Methods of Functional Analysis and Topology
Vol. 27 (2021), no. 1, pp. 25–30
https://doi.org/10.31392/MFAT-npu26_1.2021.04

ON ONE PROBLEM OF YU. M. BEREZANSKY

YURI BOGDANSKII

Abstract. In this article we prove the maximum principle for L-harmonic functions on a Hilbert space, where $(Lu)(x) = j(x)(u''(x))$ with $j(x)$ being a nonnegative functional on the space of self-adjoint bounded operators. The proposed method is then applied to a study of parabolic equations for functions on a Hilbert space.

Доведено принцип максимума для L-гармонічних функцій на гільбертовім просторі, де $(Lu)(x) = j(x)(u''(x))$, $j(x)$–невід'ємний функціонал на просторі самоспряжених обмежених операторів. Запропонований метод застосовується також до дослідження параболічних рівнянь відносно функцій на гільбертовім просторі.

Let $B_s(H)$ be the Banach space (equipped with the operator norm) of self-adjoint bounded linear operators on a real Hilbert space H, J the cone of nonnegative linear functionals on $B_s(H)$, D a bounded domain in H.

For a twice Fréchet differentiable at $x \in H$ scalar function u, the value $u''(x)$ lies in $B_s(H)$.

Given a function $u \in C^2(D) \cap C(\overline{D})$, let us consider a second order elliptic differential expression of the form

$$(Lu)(x) = j(x)(u''(x)), \tag{1}$$

where $j : D \to J$.

It is natural then to ask the question about the validity of the infinite-dimensional analog of the weak maximum principle for L-harmonic functions on D.

By convention, functionals $\alpha \in J$ of the form $C \mapsto Tr(AC)$, where A is a nonnegative nuclear operator in H, are called *regular*. In the case where H is infinite-dimensional, there also exist *singular* functionals, whose kernel contains all operators of finite rank in $B_s(H)$.

It is also known that every functional $\alpha \in J$ admits a unique decomposition $\alpha = \alpha_1 + \alpha_2$ into a sum of regular and singular functionals.

In the case where all functionals $j(x)$ in (1) are regular, the corresponding maximum principle can be obtained by applying the method of finite-dimensional approximations.

In the case where all functionals $j(x)$ in (1) are singular, the maximum principle was proved in [1, 4].

The last result more that 40 years ago was presented by the author at a seminar organized by Yu. M. Berezansky. And instantly the question from Yu. M. Berezansky followed: does a maximum principle holds in the case where the functionals $j(x)$ in (1) are of the general form?

I did not succeed in answering this question at that time. Afterwards, the statement was proved for a smaller class of functions (see [2]). And only after many years, having approached the original problem again, I managed to obtain the desired result. The summary of the proof was published in [3].

Theorem 1. *Let H be a real Hilbert space, D a bounded domain in H, $f : \overline{D} \to \mathbb{R}$ a function of class $C(\overline{D})$ such that $\inf_D f > -\infty$, $\varepsilon > 0$.*

2020 *Mathematics Subject Classification.* 35J99; 35J25.
Keywords. Hilbert space, maximum principle, infinite-dimensional elliptic operator, boundary problem, heat equation.

Then there exists a function $h(x) = a\|x\|^2 + (b,x) + c$ $(a, c \in \mathbb{R}; a > 0; b \in H)$ *for which*

$$\sup_D \|h^{(k)}(\cdot)\| \le \varepsilon, \ k = 0,1,2 \tag{2}$$

and such that for the function $g(x) = f(x) + h(x)$ *there exists a point* $x_0 \in \overline{D}$ *such that* $g(x) > g(x_0)$ *for all* $x \in \overline{D} \setminus \{x_0\}$*. If additionally* $\inf_D f < \inf_{\partial D} f$*, then one can choose a sufficiently small* $\varepsilon > 0$ *such that* $x_0 \in D$*.*

Proof. Denote $a = \inf_D f$ and

$$\delta = \min\left\{\frac{\varepsilon}{2}, \frac{\varepsilon}{2\,\mathrm{diam}\,D}, \frac{\varepsilon}{(\mathrm{diam}\,D)^2}\right\}. \tag{3}$$

In the case where $f(x) \equiv a$, we can take $g(x) = a + \delta\|x - x_0\|^2$ $(x_0 \in D)$. Thus, in the following we can assume that f is non-constant.

Take a point $x_1 \in D$ for which $\varepsilon_1 = f(x_1) - a \in (0, \frac{\delta}{8})$ and, given $\delta_1 > 0$, define the function f_1 on $\overline{D}$ by $f_1(x) = f(x) + \delta_1\|x - x_1\|^2$.

Then for all $x \in \overline{D}$ which satisfy the inequality $\|x - x_1\|^2 > \frac{\varepsilon_1}{\delta_1}$, one has $f_1(x) > f(x) + \varepsilon_1$. We also have that $f_1(x_1) = f(x_1)$, $\inf_D f_1 \ge \inf_D f$ and

$$\inf_{\overline{D}} f_1 = \inf\left\{f_1(x) : x \in \overline{D}, \ \|x - x_1\|^2 \le \frac{\varepsilon_1}{\delta_1}\right\}. \tag{4}$$

Let us construct a sequences $\{\varepsilon_n\}, \{\delta_n\}, \{x_n\}, \{f_n\}$ for $n \ge 1$ as follows: $\varepsilon_{n+1} \in (0, \varepsilon_n)$; $x_{n+1} \in \overline{D}$ has to satisfy the following two (compatible) conditions

$$\begin{cases} f_n(x_{n+1}) \le f_n(x_n) = f_{n-1}(x_n), \\ f_n(x_{n+1}) < \inf_D f_n + \varepsilon_{n+1} \end{cases} \tag{5}$$

(let $f_0 := f$).

For $\delta_{n+1} > 0$, we define the function f_{n+1} as follows:

$$f_{n+1}(x) = f_n(x) + \delta_{n+1}\|x - x_{n+1}\|^2.$$

Then for all $x \in \overline{D}$ satisfying $\|x - x_{n+1}\|^2 > \frac{\varepsilon_{n+1}}{\delta_{n+1}}$, we have $f_{n+1}(x) > f_n(x) + \varepsilon_{n+1}$ and therefore, similarly to (4), one has that

$$\inf_{\overline{D}} f_{n+1} = \inf\left\{f_{n+1}(x) : x \in \overline{D}, \ \|x - x_{n+1}\|^2 \le \frac{\varepsilon_{n+1}}{\delta_{n+1}}\right\}. \tag{6}$$

It follows from (4)–(6) that $\|x_{n+1} - x_n\|^2 \le \frac{\varepsilon_n}{\delta_n}$ for $n \ge 1$.

If $r_n = \left(\frac{\varepsilon_n}{\delta_n}\right)^{\frac{1}{2}} \le 2^{-n}$, then there exists $x_0 = \lim_{n\to\infty} x_n \in \overline{D}$.

Let $\gamma \ge 0$ be a solution of the equation $\varepsilon_1 = \frac{\delta}{8^{1+\gamma}}$. Let $\varepsilon_n = \frac{\delta}{8^{n+\gamma}}$ and $\delta_n = \frac{\delta}{2^{n+\gamma}}$. Then $r_n = \left(\frac{\varepsilon_n}{\delta_n}\right)^{\frac{1}{2}} = 2^{-n-\gamma} \le 2^{-n}$; $r_{n+1} = \frac{1}{2}r_n$, $n \ge 1$; the series $\sum_{k=1}^{\infty} \delta_k$ converges and its sum does not exceed δ.

Since $f_n(x) = f(x) + \sum_{k=1}^{n} \delta_k\|x - x_k\|^2$, the sequence of functions f_n converges uniformly on $\overline{D}$ to the function $g(x) = f(x) + h(x)$, where $h(x) = \sum_{k=1}^{\infty} \delta_k\|x - x_k\|^2 = a\|x\|^2 + (b,x) + c$.

Additionally, we have

$$0 \le h(x) \le \delta \cdot (\mathrm{diam}\,D)^2,$$

$$\|\operatorname{grad} h(x)\| = 2\left\|\sum_{k=1}^{\infty}\delta_k(x-x_k)\right\| \le 2\delta \operatorname{diam} D,$$

$$\|h''(x)\| = 2\sum_{k=1}^{\infty}\delta_k \le 2\delta,$$

from which inequality (2) follows.

Let us prove that x_0 is a strict minimum point of g.

Let $x \ne x_0$ and $r_n = \left(\frac{\varepsilon_n}{\delta_n}\right)^{\frac{1}{2}} < \frac{1}{6}\|x-x_0\|$. Then for $m \ge n$, the following inequalities hold:

$$\begin{aligned}&\|x_m - x_0\| \le 2r_m < \tfrac{1}{3}\|x-x_0\|,\\ &\|x-x_m\| \ge \|x-x_0\| - \|x_m - x_0\| > 2r_m.\end{aligned} \tag{7}$$

From (4)–(6) and inequalities (7) we obtain the inequalities

$$f_n(x_{n+1}) \le f_n(x_n) < f_n(x).$$

Suppose now that $m > n$. Then, by (7), we have

$$\begin{aligned}&\|x_m - x_{n+1}\| \le \|x_m - x_0\| + \|x_{n+1} - x_0\| \le 2r_m + 2r_{n+1} < 2r_n\\ &< \frac{1}{3}\|x-x_0\| \le \frac{1}{3}(\|x-x_m\| + \|x_m - x_0\|) < \|x-x_m\|.\end{aligned}$$

Therefore, one has the inequality

$$\begin{aligned}&f_m(x) - f_m(x_{n+1}) = (f_{m-1}(x) + \delta_m\|x-x_m\|^2) - (f_{m-1}(x_{n+1}) + \delta_m\|x_{n+1}-x_m\|^2)\\ &> f_{m-1}(x) - f_{m-1}(x_{n+1}).\end{aligned}$$

Thus, there exist $\alpha > 0$ and $n \in \mathbb{N}$ such that for any $m \ge n$, the following inequality holds:

$$f_m(x_{n+1}) < f_m(x) - \alpha.$$

Passing to the limit as $m \to \infty$ we get

$$g(x_{n+1}) \le g(x) - \alpha.$$

Passing now to the limit as $n \to \infty$ we obtain

$$g(x_0) < g(x).$$

In the case where $\inf_{\partial D} f - \inf_D f = \beta > 0$ with $\varepsilon \in \left(0, \frac{1}{2}\beta\right)$, the minimum point x_0 lies in D. □

Remark 2. In conditions of Theorem 1, with a minor modification of the proof, one can also guarantee existence of the corresponding function $h(x)$ of the form $h(x) = a\|x\|^2 + (b, x)$, where $0 < a < \varepsilon$ and $\|b\| < \varepsilon$, for which the function $g(x) = f(x) + h(x)$ has the required property.

Corollary 3. *Let H be a real Hilbert space, D a bounded domain in H, $f : \overline{D} \to \mathbb{R}$ a function of class $C^m(D) \cap C(\overline{D})$ such that $\inf_D f > -\infty$, $\varepsilon > 0$.*

Then there exists a function $g \in C^m(D) \cap C(\overline{D})$ such that

$$\sup_D \|f^{(k)}(\cdot) - g^{(k)}(\cdot)\| \le \varepsilon,\ k = 0, 1, 2$$

and

$$f^{(k)}(x) = g^{(k)}(x) \text{ in } D \text{ for } k > 2$$

and there exists $x_0 \in \overline{D}$ such that $g(x) > g(x_0)$ for all $x \in \overline{D} \setminus \{x_0\}$.

In the case $\inf_D f < \inf_{\partial D} f$, one can choose a sufficiently small $\varepsilon > 0$ such that $x_0 \in D$.

Theorem 4. *Let D be a bounded domain in a Hilbert space H, $f : \overline{D} \to \mathbb{R}$ a function of class $C^2(D) \cap C(\overline{D})$ such that $\inf\limits_D f > -\infty$, $j : D \to J$ such that $\inf\limits_D \|j(\cdot)\| > 0$. Suppose that for all $x \in D$, the following inequality holds:*

$$(Lf)(x) = j(x)(f''(x)) \leq 0.$$

Then $\inf\limits_D f = \inf\limits_{\partial D} f$.

Proof. Assume that $\inf\limits_{\partial D} f - \inf\limits_D f = \varepsilon > 0$. Pick any point $x_1 \in D$ and consider the function $h(x) = f(x) - \delta\|x - x_1\|^2$.

If $\delta < \frac{\varepsilon}{2(\operatorname{diam} D)^2}$, then $\inf\limits_{\partial D} h - \inf\limits_D h > \frac{\varepsilon}{2}$. In this case one has $h''(x) = f''(x) - 2\delta I$ and $(Lh)(x) = (Lf)(x) - 2\delta\|j(x)\|$, where $I : H \to H$ is the identity operator.

Denoting $\inf\limits_D \|j(\cdot)\| = \alpha > 0$, we get the inequality

$$(Lh)(x) < -2\alpha\delta$$

that holds for all $x \in D$.

By Corollary 3, there exists a function $g \in C^2(D) \cap C(\overline{D})$ such that for all $x \in D$, the following inequalities hold:

$$|g(x) - h(x)| < \frac{\varepsilon}{4}, \ \|g''(x) - h''(x)\| < 2\alpha\delta,$$

and which attains a strict minimum at some point $x_0 \in D$.

On one hand, for all $x \in D$, we have $(Lg)(x) < 0$, and on the other hand, $g''(x_0) \geq 0$ and thus, $(Lg)(x_0) \geq 0$, which is a contradiction. □

Corollary 5 (Maximum principle for L-harmonic functions)**.** *Let D be a bounded domain in H, $f : \overline{D} \to \mathbb{R}$ a function of class $C^2(D) \cap C(\overline{D})$ which is bounded in D, $j : D \to J$ such that $\inf\limits_D \|j(\cdot)\| > 0$. Suppose that $(Lf)(x) = j(x)(f''(x)) = 0$ for all $x \in D$.*

Then $\sup\limits_{\partial D} f = \sup\limits_D f$ and $\inf\limits_{\partial D} f = \inf\limits_D f$.

The following statement is a modification of Corollary 3 and also follows directly from Theorem 1.

Let $H = H_1 \oplus H_2$ and $D = D_1 \times D_2$, where D_k is a domain in H_k $(k = 1, 2)$. Let $\Gamma \subset \partial D$. Denote by $C^{p,q}(D; \Gamma)$ the set of all continuous on $D \cup \Gamma$ functions f, for which the partial Fréchet derivatives $\frac{\partial^k}{\partial x_1^k} f$ and $\frac{\partial^l}{\partial x_2^l} f$ exist and are continuous on D for all $k \in \{1, \dots, p\}$ and $l \in \{1, \dots, q\}$.

Lemma 6. *Suppose that the domains D_k are bounded in H_k. Let $f \in C^{p,q}(D; \Gamma)$ be such that $\inf\limits_D f > -\infty$ and $\inf\limits_\Gamma f - \inf\limits_D f = 2\varepsilon > 0$.*

Then there exists a function $g \in C^{p,q}(D; \Gamma)$ that attains a strict minimum in D, for which the following inequalities hold:

$$\sup_D |f(\cdot) - g(\cdot)| \leq \varepsilon,$$

$$\sup_D \left\| \frac{\partial^k}{\partial x_1^k} f - \frac{\partial^k}{\partial x_1^k} g \right\| \leq \varepsilon,$$

$$\sup_D \left\| \frac{\partial^l}{\partial x_2^l} f - \frac{\partial^l}{\partial x_2^l} g \right\| \leq \varepsilon$$

for $1 \leq k \leq \min\{p, 2\}$, $1 \leq l \leq \min\{q, 2\}$; and for $3 \leq k \leq p$, $3 \leq l \leq q$ one has

$$\frac{\partial^k}{\partial x_1^k} f = \frac{\partial^k}{\partial x_1^k} g, \ \frac{\partial^l}{\partial x_2^l} f = \frac{\partial^l}{\partial x_2^l} g$$

everywhere in D.

Let D be a bounded domain in a Hilbert space H, $T \in (0,+\infty)$, $P = D\times(0,T) \subset H\times\mathbb{R}$, $\widehat{P} = \overline{D} \times [0,T)$, $\Gamma = \widehat{P} \setminus P = (\overline{D} \times \{0\}) \cup (\partial D \times [0,T)) \subset \partial P$.

Theorem 7. *Let $j : P \to J$ and suppose that a function $u \in C^{2,1}(P;\Gamma)$ satisfies the inequality*

$$\frac{\partial u}{\partial t} \geq L_x u = j(x,t)(u''_x)$$

everywhere in the cylinder P. Suppose also that $a = \sup\limits_P \|j(\cdot)\| < +\infty$ and $\inf\limits_P u > -\infty$.

Then $\inf\limits_P u = \inf\limits_\Gamma u$.

Proof. Suppose that $\inf\limits_\Gamma u - \inf\limits_P u > 2\alpha > 0$. Let $\delta = \frac{\alpha}{T}$. Consider the function $v(x,t) = u(x,t) + \delta t$.

The inequality $\frac{\partial v}{\partial t} - L_x v \geq \delta > 0$ holds everywhere in P. Moreover, we have that $\inf\limits_\Gamma v \geq \inf\limits_\Gamma u$ and $\inf\limits_P v \leq \inf\limits_P u + \delta T$, and thus

$$\inf_\Gamma v - \inf_P v \geq 2\alpha - \delta T = \alpha.$$

Let w be a function that is ε-close to v in the sense of Lemma 6 and that attains a strict minimum in P at a point (x_0,t_0). Then $L_x w = j(x,t)(w''_x) \leq j(x,t)(v''_x) + a\varepsilon$ and

$$\frac{\partial w}{\partial t} - L_x w \geq \frac{\partial v}{\partial t} - \varepsilon - L_x v - a\varepsilon \geq \delta - \varepsilon(a+1).$$

Take $\varepsilon < \min\{\frac{\delta}{a+1}, \frac{\alpha}{2}\}$. Then everywhere in P one has

$$\frac{\partial w}{\partial t} - L_x w > 0,$$

while the minimum point $(x_0,t_0) \in P$ and thus, one has

$$\frac{\partial w}{\partial t}(x_0,t_0) - L_x w(x_0,t_0) \leq 0,$$

which is a contradiction. □

Corollary 8 (Maximum principle for the first boundary value problem for the heat equation)**.** *Let $j : P \to J$ be such that $\sup\limits_P \|j(\cdot)\| < +\infty$. Suppose that a function $u \in C^{2,1}(P;\Gamma)$ is bounded on P and satisfies in P the equation*

$$\frac{\partial u}{\partial t} = L_x u = j(x,t)(u''_x).$$

Then $\inf\limits_P u = \inf\limits_\Gamma u$ and $\sup\limits_P u = \sup\limits_\Gamma u$.

Theorem 9. *Let $W = H \times (0,T)$, where $T \in (0,+\infty]$, and $\Gamma = H \times \{0\}$. Let $u \in C^{2,1}(W;\Gamma)$ be bounded on W, $j : W \to J$. Suppose that $a = \sup\limits_W \|j(\cdot)\| < +\infty$ and everywhere in W the following inequality holds:*

$$\frac{\partial u}{\partial t}(x,t) \geq (L_x u)(x,t) = j(x,t)(u''_x(x,t)).$$

Then $\inf\limits_W u = \inf\limits_\Gamma u = \inf\limits_H u(\cdot,0)$.

Proof. Pick a point $(x_0,t_0) \in W$. Define the function w by $w(x,t) = 2a(t-t_0) + \|x-x_0\|^2$. Then w satisfies in W the inequality

$$\frac{\partial w}{\partial t} = 2a \geq L_x w.$$

Take $\varepsilon > 0$. For the function $v = u + \varepsilon w$, one has

$$v(x_0, t_0) = u(x_0, t_0), \ \frac{\partial v}{\partial t} \geq L_x v \text{ (in } W).$$

Given $R > 0$, consider the cylinder $P_R = \{(x, t) : \|x - x_0\| < R, \ t \in (0, T)\}$. By Theorem 7, the following inequality holds:

$$v(x_0, t_0) \geq \inf_{\Gamma_R} v, \tag{8}$$

where $\Gamma_R = \{(x, 0) : \|x - x_0\| \leq R\} \cup \{(x, t) : \|x - x_0\| = R, \ t \in [0, T)\}$.

Considering now the function u, by (8), we get

$$u(x_0, t_0) \geq \inf_{\Gamma_R} \left(u(x, t) + 2a\varepsilon(t - t_0) + \varepsilon\|x - x_0\|^2\right) \geq \inf_{\Gamma_R} \left(u(x, t) - 2a\varepsilon t_0 + \varepsilon\|x - x_0\|^2\right). \tag{9}$$

Let $M = \sup_W u$. Take $R > \sqrt{\frac{2M}{\varepsilon}}$.

Then $\|x - x_0\|^2 = R$ implies that $\varepsilon\|x - x_0\|^2 > 2M$ and thus, by (9),

$$u(x_0, t_0) \geq \inf_{\|x - x_0\| \leq R} u(x, 0) - 2a\varepsilon t_0 \geq \inf_{x \in H} u(x, 0) - 2a\varepsilon t_0,$$

which, since $\varepsilon > 0$ is arbitrary, proves the theorem. □

Corollary 10 (Maximum principle for a Cauchy problem for the heat equation)**.** *Let $j : W \to J$ be such that $\sup_W \|j(\cdot)\| < +\infty$. Suppose that a function $u \in C^{2,1}(W; H \times \{0\})$ is bounded on W and satisfies in W the equation*

$$\frac{\partial u}{\partial t}(x, t) = j(x, t)\left(u''_x(x, t)\right).$$

Then $\inf_W u = \inf_H u(\cdot, 0)$ and $\sup_W u = \sup_H u(\cdot, 0)$.

Remark 11. In an obvious way from Corollaries 5–10 one obtains corresponding uniqueness theorems and theorems about continuous dependence of a solution on the boundary-value (or initial-vlaue) conditions.

References

[1] Yu. Bogdanskii, *Cauchy problem for parabolic equations with essentially infinite-dimensional elliptic operators*, Ukr. Math. J. **29** (1977), no. 6, 781–784.

[2] Yu. Bogdanskii, *Maximum principle for irregular elliptic differential equation in countable-dimensional Hilbert space*, Ukr. Math. J. **40** (1988), no. 1, 21–25.

[3] Yu. Bogdanskii, *Maximum principle for an elliptic equation in a Hilbert space*, Spectral and Evolution Problems **10** (2000), 93–95.

[4] Yu. Bogdanskii and Yu. Daletskii, *Cauchy problem for the simplest parabolic equation with essentially infinite-dimensional elliptic operator*, (Suppl. to chapters IV, V) Yu. Daletskii, S. Fomin. Measures and differential equations in infinite-dimensional space, Kluwer Acad. Publ., 1991, pp. 309–322.

Yuri Bogdanskii: bogd__@ukr.net
Igor Sikorsky Kyiv Polytechnic Institute, 37 Peremohy Avenue, 03056 Kyiv, Ukraine

Received 21/09/2020; Revised 02/11/2020