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n-POWER-POSINORMAL OPERATORS

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Abstract. $\mathcal{B}(\mathcal{H})$ will denote the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. In [6], the authors proved that natural power of a posinormal operator is not in general posinormal. Precisely, they constructed an example of a posinormal operator with square not being posinormal. Given a positive integer n, the aim of this article is to study a class of operators in $\mathcal{B}(\mathcal{H})$ called n-power-posinormal. This class is invariant under natural power and contains any natural power of any posinormal operator and all n-power normal operators.

Позначимо через $\mathcal{B}(\mathcal{H})$ алгебру всіх обмежених лінійних операторів у комплексному гільбертовім просторі $\mathcal{H}$. У [6] доведено, що цілий степінь позінормального оператора не обов'язково є позінормальним. Зокрема, був наведений приклад позінормального оператора, квадрат якого не є позінормальним. Метою цієї статті є дослідження класу n-степенево позінормальних операторів з $\mathcal{B}(\mathcal{H})$, інваріантного відносно натуральних степенів, який містить натуральні степені позінормальних операторів та n-степенево нормальні оператори.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. Let $S, T \in \mathcal{B}(\mathcal{H})$. We will let T^* denote the adjoint of T, $\mathcal{N}(T)$ denote the null space of T and $\mathcal{R}(T)$ denote the range of T. Moreover, T is self-adjoint if $T^* = T$, T is positive ($T \geq 0$) if it is self-adjoint and $\langle Tx, x\rangle \geq 0$, for all $x \in \mathcal{H}$, $T \geq S$ if S and T are self-adjoint and $T - S \geq 0$ and T ts hyponormal if $T^*T \geq TT^*$. $[S, T] = ST - TS$ is the commutator of S, T.

Adnan Jibril [4] generalized the concept of a normal operator to the concept of the n-power normal operator ($n \in \mathbb{N}$), T is n-power normal if $T^nT^* = T^*T^n$. He showed that T is n-power normal if and only if T^n is normal.

Let $T \in \mathcal{B}(\mathcal{H})$. P is said to be an interrupter for T if it satisfies the equation

$$TT^* = T^*PT.$$

From the last equation, if T is not the zero operator, we get that the operator norm of the interrupter P satisfies $\|P\| \geq 1$. Rhaly [8] introduced a class of posinormal operators. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be posinormal if it has a positive interrupter or equivalently if there exists a positive operator P such that the self commutator $[T^*, T]$ of T verifies the equation

$$[T^*, T] = T^*(I - P)T,$$

where I stands for the identity operator. T is called coposinormal if its adjoint is posinormal.

Normal operators are obviously posinornormal. Rhaly proved (see [8], Corollary 2.1) that hyponormal operators are posinormal. From the last statement, using the fact that a hyponormal operator needs not to be normal, we get that a posinormal operator is not necessarily normal.

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It is easy to verify that if V is an isometry and T is posinormal with interrupter P, then V^*TV is posinormal with interrupter V^*PV. Consequently, two unitarily equivalent operators are both posinormal or both nonposinormal.

A natural power of a posinormal operator is not in general posinormal. Indeed, Kubrusly *et al.* (see [6], Example 1) proved that there is a posinormal operator with square not posinormal.

The class what we introduce in the present paper, namely the class of n-power-posinomal operator, contains posinormal operators as a subclass and is invariant under positive integer power.

2. Fundamental Properties of n-power-Posinormal Operators

We start with a definition of what we call an n-interrupter.

Definition 2.1. Let n be a positive integer, T and S be operators in $\mathcal{B}(\mathcal{H})$. S is said to be an n-interrupter for T if

$$T^n T^{*n} = T^* S T. \tag{2.1}$$

Note that a 1-interrupter is an interrupter as described above.

Remark 2.2. It is easy to check from (2.1) that if S serves as an n-interrupter for a nonzero operator, then $\|S\| \geq \frac{\|T^n\|^2}{\|T\|^2}$, where $\|S\|$ is the operator norm of S. In particular if $n = 1$, we have $\|S\| \geq 1$.

Proposition 2.3. *If S is n-interrupter for T, then*

$$\langle Sy, y\rangle \geq 0, \quad \forall y \in \overline{\mathcal{R}(T)}.$$

Proof. From Definition 2.1, we obtain $\|T^{*n}x\|^2 = \langle ST(x), Tx\rangle$, for all $x \in \mathcal{H}$. Thus, $\langle Sy, y\rangle \geq 0$, for all $y \in \mathcal{R}(T)$. Therefore, $\langle Sy, y\rangle \geq 0$, for all $y \in \overline{\mathcal{R}(T)}$. □

As a direct consequence of Proposition 2.3, we have the following corollary.

Corollary 2.4. *If T has dense range, then any n-interrupter for T is positive.*

Proof. Since $\overline{\mathcal{R}(T)} = \mathcal{H}$, the corollary follows from Proposition 2.3. □

Proposition 2.5. *If T has dense range, then T has at most one n-interrupter.*

Proof. Let S_1 and S_2 be n-interrupters for T. We have $T^*S_2T = T^*S_1T$ which gives $T^*(S_2 - S_1)T = 0$. Since the range of T is dense, we obtain $T^*(S_2 - S_1) = 0$. Applying again the fact that T has dense range, we get that T^* is one to one. Thus, we obtain from the later identity $S_2 - S_1 = 0$. Therefore, $S_2 = S_1$. □

We need the two following results:

Theorem 2.6. [[1], Theorem 1] *Let A and B be bounded operators on a Hilbert space $\mathcal{H}$. The following statements are equivalent:*

(1) $\mathcal{R}(A) \subseteq \mathcal{R}(B)$;
(2) $AA^* \leq \mu^2 BB^*$ *for some* $\mu \geq 0$;
(3) *There exists a bounded operator C so that $A = BC$.*

Moreover, if (1), (2) *and* (3) *hold, then there is a unique operator T such that*

(*a*) $\|T\|^2 = \inf\{\mu, AA^* \leq \mu^2 BB^*\}$;
(*b*) $\mathcal{N}(A) = \mathcal{N}(T)$;
(*c*) $\mathcal{R}(T) \subseteq \overline{\mathcal{R}(B^*)}$.

Theorem 2.7. [[8], Theorem 2.1] *For $T \in \mathcal{B}(\mathcal{H})$ the following statements are equivalent:*
(1) *T is posinormal;*
(2) *$\mathcal{R}(T) \subseteq \mathcal{R}(T^*)$;*
(3) *$TT^* \leq \lambda^2 T^*T$ or equivalently $\|T^*x\| \leq \lambda\|Tx\|$, $x \in \mathcal{H}$ for some $\lambda \geq 0$;*
(4) *There exists a bounded operator C so that $T = T^*C$.*
Moreover, if (1), (2), (3) and (4) hold, then there is a unique operator S such that
(*i*) *$\|S\|^2 = \inf\{\lambda, TT^* \leq \lambda^2 T^*T\}$;*
(*ii*) *$\mathcal{N}(T) = \mathcal{N}(S)$;*
(*iii*) *$\mathcal{R}(S) \subseteq \overline{\mathcal{R}(T)}$.*

We observe from (2) in Theorem 2.7 that if T is posinormal then $\mathcal{R}(T^n) \subseteq \mathcal{R}(T^*)$ for any positive integer n. Starting from this observation, and since our goal is to introduce a new class of operators containing posinormal operators, we take this necessary condition for posinormality as a definition of the new concept. This leads to the following definition:

Definition 2.8. Let n be a positive integer and $T \in \mathcal{B}(\mathcal{H})$. T is n-power-posinormal if $\mathcal{R}(T^n) \subseteq \mathcal{R}(T^*)$, and T is n-power-coposinormal if T^* is n-power-posinormal.

Remark 2.9. The class of 1-power-posinormal operators is the well known class of posinormal operators introduced by Rhaly in [8].

Remark 2.10. If $\mathcal{N}(T) = \{0\}$, then T^* is surjective. Consequently, T is n-power-posinormal for any positive integer n.

Remark 2.11. Let n be a positive integer. It is easy to check that the following statements hold:
(a) If T is n-power-posinormal, then T is m-power-posinormal for any integer $m \geq n$;
(b) If T is n-power-posinormal, then $\mathcal{N}(T) \subseteq \mathcal{N}({T^*}^n)$;
(c) If T is n-power-posinormal, then $\mathcal{N}(T^{n+1}) = \mathcal{N}(T^n)$;
(d) If T is n-power-posinormal, then $\mathcal{N}(T^{k+1}) = \mathcal{N}(T^k)$ for $k \geq n$;
(e) If T is n-power-posinormal, then T is $n+1$-power-posinormal ;
(f) If T is posinormal, then T^k is n-power-posinormal for any positive integer k.

Remark 2.12. In ([6], Example 1), the authors constructed a posinormal operator T for which the square T^2 is not posinormal. From (f) in Remark 2.11 (take $k = 1$), T is 2-power-posinormal. This example shows that n-power-posinormality of T does not imply that T^n is posinormal.

Theorem 2.13. *For $T \in \mathcal{B}(\mathcal{H})$, the following statements are equivalent:*
(1) *T has positive n-interrupter;*
(2) *$T^n{T^*}^n \leq \lambda^2 T^*T$ or equivalently $\|{T^*}^n x\| \leq \lambda\|Tx\|$, $x \in \mathcal{H}$ for some $\lambda > 0$;*
(3) *T is n-power-posinormal : $\mathcal{R}(T^n) \subseteq \mathcal{R}(T^*)$;*
(4) *There exists $C \in \mathcal{B}(\mathcal{H})$ such that $T^n = T^*C$.*
Moreover, if (1), (2), (3) and (4) hold, then there is a unique operator S such that
(*i*) *$\|S\|^2 = \inf\{\lambda, T^nT^{n*} \leq \lambda^2 T^*T\}$;*
(*ii*) *$\mathcal{N}(T^n) = \mathcal{N}(S)$;*
(*iii*) *$\mathcal{R}(S) \subseteq \overline{\mathcal{R}(T^n)}$.*

Proof. (1) $\implies$ (2) : If $T^n{T^*}^n = T^*PT$ with P positive, we get

$$\langle T^n{T^*}^n x, x\rangle = \langle \sqrt{P}Tx, \sqrt{P}Tx\rangle = \|\sqrt{P}Tx\|^2 \leq \|\sqrt{P}\|^2\|Tx\|^2 = \|\sqrt{P}\|^2\langle T^*Tx, x\rangle.$$

Thus (2) holds with $\lambda > \|\sqrt{P}\|$.

Applying Theorem 2.6, by taking $A = T^n$ and $B = T^*$, we obtain the equivalences (2) $\iff$ (3) $\iff$ (4). If (4) holds then (1) holds by taking $P = C^" C$.

To get (i), (ii) and (iii), take $A = T^n$ and $B = T^*$ in (a), (b) and (c), respectively, in Theorem 2.6. ∎

The following example shows that a 3-power-posinormal weighted shift is not necessarily posinormal.

Example 2.14. Let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis of a Hilbert space $\mathcal{H}$. Define an operator T on $\mathcal{H}$ by

$$Te_1 = e_2,\ Te_2 = 2e_3,\ Te_3 = 0,\ Te_k = e_{k+1},\ k \geq 4.$$

A simple calculation yields

$$T^*e_1 = 0,\ T^*e_2 = 2e_1,\ T^*e_3 = 2e_2,\ T^*e_4 = 0,\ T^*e_k = e_{k-1},\ k \geq 5. \tag{2.2}$$

Observe that $e_3 \in \mathcal{N}(T)$ but $e_3 \notin \mathcal{N}(T^*)$. Thus, T is not posinormal.

Iterating, we obtain

$$T^3e_k = 0, k = 1, 2, 3 \text{ and } T^3e_k = e_{k+3},\ k \geq 4. \tag{2.3}$$

From (2.2) and (2.3), we obtain $T^3e_k \in \mathcal{R}(T^*)$ for $k \geq 1$. Thus $\mathcal{R}(T^3) \subseteq \mathcal{R}(T^*)$. Therefore, by Theorem 2.13, T is 3-power-posinormal.

Theorem 2.15. *Let $T \in \mathcal{B}(\mathcal{H})$. If T has dense range and an n-interrupter, then T is n-power-posinormal.*

Proof. This follows from (1) in Theorem 2.13 and Corollary 2.4. □

Remark 2.16. We observe from (1) in Theorem 2.13 that if T is n-power-posinormal with P as an n-interrupter and V is an isometry, then VTV^* is n-power-posinormal with VPV^* as n-interrupter. Consequently, n-power-posinotmality is a unitary invariant.

Theorem 2.17. *T^n is k-power-posinormal if and only if T is nk-power-posinormal.*

Proof. Since $\mathcal{R}(T^{nk}) = \mathcal{R}((T^n)^k)$, we have

$$\mathcal{R}(T^{nk}) \subseteq \mathcal{R}(T^*) \iff \mathcal{R}((T^n)^k) \subseteq \mathcal{R}(T^*).$$

This yields the desired equivalence. □

Corollary 2.18. *T is n-power-posinormal if one of the following statements holds:*

(1) *T^n is hyponormal;*
(2) *T is n-power normal.*

Proof. (1) The statement follows from Corollary 2.1 [8] and Theorem 2.17.

(2) If T is n-power normal then T^n is normal (see [4]) and thus T^n is posinormal. Applying Theorem 2.17, we obtain that T is n-power-posinormal. □

Proposition 2.19. *If T is n-power-posinormal and $\mathcal{R}(T) = \mathcal{R}(T^n)$, then T is posinormal.*

Proof. If T is n-power-posinormal, we have $\mathcal{R}(T^n) \subseteq \mathcal{R}(T^*)$. Since $\mathcal{R}(T) = \mathcal{R}(T^n)$ we obtain $\mathcal{R}(T) \subseteq \mathcal{R}(T^*)$. Thus, from (2) in Theorem 2.7, T is posinormal. □

Proposition 2.20. *If T is $k+n$-power-posinormal such that ${T^*}^n$ is an isometry, then T is k-power-posinormal.*

Proof. Let P be a positive $n+k$-interrupter for T. We have

$$T^{k+n}{T^*}^{n+k} = T^*PT.$$

Since $T^n{T^*}^n = I$, we have $T^{k+n}{T^*}^{n+k} = T^k{T^*}^k$. Thus

$$T^k{T^*}^k = T^*PT.$$

Therefore, T is k-power-posinormal. □

Corollary 2.21. *If T^n is posinormal and ${T^*}^{n-1}$ is an isometry, then T is posinormal.*

Proof. Straightforward from Theorem 2.17 and Proposition 2.20. □

Proposition 2.22. *If T is 2-power-posinormal and T^* is an isometry, then T is unitary.*

Proof. From Proposition 2.20, we obtain that T is posinormal. Let P be an interrupter for T. We have,

$$TT^* = T^*PT.$$

Since T^* is an isometry, we have $TT^* = I$. This gives

$$I = T^*PT$$

Multiplying the later identity from the left by T and from the right by T^*, we obtain

$$I = TT^* = P$$

Thus, we have $I = TT^* = T^*T$. This achieves the proof. □

3. Other Characterization of n-power-Posinormal Operators

Masuo Ito obtained the following characterization of posinormal operators:

Theorem 3.1. [[3] , Theorem 2] *T is posinormal if and only if , there exists $\lambda > 0$ such that*

$$\left|\left\langle T|T|x, y\right\rangle\right| \leq \lambda \||T|x\|\||T|y\|, \quad x, y \in \mathcal{H}. \tag{3.4}$$

We give a generalization of Theorem 3.1 to the class of n-power-posinormal operators. For that we need the following result due to Fuji *et al.* (see [2]).

Theorem 3.2. [2] *Let $A \geq 0$ and $B \geq 0$. If $T^*T \leq A^2$ and $TT^* \leq B^2$ the inequality*

$$\left|\left\langle T|T|^{p+q-1}x, y\right\rangle\right| \leq \lambda \||A^p|x\|\||B^q|y\| \tag{3.5}$$

holds for all $x, y \in \mathcal{H}$, $0 \leq p, q \leq 1$ with $p + q \geq 1$

Proposition 3.3. *If T is n-power-posinormal operator, then there exists $\lambda > 0$ such that*

$$\left|\left\langle T^n|T^n|^{p+q-1}x, y\right\rangle\right| \leq \lambda \||T^n|^p x\|\|\lambda^q|T|^q y\| \tag{3.6}$$

holds for all $x, y \in \mathcal{H}$, $0 \leq p, q \leq 1$ with $p + q \geq 1$

Proof. Taking in (3.5), $A = |T^n|$ and $B = \lambda|T|$ and $\lambda > 0$ as in the statement (2) of Theorem (2.13), we obtain

$$\left|\left\langle T^n|T^n|^{p+q-1}x, y\right\rangle\right| \leq \lambda \||T^n|^p x\|\|\lambda^q|T|^q y\|$$

for all $x, y \in \mathcal{H}$, $0 \leq p, q \leq 1$ with $p + q \geq 1$. □

Corollary 3.4. *T is n-power-posinormal if and only if, there exists $\lambda > 0$ such that*

$$\left|\left\langle T^n|T^n|x, y\right\rangle\right| \leq \lambda \||T^n|x\|\||T|y\|, \quad \forall\, x, y \in \mathcal{H}. \tag{3.7}$$

Proof. The direct statement follows immediately from Proposition 3.3 by putting $p = q = 1$ in (3.6).

Conversely, suppose that (3.7) holds. Let $T^n = U|T^n|$ be the polar decomposition of T^n. Let $x \in \mathcal{H}$. Applying (3.7) to vectors U^*x and x , we obtain

$$\left|\left\langle T^n|T^n|U^*x, x\right\rangle\right| \leq \lambda \||T^n|U^*x\|\||T|x\|$$

which can be written

$$\left|\left\langle T^nT^{*n}x, x\right\rangle\right| \leq \lambda \|T^{*n}x\|\||T|x\|.$$

Thus, $\|T^{*n}x\|^2 \leq \lambda \|T^{*n}x\|\||T|x\|$. This yields $\|T^{*n}x\|^2 \leq \lambda \||T|x\|^2$. Since $\||T|x\|^2 = \|Tx\|^2$ for $x \in \mathcal{H}$, we have then proved that there exists $\lambda > 0$ such that

$$\|T^{*n}x\|^2 \leq \lambda^2 \|Tx\|^2, \quad x \in \mathcal{H}.$$

Therefore, by Theorem 2.13, T is n-power-posinormal. This completes the proof. □

4. Examples

This section is devoted to give examples which illustrate various aspects of n-power-posinormality. The following proposition allows to calculate an n-interrupter for the power of posinormal operator.

Proposition 4.1. *If T is posinormal and P is an interrupter for T then $(PT)^{n-1}P(T^*P)^{n-1}$ is an n-interrupter for T.*

Proof. By induction : It is obvious that the statement is true for $n = 1$. Suppose that is true for n. We have

$$T^{n+1}T^{*n+1} = TT^nT^*T^* = TT^*(PT)^{n-1}P(T^*P)^{n-1}TT^*.$$

Since $TT^* = T^*PT$, we obtain

$$T^{n+1}T^{*n+1} = TT^nT^*T^* = T^*PT(PT)^{n-1}P(T^*P)^{n-1}T^*PT = T^*(PT)^nP(T^*P)^nT.$$

Thus the property is true for $n + 1$. □

Example 4.2. Let $\mathcal{H} = \ell^2$ and $\{e_n,\ n = 0, 1, \ldots\}$ its standard basis. The Cesàro operator on $\mathcal{H}$ is defined by

$$Ce_n = \sum_{k=n}^{\infty} \frac{1}{k+1} e_k, \quad n = 0, 1, \ldots$$

By routine computation, one gets

$$C^*e_n = \frac{1}{n+1} \sum_{k=0}^{n} e_k, \quad n = 0, 1, \ldots$$

C is posinormal (see [8]) with interrupter the diagonal operator P with diagonal entries $a_{nn} = (n + 1)/(n + 2)$ for $n = 0, 1, \cdots$ Applying the statement (f) of Remark 2.11 (take $k = 1$) and the Proposition 4.1, we obtain that C is n-power-posinormal for any n with n-interrupter $(PC)^{n-1}P(C^*P)^{n-1}$.

Example 4.3. Unilateral weighted shifts are posinormal (see Proposition 1.1 [8]). Thus these operators are n-power-posinormal operator for any positive integer n.

Let $T \in \mathcal{B}(\mathcal{H})$. The hereditary functional calculus defines $p(T) = \sum_{m,n\geq 0} c_{m,n}T^{*n}T^m$ for a polynomial $p(x, y) = \sum_{m,n\geq 0} c_{m,n}x^my^n \in \mathbb{C}[x, y]$,
where $c_{m,n}$ is the coefficient of x^my^n in p (see [7, 9]).

Proposition 4.4. *Let T be a root of $p(x, y) = x^n+yq(x, y)$, then T is n-power-posinormal.*

Proof. If T is a root of p, we get, from $p(T) = 0$, that $T^n = TC$, where $C = q(T)$. Therefore, by the statement (4) of Theorem 2.13, T is n-power-posinormal. □

As a consequences of Proposotion 4.4, we get that If T is (m, n)-isosymetry (T is a root of $p(x, y) = (yx - 1)^m(y - x)^n$), then T is n-power-posinormal. In particular, if T is n-symetry (T is a root of $p(x, y) = (y - x)^n$), then T is n-power-posinormal.

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