.$$

Remark 3.8. The power boundedness does not imply Ritt's condition, even in Banach spaces. For example, let

$$Vf(t) := \int_0^t f(s)ds$$

be the Volterra operator on $L^2[0;1]$. Then, the operator $I - V$ is power bounded [17, Theorem 5], but does not Ritt operator [7, Remark 2.7].

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