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A NEW REPRESENTATION OF LEFT AND RIGHT GENERALIZED DRAZIN INVERTIBLE OPERATORS

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Abstract. The purpose of this paper is to study the relationship between spectral properties of a bounded operator and its left and right generalized Drazin inverses. The description of the associated spectral projections allows us to find some new representation results and certain generalizations on left and right generalized Drazin invertible bounded operators.

Метою статті є дослідження співвідношення між спектральними властивостями обмеженого оператора і його лівого та правого узагальненого оберненого в сенсі Дразіна. Опис відповідних спектральних проєкторів дозволяє знайти нові теореми представлення, а також певні узагальнення класу операторів, оборотних у сенсі Дразіна.

1. Introduction and Preliminaries

Throughout this paper, let $\mathcal{B}(\mathcal{H})$ denote the Banach algebra of all bounded linear operators acting on a non trivial complex Hilbert space $\mathcal{H}$. For $A \in \mathcal{B}(\mathcal{H})$, we denote the spectrum of A, null space of A and range of A by $\sigma(A)$, $\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively. I is the identity operator on $\mathcal{H}$. The ascent denoted by $a(A)$ is the smallest nonnegative integer p such that $\mathcal{N}(A^p) = \mathcal{N}(A^{p+1})$. If no such integer exists we set $a(A) = \infty$. The descent of A, denoted by $d(A)$, is the smallest non negative integer q such that $\mathcal{R}(A^q) = \mathcal{R}(A^{q+1})$. If no such integer exists we set $d(A) = \infty$. If both $a(A)$ and $d(A)$ are finite then $a(A) = d(A)$. If $\mathcal{R}(A)$ is closed then $a(A) = d(A^*)$ and $d(A) = a(A^*)$, see [1]. For a subset $\mathcal{U}$ of the complex plane $\mathbb{C}$, the set of accumulation points of $\mathcal{U}$ is denoted by $acc(\mathcal{U})$. If 0 is an isolated point of $\sigma(A)$, then the spectral projection of A associated with $\{0\}$ is defined by:

$$P(A) = \frac{1}{2\pi i} \oint_{\Gamma} (\lambda I - A)^{-1} \, d\lambda \tag{1.1}$$

where Γ is a small circle surrounding 0 and separating $\{0\}$ from $\sigma(A)\setminus\{0\}$. An element $A \in \mathcal{B}(\mathcal{H})$ whose spectrum $\sigma(A)$ consists of the set $\{0\}$ is said to be quasinilpotent. The set of all quasinilpotent operators of $\mathcal{B}(\mathcal{H})$ is denoted by $QNIL(\mathcal{H})$. For an A-invariant subspace M of $\mathcal{H}$, we define $A_M : M \longrightarrow M$ by $A_M x = Ax$ for all $x \in M$. We say A is completely reduced by the pair (M, N) (denoted by $(M, N) \in Red(A)$) if M and N are two closed A-invariant subspaces of $\mathcal{H}$ such that $\mathcal{H} = M \oplus N$ (that is, $\mathcal{H} = M + N$ and $M \cap N = \{0\}$). In such a case we have $\mathcal{N}(A) = \mathcal{N}(A_M) \oplus \mathcal{N}(A_N)$, $\mathcal{R}(A) = \mathcal{R}(A_M) \oplus \mathcal{R}(A_N)$ and $A^n = A_M^n \oplus A_N^n$, for all $n \in \mathbb{N}$. An operator is said to be bounded below if it is injective with closed range, boundedness from below is equivalent to left invertibility of A in the Hilbert spaces setting. If A is surjective on $\mathcal{H}$ it is right invertible and vice versa. Note that the properties to be bounded below or to be surjective are dual to each other, then A is bounded below if and only if A^* is surjective, where A^* is the adjoint operator of A.

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$A \in \mathcal{B}(\mathcal{H})$ is called Drazin invertible if and only if it has finite ascent and descent, which is equivalent to 0 is a finite order pole of the resolvent operator $(\lambda I - A)^{-1}$, say of order k. In such case $a(A) = d(A) = k$. In fact, it is well known that A is Drazin invertible if there exists a unique operator $A^D \in \mathcal{B}(\mathcal{H})$ such that $AA^D = A^D A$, $A^D AA^D = A^D$ and $AA^D A - A$ is nilpotent of degree k. Koliha generalized this concept in [3] by replacing the third condition by $AA^{GD}A - A \in QNIL(\mathcal{H})$, where the generalized Drazin inverse of A is now noted A^{GD} instead of A^D. Precisely, a necessary and sufficient condition for $A \in \mathcal{B}(\mathcal{H})$ to be generalized Drazin invertible is that $0 \notin acc\sigma(A)$. Equivalently, $(\mathcal{K}(A), \mathcal{H}_0(A)) \in Red(A)$, the restriction $A_{\mathcal{K}(A)}$ of A to $\mathcal{K}(A)$ is invertible and the spectrum $\sigma\left(A_{\mathcal{H}_0(A)}\right)$ of the restriction of A to $\mathcal{H}_0(A)$ is contained in $\{0\}$, where:

$$\mathcal{K}(A) = \left\{ \begin{array}{c} x \in \mathcal{H} : \exists (x_n)_{n\in\mathbb{N}} \subset \mathcal{H},\ \exists \delta_x > 0 \text{ such that } Ax_1 = x, \\ Ax_{n+1} = x_n \text{ and } \|x_n\| \leq \delta_x^n \|x\| \text{ for all} n \in \mathbb{N} \end{array} \right\}$$

is the analytical core of A and

$$\mathcal{H}_0(A) = \left\{ x \in \mathcal{H} : \lim_{n\to\infty} \|A^n x\|^{1/n} = 0 \right\}$$

is the quasinilpotent part of A.

$\mathcal{K}(A)$ and $\mathcal{H}_0(A)$ are generally not necessarily closed subspaces of $\mathcal{H}$, we always have $A(\mathcal{H}_0(A)) \subset \mathcal{H}_0(A)$ and $A(\mathcal{K}(A)) = \mathcal{K}(A)$. It is clear that $A \in QNIL(\mathcal{H})$ if and only if $\mathcal{H}_0(A) = \mathcal{H}$. Moreover, if A is bounded below, then $\mathcal{H}_0(A) = \{0\}$. $\mathcal{K}(A) = \{0\}$ if $A \in QNIL(\mathcal{H})$ and $\mathcal{K}(A) = \mathcal{H}$ if A is surjective. Fore more details on the analytical core and the quasinilpotent part see e.g. [1].

Recently, using these two subspaces Ounadjela, Miloud and Messirdi introduced in [5] the concept of left and right generalized Drazin invertible operators as a generalization of Drazin invertible operators. $A \in \mathcal{B}(\mathcal{H})$ is called left (resp. right) generalized Drazin invertible if $\mathcal{H}_0(A)$ (resp. $\mathcal{K}(A)$) is closed and complemented with a subspace M (resp. N) in $\mathcal{H}$ such that $A(M) \subset M$ and $A(M)$ is closed (resp. $A(N) \subset N \subseteq \mathcal{H}_0(A)$). If A is left (resp. right) generalized Drazin invertible, the pair $(M, \mathcal{H}_0(A))$ (resp. $(\mathcal{K}(A), N)$) is called a left (resp. right) generalized Drazin decomposition of A and denoted by $(M, \mathcal{H}_0(A)) \in lRed(A)$ (resp. $(\mathcal{K}(A), N) \in rRed(A)$).

$A \in \mathcal{B}(\mathcal{H})$ is generalized Drazin invertible if and only if it is both left and right generalized Drazin invertible. It follows from the definition that if A is left (resp. right) generalized invertible, it generally admits several left (resp. right) generalized Drazin inverses. $\{A^{lGD}\}$ (resp. $\{A^{rGD}\}$) denotes the set of all left (resp. right) generalized Drazin inverses of A. Notice that the set of all left generalized Drazin invertible operators in $\mathcal{B}(\mathcal{H})$ is nonempty, for instance it contains $I \in \mathcal{B}(\mathcal{H})$.

The basic existence results of left and right generalized Drazin inverses and their relations to the quasinilpotent part and the analytical core are summarized in the following theorem.

Theorem 1.1. *([2]) Let $A \in \mathcal{B}(\mathcal{H})$, then the following assertions are equivalent:*

1) A is left (resp. right) generalized Drazin invertible,

2) $A = A_1 \oplus A_2$, where $A_1 = A_M$ is bounded below (resp. $A_1 = A_{\mathcal{K}(A)}$ is surjective) and $\sigma(A_2) \subset \{0\}$ where $A_2 = A_{\mathcal{H}_0(A)}$ (resp. $A_2 = A_N$).

3) There exists a bounded projection $P_l(A)$ (resp. $P_r(A)$) on $\mathcal{H}$ such that $AP_l(A) = P_l(A)A$ (resp. $AP_r(A) = P_r(A)A$), $A + P_l(A)$ is bounded below (resp. $A + P_r(A)$ is surjective), $AP_l(A)$ is quasinilpotent (resp. $AP_r(A)$ is quasinilpotent) and $\mathcal{R}(P_l(A)) = \mathcal{H}_0(A)$ (resp. $\mathcal{N}(P_r(A)) = \mathcal{K}(A)$).

It follows that if $A = A_1 \oplus A_2$ is the decomposition of a left or a right generalized Drazin invertible operator $A \in \mathcal{B}(\mathcal{H})$, described in the assertion (2) of Theorem 1.1, we must have:

$$\left\{A^{lGD}\right\} = \left\{A_1^{-1,left} \oplus 0_{\mathcal{H}_0(A)} : A_1^{-1,left} \text{ is a left inverse of } A_M \text{ and } (M, \mathcal{H}_0(A)) \in lRed(A)\right\},$$

$$\left\{A^{rGD}\right\} = \left\{A_1^{-1,right} \oplus 0_N : A_1^{-1,right} \text{ is a right inverse of } A_{\mathcal{K}(A)} \text{ and } (\mathcal{K}(A), N) \in rRed(A)\right\}.$$

We will justify these formulations in section 2.

As a consequence of Theorem 1.1, if A is generalized Drazin invertible there exists a unique bounded projection $P(A)$ on $\mathcal{H}$ such that $AP(A) = P(A)A$, $A + P(A)$ is invertible, $AP(A)$ is quasinilpotent, $\mathcal{R}(P(A)) = \mathcal{H}_0(A)$ and $\mathcal{N}(P(A)) = \mathcal{K}(A)$, this result was first shown by Koliha in [3]. In this case the generalized Drazin inverse $A^{GD} \in \mathcal{B}(\mathcal{H})$ is given by:

$$A^{GD} = (A + P(A))^{-1} (I - P(A)) \tag{1.2}$$

where $P(A)$ is the spectral projection (1.1) of A associated with $\{0\}$ and $\left\{A^{lGD}\right\} = \left\{A^{rGD}\right\} = A^{GD}$.

When A is left or right generalized Drazin invertible, it's clear that the bounded projection $P_l(A)$ or $P_r(A)$ associated to A, satisfying assertion (3) of Theorem 1.1 is not necessarily unique. Moreover, knowing $P_l(A)$ (resp. $P_r(A)$) we do not have a relation, such as (1.2), which expresses the left (resp. right) generalized Drazin inverses of A via these projections. So, Theorem 1.1 motivates us to put the following natural question : Suppose that $A \in \mathcal{B}(\mathcal{H})$ is left or right generalized Drazin invertible, does it exist expressions as (1.1) and (1.2) to describe the projections $P_l(A)$ and $P_r(A)$ and the operators A^{lGD} and A^{rGD}. The purpose of this article is to provide an explicit representation of the projections $P_l(A)$ and $P_r(A)$ and that of the corresponding left and right generalized Drazin inverses. So, we generalize the formula (1.2) obtained in [3] and we also improve certain characterization results obtained in the paper [5].

2. Main results

We will have to recall the following preparatory results.

Proposition 2.1 ([4]). *$A \in \mathcal{B}(\mathcal{H})$ is left (resp. right) generalized Drazin invertible if and only if there exist two operators $L_l, Q_l \in \mathcal{B}(\mathcal{H})$ (resp. $L_r, Q_r \in \mathcal{B}(\mathcal{H})$) such that $Q_l \in QNIL(\mathcal{H})$ (resp. $Q_r \in QNIL(\mathcal{H})$), $AL_lA = L_lA^2 = A - Q_l$ (resp. $AL_rA = A^2L_r = A - Q_r$) and $L_lAL_l = L_l^2A = L_l$ (resp. $L_rAL_r = AL_r^2 = L_r$). L_l (resp. L_r) is a left (resp. right) generalized Drazin inverse of A (not necessarily unique).*

Proposition 2.2 ([2]). *Let $A \in \mathcal{B}(\mathcal{H})$, then A is left (resp. right) generalized Drazin invertible, if and only if A^* is right (resp. left) generalized Drazin invertible, and if $(M, N) \in lRed(A)$ (resp. $(M, N) \in rRed(A)$) then $\left(N^\perp, M^\perp\right) \in rRed(A^*)$ (resp. $\left(N^\perp, M^\perp\right) \in lRed(A^*)$), in particular, $\mathcal{K}(A^*) = \mathcal{H}_0(A)^\perp$ (resp. $\mathcal{H}_0(A^*) = \mathcal{K}(A)^\perp$).*

Our first main result essentially gives the explicit expressions of the bounded projections $P_l(.)$ and $P_r(.)$ involved in Theorem 1.1.

Theorem 2.3. *Let $A \in \mathcal{B}(\mathcal{H})$ be left generalized Drazin invertible with $(M, \mathcal{H}_0(A)) \in lRed(A)$ and L_l a left generalized Drazin inverse of A (resp. right generalized Drazin invertible with $(\mathcal{K}(A), N) \in rRed(A)$ and L_r a right generalized Drazin inverse of A). Then, $P_l(A) = I - L_lA$ (resp. $P_r(A) = I - AL_r$) is a bounded projection on $\mathcal{H}$ such that:*

1) $AP_l(A) = P_l(A)A$ (resp. $AP_r(A) = P_r(A)A$),
2) $A + P_l(A)$ is bounded below (resp. $A + P_r(A)$ is surjective),
3) $AP_l(A) \in QNIL(\mathcal{H})$ (resp. $AP_r(A) \in QNIL(\mathcal{H})$),
4) $\mathcal{R}(P_l(A)) = \mathcal{H}_0(A)$ (resp. $\mathcal{N}(P_r(A)) = \mathcal{K}(A)$).

Proof. $P_l(A) \in \mathcal{B}(\mathcal{H})$ (resp. $P_r(A) \in \mathcal{B}(\mathcal{H})$). From Proposition 2.1, we obtain:

$$(P_l(A))^2 = I - 2L_lA + (L_lAL_l)\,A = I - 2L_lA + L_lA = P_l(A),$$
$$(\text{resp. } (P_r(A))^2 = I - 2AL_r + A\,(L_rAL_r) = I - 2AL_r + AL_r = P_r(A)).$$

1) Proposition 2.1 also shows that:

$$AP_l(A) = A\,(I - L_lA) = A - AL_lA = A - L_lA^2 = (I - L_lA)\,A = P_l(A)A,$$
$$(\text{resp. } AP_r(A) = A\,(I - AL_r) = A - A^2L_r = A - AL_rA = (I - AL_r)\,A = P_r(A)A).$$

2) In fact,

$$(L_l + P_l(A))\,(A + P_l(A)) = A - L_lAA + I,$$

then $(L_l + P_l(A))\,(A + P_l(A))$ is invertible (commutative sum of the indentity operator and a quasinilopotent operator), this involves that $A + P_l(A)$ is bounded below.

On the other hand,

$$(A + P_r(A))\,(L_r + P_r(A)) = A - AAL_r + I,$$

then $(A + P_r(A))\,(L_r + P_r(A))$ is invertible (commutative sum of the identity operator and a quasinilopotent operator), this implies that $A + P_r(A)$ is surjective.

3) We have:

$$\begin{aligned} AP_l(A) &= A - AL_lA = Q_l \\ (\text{resp. } AP_r(A) &= A - AAL_r = Q_r) \end{aligned}$$

4) Let $y = P_l(A)x$, then for every $n \in \mathbb{N}^*$ we have:

$$\begin{aligned} A^ny &= A^n\,(I - L_lA)\,x \\ &= (A^n - A^nL_lA)\,x \\ &= \left(A^n - A^{n-1}L_lA^2\right)x \\ &\;\;\vdots \\ &= \left(A^n - L_lA^{n+1}\right)x \\ &= A^{n-1}\left(A - L_lA^2\right)x \\ &= A^{n-1}Q_lx. \end{aligned}$$

But

$$\begin{aligned} Q_l^2 &= A^2 + AL_lAAL_lA - AL_lAA - L_lAAA \\ &= A^2 - AL_lA^2 \\ &= AQ_l, \end{aligned}$$

then:

$$A^ny = Q_l^nx$$

which means that $y \in \mathcal{H}_0(A)$. Conversely, suppose that $y \in \mathcal{H}_0(A)$, that is $\lim \|A^ny\|^{\frac{1}{n}} = 0$. Thus,

$$\begin{aligned} P_l(A)y &= y - L_lAy \\ &= y - L_l^nA^ny. \end{aligned}$$

But

$$\lim \|L_l^nA^ny\| \leq \lim \|L_l\|^n\,\|A^ny\| = 0.$$

So,

$$P_l(A)y = y$$

which means that $y \in \mathcal{R}(P_l(A))$.

The corresponding formula for the right case is obtained by applying duality (see Proposition 2.2). □

Remark 2.4. If $L_l, Q_l \in \mathcal{B}(\mathcal{H})$ (resp. $L_r, Q_r \in \mathcal{B}(\mathcal{H})$) are such that $AL_lA = L_lA^2 = A - Q_l$ (resp. $AL_rA = A^2L_r = A - Q_r$) and $L_lAL_l = L_l^2A = L_l$ (resp. $L_rAL_r = AL_r^2 = L_r$) where $Q_l \in QNIL(\mathcal{H})$ (resp. $Q_r \in QNIL(\mathcal{H})$), then $(M, \mathcal{H}_0(A)) \in lRed(A)$ and $L_l = L_M \oplus 0_{\mathcal{H}_0(A)}$, L_M is a left inverse of A_M (resp. $(\mathcal{K}(A), N) \in rRed(A)$ and $L_r = L_{\mathcal{K}(A)} \oplus 0_N$, $L_{\mathcal{K}(A)}$ is a right inverse of $A_{\mathcal{K}(A)}$) where $M = \mathcal{R}(I - P_l(A)) = \mathcal{R}(L_lA)$ (resp. $N = \mathcal{N}(I - P_l(A)) = \mathcal{N}(AL_r)$).

Consequently, we obtain the following second main result.

Theorem 2.5. *Let $A \in \mathcal{B}(\mathcal{H})$ be left generalized Drazin invertible with $(M, \mathcal{H}_0(A)) \in lRed(A)$ (resp. right generalized Drazin invertible with $(\mathcal{K}(A), N) \in rRed(A)$). Then,*

$$\left\{L_l = L_M \oplus 0_{\mathcal{H}_0(A)} : L_M \text{ is a left inverse of } A_M\right\} \subset \left\{A^{lGD}\right\}$$
$$\text{(resp. } \left\{L_r = L_{\mathcal{K}(A)} \oplus 0_N : L_{\mathcal{K}(A)} \text{ is a right inverse of } A_{\mathcal{K}(A)}\right\} \subset \left\{A^{rGD}\right\}\text{)},$$

but the opposite inclusion does not hold in general.

Moreover,

$$\left\{A^{lGD}\right\} = \left\{L_l = L_M \oplus 0_{\mathcal{H}_0(A)} : L_M \text{ is a left inverse of } A_M \text{ and } (M, \mathcal{H}_0(A)) \in lRed(A)\right\},$$
$$\left\{A^{rGD}\right\} = \left\{L_r = L_{\mathcal{K}(A)} \oplus 0_N : L_{\mathcal{K}(A)} \text{ is a right inverse of } A_{\mathcal{K}(A)} \text{ and } (\mathcal{K}(A), N) \in rRed(A)\right\}.$$

Proof. The inclusion $\left\{L_l = L_M \oplus 0_{\mathcal{H}_0(A)} : L_M \text{ is a left inverse of } A_M\right\} \subset \left\{A^{lGD}\right\}$ (resp. $\left\{L_r = L_{\mathcal{K}(A)} \oplus 0_N : L_{\mathcal{K}(A)} \text{ is a right inverse of } A_{\mathcal{K}(A)}\right\} \subset \left\{A^{rGD}\right\}$) follows from the previous constructions and is generally strict since M (resp. N are not uniquely determined such that $(M, \mathcal{H}_0(A)) \in lRed(A)$ (resp. $(\mathcal{K}(A), N) \in rRed(A)$).

Indeed, let $l^2(\mathbb{N})$ be the space of the square-summable complex sequences and $(e_n)_{n\in\mathbb{N}}$ its canonical basis. Consider:

1) $A \in \mathcal{B}\left(l^2(\mathbb{N})\right)$ defined by:

$$Ax = y \text{ such that } \begin{cases} y_0 = y_1 = 0, \\ y_n = x_{n-1} \text{ if } n \geq 2 \end{cases}$$

$x = (x_n)_{n\in\mathbb{N}} \in l^2(\mathbb{N})$. A is left generalized Drazin invertible with $M = span\{e_1, e_2, e_3, ...\}$ and $N = span\{e_0\}$ as first possible left decomposition, but there is an other possible choice for M. We can take for example $M = span\{e_0 + e_1, e_2, e_3, e_4, ...\}$ or any $M = span\{ae_0 + e_1, e_2, e_3, e_4, ...\}$ with $a \in \mathbb{C}$.

2) $A \in \mathcal{B}\left(l^2(\mathbb{N})\right)$ defined by:

$$Ax = y \text{ such that } \begin{cases} y_0 = 0, \\ y_n = x_{n+1} \text{ if } n \geq 1 \end{cases}$$

$x = (x_n)_{n\in\mathbb{N}} \in l^2(\mathbb{N})$. Then A is right generalized Drazin invertible with $M = span\{e_1, e_2, e_3, ...\}$ and $N = span\{e_0\}$ as first possible right decomposition. There are other possible choices of N, for example we can take $N = span\{e_0 + e_1\}$ or any $N = span\{e_0 + ae_1\}$ with $a \in \mathbb{C}$.

Finally,

$$\left\{A^{lGD}\right\} = \bigcup_{\substack{L_M \text{ is a left inverse of } A_M,\\ (M,\mathcal{H}_0(A))\in lRed(A)}} \left\{L_l = L_M \oplus 0_{\mathcal{H}_0(A)}\right\},$$

and

$$\left\{A^{rGD}\right\} = \bigcup_{\substack{L_{\mathcal{K}(A)} \text{ is a right inverse of } A_{\mathcal{K}(A)},\\ (\mathcal{K}(A),N)\in rRed(A)}} \left\{L_r = L_{\mathcal{K}(A)} \oplus 0_N\right\}.$$

□

Conjecture 2.6. *If $A \in \mathcal{B}(\mathcal{H})$ is left (resp. right) generalized Drazin invertible with unique left (resp. right) reduction, then A is generalized Drazin invertible.*

We hope show this conjecture in a future work.

By using the characterizations given in Theorem 2.3 and Theorem 2.5, we provide the following illustrative examples.

Example 2.7. 1) Let

$$\mathcal{X} = \left\{ (x_n)_{n\in\mathbb{N}^*} \in l^2(\mathbb{N}) : x_{2n-1} = 0 \right\},$$
$$\mathcal{Y} = \left\{ (x_n)_{n\in\mathbb{N}^*} \in l^2(\mathbb{N}) : x_{2n} = 0 \right\}.$$

Let $S : \mathcal{X} \longrightarrow \mathcal{X}$, and $T : \mathcal{Y} \longrightarrow \mathcal{Y}$ be the bounded linear operators defined by:

$$S(x_1, x_2, x_3, ...) = (0, \mu_2 x_2, 0, \mu_4 x_4, 0, \mu_6 x_6, ...),$$
$$T(x_1, x_2, x_3, ...) = \left(\frac{x_3}{3}, 0, \frac{x_5}{5}, 0, \frac{x_7}{7}, ... \right)$$

where $(\mu_n)_{n\in\mathbb{N}^*}$ is a bounded sequence of nonzero complex numbers.

Clearly, $l^2(\mathbb{N}) = \mathcal{X} \oplus \mathcal{Y}$. S is injective, in fact invertible on $\mathcal{X}$, with closed range of infinite-codimension in $l^2(\mathbb{N})$. Hence, the operator S is left invertible with left inverse $S^{-1,left}(x_n)_{n\in\mathbb{N}^*} = \left(\frac{x_n}{\mu_n}\right)_{n\in\mathbb{N}^*}$. Moreover; for $k \in \mathbb{N}$, $k \geq 1$,

$$T^k(x_1, x_2, x_3, ...) = \left(\frac{x_{2k+1}}{357...(2k+1)}, 0, \frac{x_{2k+3}}{579...(2k+3)}, 0, \frac{x_{2k+5}}{7911...(2k+5)}, ... \right).$$

So, $\left\| T^k (x_n)_{n\in\mathbb{N}^*} \right\| \leq \frac{\left\| (x_n)_{n\in\mathbb{N}^*} \right\|}{k!}$ and then $\left\| T^k \right\|^{1/k} \leq \frac{1}{\sqrt[k]{k!}} \underset{k\to\infty}{\longrightarrow} 0$, since if $n \in \mathbb{N}$ is odd we have $\prod_{m=0}^{k-1} \frac{1}{2(m+1)+n} \leq \frac{1}{k!}$. Therefore, T is a quasinilpotent operator and $\mathcal{H}_0(T) = \mathcal{Y}$.

Then, $A : l^2(\mathbb{N}) \longrightarrow l^2(\mathbb{N})$ defined by:

$$A(x_1, x_2, x_3, ...) = \left(\frac{x_3}{3}, x_1, \frac{x_5}{5}, x_2, \frac{x_7}{7}, ... \right)$$

is a bounded linear operator that satisfies $A = S \oplus T$. Thus, A is left generalized Drazin invertible operator with:

$$M = \mathcal{X},\ \mathcal{H}_0(A) = \mathcal{Y},\ L_l = S^{-1,left} \oplus 0_{\mathcal{Y}},\ P_l(A) = 0_{\mathcal{X}} \oplus I_{\mathcal{Y}} \text{ and } \left\{ A^{lGD} \right\} = \left\{ S^{-1,left} \oplus 0_{\mathcal{Y}} \right\}.$$

It is clear that A is also right generalized Drazin invertible since $S^{-1,left} = S^{-1,right} = S^{-1}$ with:

$$N = \mathcal{Y},\ \mathcal{K}(A) = \mathcal{X},\ L_r = S^{-1,right} \oplus 0_{\mathcal{Y}},\ P_r(A) = 0_{\mathcal{X}} \oplus I_{\mathcal{Y}} \text{ and } \left\{ A^{lGD} \right\} = \left\{ S^{-1,left} \oplus 0_{\mathcal{Y}} \right\}.$$

Therefore, A is generalized invertible with generalized Drazin invertible $A^{GD} = S^{-1} \oplus 0_{\mathcal{Y}}$ and the corresponding spectral projection $P(A) = 0_{\mathcal{X}} \oplus I_{\mathcal{Y}}$.

2) Let S and T be the linear operators defined on $l^2(\mathbb{N})$ by:

$$S(x_1, x_2, x_3, ...) = (x_2, x_3, ...),$$
$$T(x_1, x_2, x_3, ...) = \left(\frac{x_2}{2}, \frac{x_3}{3}, ..., \frac{x_n}{n}, ... \right).$$

S is surjective, but not injective, then S is right invertible with right inverse the right shift operator $S^{-1,right}(x_1, x_2, x_3, ...) = (0, x_1, x_2, x_3, ...)$, and T is quasinilpotent. So, $A = S \oplus T$ is right generalized Drazin invertible operator on $l^2(\mathbb{N}) \oplus l^2(\mathbb{N})$ with $S^{-1,right} \oplus 0_{l^2(\mathbb{N})} \subsetneq \left\{ A^{rGD} \right\}$.

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