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EQUALITY BETWEEN DIFFERENT TYPES OF INVERTIBILITY

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Dedicated to our Professor Abdelaziz Tajmouati on the occasion of his retirement.

ABSTRACT. Necessary and sufficient conditions for the between different types of invertibility are established.

Встановлені необхідні та достатні умови співпадіння різних типів оборотності.

1. Introduction and Preliminaries

Throughout, X denotes a complex Banach space, $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators on X, let I be the identity operator, and for $T \in \mathcal{B}(X)$ we denote by T^*, $N(T)$, $R(T)$, $R^\infty(T) = \bigcap_{n\geq 0} R(T^n)$, $\rho(T)$, $\sigma(T)$ respectively the adjoint, the null space, the range, the hyper-range, the resolvent set and the spectrum of T.

Let E be a subset of X. E is said to be T-invariant if $T(E) \subseteq E$. We say that T is completely reduced by a pair (E, F) and we denote $(E, F) \in Red(T)$ if E and F are two closed T-invariant subspaces of X such that $X = E \oplus F$. In this case we write $T = T_{|E} \oplus T_{|F}$ and we say that T is a direct sum of $T_{|E}$ and $T_{|F}$. An operator $T \in \mathcal{B}(X)$ is said to be semi-regular, if $R(T)$ is closed and $N(T) \subseteq R^\infty(T)$ [1].

On the other hand, recall that an operator $T \in \mathcal{B}(X)$ admits a generalized Kato decomposition, (GKD for short), if there exists $(X_1, X_2) \in Red(T)$ such that $T_{|X_1}$ is semi-regular and $T_{|X_2}$ is quasi-nilpotent, in this case T is said tobe a pseudo Fredholm operator. If we assume in the definition above that $T_{|X_2}$ is nilpotent, then T is said to be of Kato type. Clearly, every semi-regular operator is of Kato type and a quasi-nilpotent operator has a GKD, see [5, 7] for more information about generalized Kato decomposition.

For every bounded operator $T \in \mathcal{B}(X)$, let us define a semi-regular spectrum, Kato type spectrum and generalized Kato spectrum respectively by:

$$\sigma_{se}(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not semi-regular }\};$$

$$\sigma_{tk}(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ does not admit a Kato decomposition}\};$$

$$\sigma_{gk}(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ does not admit a generalized Kato decomposition}\}.$$

Recall that a bounded operator $T \in \mathcal{B}(X)$ is Riesz, if $T - \lambda I$ is Fredholm in the usual sense for every $\lambda \in \mathbb{C}\backslash\{0\}$ [1]. Of course compact and quasi-nilpotent operators are particular cases of Riesz operators.

In [11], Živković-Zlatanović SČ and M D. Cvetković introduced and studied a new concept of Kato decomposition to extend the Mbektha concept to "generalized Kato-Riesz decomposition". In fact, an operator $T \in \mathcal{B}(X)$ admits a generalized Kato-Riesz decomposition, (GKRD for short), if there exists $(X_1, X_2) \in Red(T)$ such that $T_{|X_1}$ is semi-regular and $T_{|X_2}$ is Riesz. The generalized Kato-Riesz spectrum is defined by

$$\sigma_{gKR}(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ does not admit a generalized Kato-Riesz decomposition}\}.$$

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Let $T \in \mathcal{B}(X)$, T is said to be Drazin invertible if there exist a positive integer k and an operator $S \in \mathcal{B}(X)$ such that

$$ST = TS, \ \ T^{k+1}S = T^k \ \ \textit{and} \ S^2T = S.$$

Which is also equivalent to the fact that $T = T_1 \oplus T_2$; where T_1 is invertible and T_2 is nilpotent. The Drazin spectrum is defined by

$$\sigma_D(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Drazin invertible } \}.$$

The concept of Drazin invertible operators has been generalized by Koliha [4]. In fact, $T \in \mathcal{B}(X)$ is generalized Drazin invertible if and only if $0 \notin acc(\sigma(T))$, where $acc(\sigma(T))$ is the set of accumulation points of $\sigma(T)$. This is also equivalent to the fact that there exists $(X_1, X_2) \in Red(T)$ such that $T_{|X_1}$ is invertible and $T_{|X_2}$ is quasi-nilpotent. The generalized Drazin spectrum is defined by

$$\sigma_{gD}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not generalized Drazin invertible } \}.$$

The concept of analytical core for an operator has been introduced by Vrbova in [10] and study by Mbekhta [7, 8], that is the following set:

$$K(T) = \{x \in X \ : \ \exists (x_n)_{n\geq 0} \subset X \ \text{ and } \ \delta > 0 \ \text{ such that } \ x_0 = x, \ Tx_n = x_{n-1} \ \forall n \geq 1 \ \text{ and } \ \|x_n\| \leq \delta^n \|x\|\}$$

The quasi-nilpotent part of T, $H_0(T)$ is given by :

$$H_0(T) := \{x \in X; r_T(x) = 0\} \text{ where } r_T(x) = \lim_{n\to+\infty} ||T^n x||^{\frac{1}{n}}.$$

In [3], M D. Cvetković and SČ. Živković-Zlatanović introduced and studied a new concept of generalized Drazin invertibility of bounded operators as a generalization of generalized Drazin invertible operators. In fact, an operator $T \in \mathcal{B}(X)$ is said to be generalized Drazin bounded below if $H_0(T)$ is closed and complemented with a subspace M in X such that $(M, H_0(T)) \in Red(T)$ and $T(M)$ is closed which is equivalent to there exists $(M, N) \in Red(T)$ such that $T_{|M}$ is bounded below and $T_{|N}$ is quasi-nilpotent, see [3, Theorem 3.6]. An operator $T \in \mathcal{B}(X)$ is said to be generalized Drazin surjective if $K(T)$ is closed and complemented with a subspace N in X such that $N \subseteq H_0(T)$ and $(K(T), N) \in Red(T)$ which is equivalent to there exists $(M, N) \in Red(T)$ such that $T_{|M}$ is surjective and $T_{|N}$ is quasi-nilpotent, see [3, Theorem 3.7].
The generalized Drazin bounded below and surjective spectra of $T \in \mathcal{B}(X)$ are defined respectively by:

$$\sigma_{gD\mathcal{M}}(T) = \{\lambda \in \mathbb{C}, \ T - \lambda I \text{ is not generalized Drazin bounded below}\};$$

$$\sigma_{gD\mathcal{Q}}(T) = \{\lambda \in \mathbb{C}, \ T - \lambda I \text{ is not generalized Drazin surjective}\}.$$

From [3], we have:

$$\sigma_{gD}(T) = \sigma_{gD\mathcal{M}}(T) \cup \sigma_{gD\mathcal{Q}}(T).$$

Recently, Živković-Zlatanović SČ and M D. Cvetković [11] introduced and studied a new concept of pseudo-inverse to extend the Koliha concept, generalized Drazin bounded below, and generalized Drazin surjective to "generalized Drazin-Riesz invertible", "generalized Drazin-Riesz bounded below" and "generalized Drazin-Riesz surjective" respectively. In fact, an operator $T \in \mathcal{B}(X)$ is said to be generalized Drazin-Riesz invertible, if there exists $S \in \mathcal{B}(X)$ such that

$$TS = ST, \quad STS = S \quad \text{and } TST - T \text{ is Riesz}$$

Živković-Zlatanović SČ and M D. Cvetković also showed that T is generalized Drazin-Riesz invertible iff it has a direct sum decomposition $T = T_1 \oplus T_0$ with T_1 is invertible and T_0 is Riesz. If we assume in the characterization above that T_1 is bounded below (surjective), then T is said to be generalized Drazin-Riesz bounded below(generalized Drazin-Riesz surjective). The generalized Drazin-Riesz, generalized Drazin-Riesz bounded below and generalized Drazin-Riesz surjective spectra of $T \in \mathcal{B}(X)$ are defined respectively by:

$$\sigma_{gDR}(T) = \{\lambda \in \mathbb{C}, \quad T - \lambda I \text{ is not generalized Drazin-Riesz invertible}\}$$

$$\sigma_{gDR\mathcal{M}}(T) = \{\lambda \in \mathbb{C}, \quad T - \lambda I \text{ is not generalized Drazin-Riesz bounded below }\}$$

$$\sigma_{gDR\mathcal{Q}}(T) = \{\lambda \in \mathbb{C}, \quad T - \lambda I \text{ is not generalized Drazin-Riesz surjective}\}$$

2. Main Results

Recall that $T \in \mathcal{B}(X)$ is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (SVEP for short) if for every open neighbourhood $U \subseteq \mathbb{C}$ of λ_0, the only analytic function $f : U \longrightarrow X$ which satisfies the equation $(T - zI)f(z) = 0$ for all $z \in U$ is the function $f \equiv 0$. An operator T is said to have the SVEP if T has the SVEP for every $\lambda \in \mathbb{C}$. Obviously, every operator $T \in \mathcal{B}(X)$ has the SVEP at every $\lambda \in \rho(T) = \mathbb{C} \setminus \sigma(T)$, hence T and T^* have the SVEP at every point of the boundary $\partial(\sigma(T))$ of the spectrum.

Let $T \in \mathcal{B}(X)$, the ascent of T is defined by $a(T) = min\{p \in \mathbb{N} : N(T^p) = N(T^{p+1})\}$, if such p does not exists we let $a(T) = \infty$. Analogously the descent of T is $d(T) = min\{q \in \mathbb{N} : R(T^q) = R(T^{q+1})\}$, if such q does not exists we let $d(T) = \infty$ [6]. It is well known that if both $a(T)$ and $d(T)$ are finite then $a(T) = d(T)$ and we have the decomposition $X = R(T^p) \oplus N(T^p)$ where $p = a(T) = d(T)$. The descent and ascent spectra of $T \in \mathcal{B}(X)$ are defined by :

$$\sigma_{des}(T) = \{\lambda \in \mathbb{C},\ T - \lambda I \text{ has not finite descent}\}$$

$$\sigma_{ac}(T) = \{\lambda \in \mathbb{C},\ T - \lambda I \text{ has not finite ascent }\}.$$

We start with the following example.

Example 2.1. Let $T : l^2(\mathbb{N}) \longrightarrow l^2(\mathbb{N})$ be the unilateral right shift operator defined by

$$T(x_1, x_2, \cdots) = (0, x_1, x_2, \cdots) \quad \text{for all} \quad (x_n) \in l_2(\mathbb{N}).$$

$$\sigma(T) = \{\lambda \in \mathbb{C} : 0 \leq |\lambda| \leq 1\},$$

$$\sigma_a(T) = \sigma_s(T^*) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}.$$

So, $\sigma_a(T) \subset \sigma(T)$ is strict. Also, $\sigma_s(T^*) \subset \sigma(T)$ and $\sigma_{se}(T) \subseteq \sigma_a(T) \subset \sigma(T)$ are strict.

In the following result we give several necessary and sufficient conditions for T to have some equalities between different types of spectra.

Theorem 2.2. *Let $T \in \mathcal{B}(X)$. The statements are equivalent:*

1. *T and T^* have SVEP at every $\lambda \notin \sigma_s(T)$;*
2. *$\sigma(T) = \sigma_s(T)$;*
3. *$\sigma_{gD}(T) = \sigma_{gD\mathcal{Q}}(T)$;*
4. *T and T^* have SVEP at every $\lambda \notin \sigma_{gD\mathcal{Q}}(T)$;*
5. *T and T^* have SVEP at every $\lambda \notin \sigma_{gDR\mathcal{Q}}(T)$;*
6. *$\sigma_{gDR}(T) = \sigma_{gDR\mathcal{Q}}(T)$;*
7. *$\sigma_D(T) = \sigma_{des}(T)$.*

Proof. 1) $\Longleftrightarrow$ 2): Suppose that $\sigma(T) = \sigma_s(T)$. $\lambda \notin \sigma_s(T) = \sigma(T)$, then $T - \lambda I$ is invertible, so T and T^* have SVEP at λ. Conversely, if $\lambda \notin \sigma_s(T)$, then $T - \lambda I$ is surjective and by hypothesis T has SVEP at λ, so $T - \lambda I$ is invertible. Hence, $\lambda \notin \sigma(T)$.

2) $\Longrightarrow$ 3) If $\sigma(T) = \sigma_s(T)$, according to [3, Theorems 2.5 and 2.6] we have

$$\begin{aligned}\lambda \notin \sigma_{gD\mathcal{Q}}(T) &\iff T-\lambda I \text{ is generalized Drazin surjective}\\ &\iff T-\lambda I \text{ admits a GKD and } \lambda \notin acc\sigma_s(T)\\ &\iff T-\lambda I \text{ admits a GKD and } \lambda \notin acc\sigma(T)\\ &\iff T-\lambda I \text{ is generalized Drazin invertible}\\ &\iff \lambda \notin \sigma_{gD}(T).\end{aligned}$$

Hence $\sigma_{gD\mathcal{Q}}(T) = \sigma_{gD}(T)$.

3) $\Longrightarrow$ 4): Suppose that $\sigma_{gD\mathcal{Q}}(T) = \sigma_{gD}(T)$, if $\lambda \notin \sigma_{gD\mathcal{Q}}(T)$ then $T-\lambda I$ is generalized Drazin invertible. So, T and T^* have the SVEP at λ.

4) $\Longrightarrow$ 2): If T and T^* have SVEP at every $\lambda \notin \sigma_{gD\mathcal{Q}}(T)$, then T and T^* have SVEP at every $\lambda \notin \sigma_s(T)$ which gives $\sigma(T) = \sigma_s(T)$.

2) $\Longrightarrow$ 6): Assume that $\sigma(T) = \sigma_s(T)$, according to [11, Theorems 2.5 and 2.3] we have

$$\begin{aligned}\lambda \notin \sigma_{gDR\mathcal{Q}}(T) &\iff T-\lambda I \text{ is generalized Drazin Riesz surjective}\\ &\iff T-\lambda I \text{ admits a GKRD and } \lambda \notin int(\sigma_s(T))\\ &\iff T-\lambda I \text{ admits a GKRD and } \lambda \notin int(\sigma(T))\\ &\iff T-\lambda I \text{ is generalized Drazin Riesz invertible}\\ &\iff \lambda \notin \sigma_{gDR}(T).\end{aligned}$$

Hence $\sigma_{gDR\mathcal{Q}}(T) = \sigma_{gDR}(T)$.

6) $\Longrightarrow$ 5): Suppose that $\sigma_{gDR\mathcal{Q}}(T) = \sigma_{gDR}(T)$. If $\lambda \notin \sigma_{gDR\mathcal{Q}}(T)$, then $T-\lambda I$ is generalized Drazin Riesz invertible, so T and T^* have SVEP at λ, by [11, Theorem 2.3].

5) $\Longrightarrow$ 2): If T and T^* have SVEP at every $\lambda \notin \sigma_{gDR\mathcal{Q}}(T)$, then T and T^* have SVEP at every $\lambda \notin \sigma_s(T)$ hence $\sigma(T) = \sigma_s(T)$.

2) $\iff$ 7): Suppose that $\sigma(T) = \sigma_s(T)$. $\lambda \notin \sigma_{des}(T)$, then $T-\lambda I$ has finite descent. Therefore, we may apply [2, Proposition 1.1] to find $\delta > 0$ such that, for every $\mu \in \mathbb{C}$ with $0 < |\mu - \lambda| < \delta$, $T-\mu I$ becomes surjective. By hypothesis $T-\mu I$ is bijective. Since $T-\mu I$ is invertible for every $\mu \in \mathbb{C}$ with $0 < |\mu-\lambda| < \delta$, then λ is not an accumulation point of the spectrum of T i.e. $T-\lambda I$ is generalized Drazin invertible: $T-\lambda I = T_1 \oplus T_2$, where T_1 is invertible and T_2 is quasinilpotent. Since the descent of $T-\lambda I$ is finite, the descent of T_2 is also finite. Now, T_2 is quasinilpotent with finite descent and hence is nilpotent. It follows that $T-\lambda I$ is Drazin invertible, i.e. $\lambda \notin \sigma_D(T)$. Conversely, if $\lambda \notin \sigma_s(T)$, then $T-\lambda I$ has finite descent, by hypothesis $T-\mu I$ is Drazin invertible. So $T-\lambda I$ has finite ascent and $a(T-\lambda I) = d(T-\lambda I) = 0$, which implies that $T-\lambda I$ is bijective. □

In the same way we have the following result.

Theorem 2.3. *Let $T \in \mathcal{B}(X)$. The statements are equivalent:*

(1) *T and T^* have SVEP at every $\lambda \notin \sigma_a(T)$;*
(2) *$\sigma(T) = \sigma_a(T)$;*
(3) *$\sigma_{gD}(T) = \sigma_{gD\mathcal{M}}(T)$;*
(4) *T and T^* have SVEP at every $\lambda \notin \sigma_{gD\mathcal{M}}(T)$;*
(5) *T and T^* have SVEP at every $\lambda \notin \sigma_{gDR\mathcal{M}}(T)$;*
(6) *$\sigma_{gDR}(T) = \sigma_{gDR\mathcal{M}}(T)$.*

Proposition 2.4. *Let $T \in \mathcal{B}(X)$. The statements are equivalent:*

(1) *T^* has SVEP at every $\lambda \notin \sigma_{gk}(T)$;*
(2) *T^* has SVEP at every $\lambda \notin \sigma_{se}(T)$;*
(3) *$\sigma_s(T) = \sigma_{se}(T)$;*

(4) $\sigma_{gD\mathcal{Q}}(T) = \sigma_{gk}(T)$.

Proof. 1) $\Longrightarrow$ 2): Since $\sigma_{gk}(T) \subset \sigma_{se}(T)$ we have the result.

2) $\Longrightarrow$ 3): Let $\lambda \notin \sigma_{se}(T)$, then $T - \lambda I$ is semi-regular and since T^* has SVEP at every $\lambda \notin \sigma_{se}(T)$, so $T - \lambda I$ is surjective, hence $\lambda \notin \sigma_s(T)$.

3) $\Longrightarrow$ 4): Let $\lambda \notin \sigma_{gk}(T)$, $T - \lambda I$ admit a GKD, then there exists $(M, N) \in Red(T)$ such that $(T - \lambda I)_{|M}$ is semi-regular and $(T - \lambda I)_{|N}$ is quasi-nilpotent. $(T - \lambda I)_{|M}$ is semi-regular implies that $0 \in \mathbb{C} \setminus \sigma_{se}((T - \lambda I)_{|M})$, since $\sigma_{se}((T - \lambda I)_{|M})$ is closed, there exists $\epsilon > 0$ such that $D(0, \epsilon) \subset \mathbb{C} \setminus \sigma_{se}((T - \lambda I)_{|M})$.

$(T - \lambda I)_{|N}$ is quasi-nilpotent, so $0 \in acc(\rho((T - \lambda I)_{|N}) \subset acc(\mathbb{C} \setminus \sigma_{se}((T - \lambda I)_{|N}))$. Consequently,

$$0 \in acc(\mathbb{C}\setminus\sigma_{se}((T-\lambda I)_{|M})\cap\mathbb{C}\setminus\sigma_{se}((T-\lambda I)_{|N})) = acc(\mathbb{C}\setminus\sigma_{se}(T-\lambda I)) = acc(\mathbb{C}\setminus\sigma_s(T-\lambda I)).$$

Therefore, $0 \notin int(\sigma_s(T - \lambda I))$. According to [3, Theorem 3.7], $T - \lambda$ is generalized Drazin surjective, $\lambda \notin \sigma_{gD\mathcal{Q}}(T)$.

4) $\Longrightarrow$ 1): Suppose that $\sigma_{gD\mathcal{Q}}(T) = \sigma_{gk}(T)$, let $\lambda \notin \sigma_{gk}(T)$ then $T - \lambda I$ is generalized Drazin invertible, so T and T^* have the SVEP. □

By duality, we have the following.

Proposition 2.5. *Let $T \in \mathcal{B}(X)$. The statements are equivalent:*

(1) *T has SVEP at every $\lambda \notin \sigma_{gk}(T)$;*
(2) *T has SVEP at every $\lambda \notin \sigma_{se}(T)$;*
(3) *$\sigma_a(T) = \sigma_{se}(T)$;*
(4) *$\sigma_{gD\mathcal{M}}(T) = \sigma_{gk}(T)$.*

Proposition 2.6. *Let $T \in \mathcal{B}(X)$. The statements are equivalent:*

(1) *T and T^* have SVEP at every $\lambda \notin \sigma_{gk}(T)$;*
(2) *T and T^* have SVEP at every $\lambda \notin \sigma_{se}(T)$;*
(3) *$\sigma(T) = \sigma_{se}(T)$;*
(4) *$\sigma_{gD}(T) = \sigma_{gk}(T)$.*

Proof. 1) $\Longrightarrow$ 2): Since $\sigma_{gk}(T) \subset \sigma_{se}(T)$ we have the result.

2) $\Longrightarrow$ 3): Let $\lambda \notin \sigma_{se}(T)$, then $T - \lambda I$ is semi-regular and since T and T^* have SVEP at every λ, so $T - \lambda I$ is invertible, hence $\lambda \notin \sigma(T)$.

3) $\Longrightarrow$ 4): Let $\lambda \notin \sigma_{gk}(T)$, $T - \lambda I$ admit a GKD, then there exists $(M, N) \in Red(T)$ such that $(T - \lambda I)_{|M}$ is semi-regular and $(T - \lambda I)_{|N}$ is quasi-nilpotent. $(T - \lambda I)_{|M}$ is semi-regular implies that $0 \in \mathbb{C} \setminus \sigma_{se}((T - \lambda I)_{|M})$, since $\sigma_{se}((T - \lambda I)_{|M})$ is closed, there exists $\epsilon > 0$ such that $D(0, \epsilon) \subset \mathbb{C} \setminus \sigma_{se}((T - \lambda I)_{|M})$.

$(T - \lambda I)_{|N}$ is quasi-nilpotent, so $0 \in acc(\rho((T - \lambda I)_{|N}) \subset acc(\mathbb{C} \setminus \sigma_{se}((T - \lambda I)_{|N}))$. Consequently,

$$0 \in acc(\mathbb{C}\setminus\sigma_{se}((T-\lambda I)_{|M})\cap\mathbb{C}\setminus\sigma_{se}((T-\lambda I)_{|N})) = acc(\mathbb{C}\setminus\sigma_{se}(T-\lambda I)) = acc(\mathbb{C}\setminus\sigma(T-\lambda I)).$$

Therefore, $0 \notin int(\sigma(T - \lambda I))$. According to [3, Theorem 3.9], $T - \lambda I$ is generalized Drazin invertble, $\lambda \notin \sigma_{gD}(T)$.

4) $\Longrightarrow$ 1): Suppose that $\sigma_{gD}(T) = \sigma_{gk}(T)$, let $\lambda \notin \sigma_{gk}(T)$ then $T - \lambda I$ is generalized Drazin invertible, so T and T^* have the SVEP. □

Theorem 2.7. *Let $T \in \mathcal{B}(X)$. The statements are equivalent:*

(1) *T and T^* have SVEP at every $\lambda \notin \sigma_{gk}(T)$;*
(2) *T and T^* have SVEP at every $\lambda \notin \sigma_{tk}(T)$;*
(3) *T and T^* have SVEP at every $\lambda \notin \sigma_{se}(T)$;*
(4) *$\sigma_D(T) = \sigma_{tk}(T)$;*
(5) *$\sigma(T) = \sigma_{se}(T)$;*

(6) $\sigma_{gD}(T) = \sigma_{gk}(T)$.

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References

[1] P. Aiena, *Fredholm and Local Spectral Theory with Applications to Multipliers,* Kluwer. Acad. Press, 2004.

[2] M. Burgos, A. Kaidi, M. Mbekhta and M. Oudghiri, *The descent spectrum and perturbations,* Journal of Operator Theory 56 (2006), 259-271.

[3] M. D. Cvetković, SČ. Živković-Zlatanović, *Generalized Kato decomposition and essential spectra,* Complex Anal Oper Theory, 11 (6)(2017), 1425-1449.

[4] J. J. Koliha, *Isolated spectral points,* Proc. Amer. Math. Soc. 124(1996), 3417-3424.

[5] J. P. Labrousse, *Les opérateurs quasi-Fredholm: une généralisation des opérateurs semi-Fredholm,* Rend. Circ. Math. Palermo, 29(2)(1980), 161-258.

[6] D. Lay, A. Taylor, *Introduction to functional analysis,* J. Wiley and Sons, N. York. 1980.

[7] M. Mbekhta, *Généralisation de la décomposition de Kato aux opérateurs paranormaux et spectraux,* Glasgow Math J, 29 (1987) 159-175.

[8] M. Mbekhta, *Sur la théorie spectrale locale et limite des nilpotents,* Proc. Amer. Math. Soc. 110 (1990), 621-631.

[9] V. Muller, *Spectral Theory of Linear Operators and Spectral Systems in Banach Algebras,* 2nd edition. Oper. Theory Advances and Applications, vol. 139 (2007).

[10] P. Vrbová, *On local spectral properties of operators in Banach spaces,* Czechoslovak Math. J. 23(98)(1973), 483-492.

[11] SČ. Živković-Zlatanović, M D. Cvetković *Generalized Kato-Riesz decomposition and generalized Drazin-Riesz invertible operators,* Linear Multilinear Algebra, 65(6) (2017), 1171-1193.

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