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NEW RESULTS ON THE EXISTENCE OF PERIODIC SOLUTIONS FOR A HIGHER-ORDER P-LAPLACIAN NEUTRAL DIFFERENTIAL EQUATION WITH MULTIPLE DEVIATING ARGUMENTS

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Abstract. In this article, we consider the following high-order p-Laplacian neutral differential equation with multiple deviating arguments:

$$(\varphi_p(x(t)-cx(t-r))^{(n)}(t)))^{(m)} = f(x(t))x'(t)+g(t,x(t),x(t-\tau_1(t)),...,x(t-\tau_k(t)))+e(t).$$

By applying the continuation theorem and some analytic techniques, sufficient conditions for the existence of periodic solutions are established. It is interesting that the equations not only depend on the constant c but are also dependent on the deviating arguments $\tau_i, i=(1,\dots,k)$.

Розглядаються нейтральні диференціальні рівняння з p-лапласіаном і кратними відхиленнями аргументів:

$$(\varphi_p(x(t)-cx(t-r))^{(n)}(t)))^{(m)} = f(x(t))x'(t)+g(t,x(t),x(t-\tau_1(t)),...,x(t-\tau_k(t)))+e(t).$$

Застосовуючи теорему продовження та певні аналітичні методи, отримуються достатні умови існування періодичних розв'язків. Рівняння залежать не тільки від константи c, але й від аргументів із відхиленнями $\tau_i, i=(1,\dots,k)$.

1. Introduction

In the past years, there has been a considerable interest in the existence of periodic solutions of higher order neutral differential equations because of its background in applied sciences. For example Wang and Lu [2] studied a kind of high-order neutral functional differential equation with distributed delay as follows:

$$(x(t)-cx(t-\sigma))^{(n)} = f(x(t))x'(t)+g(\int_{-r}^{0} x(t+s)d\alpha(s)) = p(t).$$

In 2010, Wang and Zhu [1] further discussed existence of periodic solutions for a fourth-order $p-$Laplacian neutral functional differential equation of the form:

$$(\varphi_p(x(t)-cx(t-\delta))''(t))'' = f(x(t))x'(t)+g(t,x(t-\tau(t,|x|_\infty)))+e(t).$$

Inspired by the above fact and other great articles, see [3, 5] and [6], in this paper, we aim at studying the existence of periodic solutions for the following higher-order $p-$Laplacian neutral differential equation with multiple deviating arguments:

$$(\varphi_p(x(t)-cx(t-r))^{(n)}(t))^{(m)} = f(x(t))x'(t)+g(t,x(t),x(t-\tau_1(t)),...,x(t-\tau_k(t)))+e(t), \tag{1.1}$$

where $p\geq 2$ is a fixed real number. The conjugate exponent of p is denoted by q, i.e $\frac{1}{p}+\frac{1}{q}=1$. Let $\varphi_p:\mathbb{R}\to\mathbb{R}$ be the mapping defined by $\varphi_p(s)=|s|^{p-2}s$ for $s\neq 0$, and $\varphi_p(0)=0$, $f\in C(\mathbb{R},\mathbb{R})$, $g\in C(\mathbb{R}^{k+2},\mathbb{R})$ with $g(t+T,u_0,u_1,...,u_k)=g(t,u_0,u_1,...,u_k)$,

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$\forall(t, u_0, u_1, ..., u_k) \in \mathbb{R}^{k+2}$, e is continuous T-periodic functions defined on $\mathbb{R}$, $\tau_i \in C^1(\mathbb{R}, \mathbb{R})(i = 1, 2, ..., k)$ with $\tau_i(t + T) = \tau_i(t)$, T is positive constant, c is a constant and $|c| \neq 1$, n, m are a positive integer.

Therefore, in this paper, based on the Mawhin continuation theorem and some analysis skill, without assumption of $\int_0^T e(t)dt = 0$, some new sufficient conditions for the existence of T-periodic solution of $p-$Laplacian equation (1.1) will be established. The rest of this paper is organized as follows. Section 2 is devoted to introducing some definitions and recalling some preliminary results that will be extensively used. The existence results will be obtained in Section 3. Finally, an example is given to illustrate the effectiveness of our result in Section 4. Our results are different from those of bibliographies listed in the previous texts. Therefore, we improve the corresponding results in literature to the multiple deviating arguments case.

2. Preliminaries

For convenience, define $C_T = \{x | x \in C(\mathbb{R}, \mathbb{R}), x(t + T) = x(t)\}$ with the norm $|x|_0 = \max_{t\in[0,T]} |x(t)|$, and $C_T^1 = \{x | x \in C^1(\mathbb{R}, \mathbb{R}), x(t + T) = x(t)\}$ with the norm $\|x\| = \max_{t\in[0,T]}\{|x|_0, |x'|_0\}$. We also define a linear operator A as follows:

$$A : C_T \to C_T, \qquad (Ax)(t) = x(t) - cx(t - r).$$

Lemma 2.1 ([9]). *If $|c| \neq 1$, then A has continuous bounded inverse A^{-1} on C_T, which satisfies*

$$(1)\ \ (A^{-1}x)(t) = \begin{cases} x(t) + \sum_{j=1}^{\infty} c^j x(t - jr), & for |c| < 1,\ \forall x \in C_T; \\ -\dfrac{x(t + r)}{c} - \sum_{j=1}^{\infty} \dfrac{x(t + (j + 1)r)}{c^{j+1}}, & for |c| > 1,\ \forall x \in C_T. \end{cases}$$

$$(2)\ \ \|(A^{-1}x)\| \leq \frac{|x|_0}{|1 - |c||}, \qquad \forall x \in C_T;$$

$$(3)\ \ \int_0^T |(A^{-1}x)(t)|dt \leq \frac{1}{|1 - |c||} \int_0^T |x(t)|dt, \qquad \forall x \in C_T.$$

Lemma 2.2. *Let $k > 0, T > 0$ be two constants, $s \in C_T(\mathbb{R}, \mathbb{R})$, $\tau_i \in C_T^1(\mathbb{R}, \mathbb{R})$ and $|\tau_i'|_0 < 1$. Then*

$$\int_0^T |s(t - \tau_i(t))|^k dt \leq \delta_i \int_0^T |s(t)|^k dt;$$

where $\delta_i = \dfrac{1}{1 - |\tau_i'|_0}$, $|\tau_i'|_0 = \max_{t\in[0,T]} |\tau_i'(t)|$.

Proof. It is easy to see that

$$\int_0^T |s(t - \tau_i(t))|^k dt = \int_0^T |s(t - \tau_i(t))|^k d(t - \tau_i(t)) + \int_0^T \tau_i'(t)|s(t - \tau_i(t))|^k dt,$$

i.e.,

$$(1 - |\tau_i'|_0) \int_0^T |s(t - \tau_i(t))|^k dt \leq \int_0^T |s(t)|^k dt$$

and thus

$$\int_0^T |s(t - \tau_i(t)|^k dt \leq \frac{1}{1 - |\tau_i'|_0} \int_0^T |s(t)|^k dt.$$

This completes the proof. ∎

Lemma 2.3 (Borsuk [15]). *$\Omega \subset \mathbb{R}^n$ is an open bounded set, and symmetric with respect to $0 \in \Omega$. If $f \in C(\overline{\Omega}, \mathbb{R}^n)$ and $f(x) \neq \mu f(-x), \forall x \in \partial\Omega, \forall \mu \in [0,1]$, then $deg(f, \Omega, 0)$ is an odd number.*

Now, we recall Mawhin's continuation theorem which our study is based upon.

Let X and Y be real Banach spaces and $L : D(L) \subset X \to Y$ be a Fredholm operator with index zero. Here $D(L)$ denotes the domain of L. This means that ImL is closed in Y and $\dim KerL = \dim(Y/ImL) < +\infty$. Consider the supplementary subspaces X_1 and Y_1 and such that $X = KerL \oplus X_1$ and $Y = ImL \oplus Y_1$ and let $P : X \to KerL$ and $Q : Y \to Y_1$ be natural projections. Clearly, $KerL \cap (D(L) \cap X_1) = \{0\}$, thus the restriction $L_p := L|_{D(L)\cap X_1}$ is invertible. Denote the inverse of L_p by K.

Now, let Ω be an open bounded subset of X with $D(L) \cap \Omega \neq \emptyset$, a map $N : \overline{\Omega} \to Y$ is said to be L-compact on $\overline{\Omega}$, if $QN(\overline{\Omega})$ is bounded and the operator $K(I - Q)N : \overline{\Omega} \to Y$ is compact.

Lemma 2.4 (Mawhin [13]). *Suppose that X and Y are two Banach spaces, and $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero. Furthemore, $\Omega \subset X$ is an open bounded set, and $N : \overline{\Omega} \to Y$ is L-compact on $\overline{\Omega}$. If all of the following conditions hold:*

(1) $Lx \neq \lambda Nx, \forall x \in \partial\Omega \cap D(L), \lambda \in]0, 1[$;

(2) $Nx \notin ImL, \forall x \in \partial\Omega \cap KerL$; *and*

(3) $deg\{JQN, \Omega \cap KerL, 0\} \neq 0$, *where $J : ImQ \to KerL$ is an isomorphism.*

Then the equation $Lx = Nx$ has at least one solution on $\overline{\Omega} \cap D(L)$.

In order to use Mawhin's continuation theorem to study the existence of T-periodic solution for Eq (1.1) , we rewrite Eq (1.1) in the following system

$$\begin{cases} (Ax_1)^{(n)}(t) = \varphi_q(x_2(t)), \\ x_2^{(m)}(t) = f(x_1(t))x_1'(t) + g(t, x_1(t), x_1(t - \tau_1(t)), ..., x_1(t - \tau_k(t))) + e(t). \end{cases} \tag{2.2}$$

Where $q \geq 2$ is constant with $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Clearly, if $x(t) = (x_1(t), x_2(t))^\top$ is a T-periodic solution to equation set (2.2), then $x_1(t)$ must be a T-periodic solution to equation (1.1). Thus, in order to prove that Eq (1.1) has a T-periodic solution, it suffices to show that equation set (2.2) has a T-periodic solution

$$X = \{x = (x_1(t), x_2(t))^\top \in C^1(\mathbb{R}, \mathbb{R}^2) : x(t + T) = x(t)\}$$

with the norm $\|x\|_X = max\{\|x_1\|, \|x_2\|\}$, $Y = \{x = (x_1(t), x_2(t))^\top \in C(\mathbb{R}, \mathbb{R}^2) : x(t + T) = x(t)\}$ with the norm $\|x\|_Y = max\{|x_1|_0, |x_2|_0\}$, obviously, X and Y are two Banach spaces. Meanwhile, let

$$L : D(L) \subset X \to Y, \ (Lx)(t) = \begin{pmatrix} (Ax_1)^{(n)}(t) \\ x_2^{(m)}(t) \end{pmatrix}, \tag{2.3}$$

where $D(L) = \{x \in C^{n+m}(\mathbb{R}, \mathbb{R}^2) : x(t + T) = x(t)\}$,

$$N : X \to Y,$$

$$[Nx](t) = \begin{pmatrix} \varphi_q(x_2(t)) \\ f(x_1(t))x_1'(t) + g(t, x_1(t), x_1(t - \tau_1(t)), ..., x_1(t - \tau_k(t))) + e(t) \end{pmatrix}. \tag{2.4}$$

It is easy to see that equation set (2.2) can be converted to the abstract equation $Lx = Nx$. Moreover, from the definition of L, we see that $KerL = \mathbb{R}^2$, $ImL = \{y : y \in Y, \int_0^T y(s)ds = 0\}$. So L is a Fredholm operator with index zero.

Let projections $P : X \to KerL$ and $Q : Y \to ImQ$ be defined by

$$Px = \begin{pmatrix} (Ax_1)(0) \\ x_2(0) \end{pmatrix}, \qquad Qy = \frac{1}{T}\int_0^T \begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix} ds,$$

Obviously $KerL = ImQ = \mathbb{R}^2$. Denote the inverse of $L|_{KerP\cap D(L)}$ by L_p^{-1} then

$$\left[L_p^{-1}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right](t) = \begin{pmatrix} (A^{-1}Gy_1)(t) \\ (Gy_2)(t) \end{pmatrix} \tag{2.5}$$

where

$$(Gy_1)(t) = \sum_{i=1}^{n-1} \frac{1}{i!}(Ax_1)^{(i)}(0)t^i + \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}y_1(s)ds,$$
$$(Gy_2)(t) = \sum_{i=1}^{m-1} \frac{1}{i!}x_2^{(i)}(0)t^i + \frac{1}{(m-1)!}\int_0^t (t-s)^{m-1}y_2(s)ds,$$

and $(Ax_1^{(i)})(0)$ $(i = 1, 2..., n-1)$ are defined by $E_1 Z = B$, where

$$E_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ c_1 & 1 & 0 & \cdots & 0 & 0 \\ c_2 & c_1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-3} & c_{n-4} & c_{n-5} & \cdots & 1 & 0 \\ c_{n-2} & c_{n-3} & c_{n-4} & \cdots & c_1 & 1 \end{pmatrix}_{(n-1)\times(n-1)}$$
$$Z = ((Ax_1)^{(n-1)}(0), (Ax_1)^{(n-2)}(0), \cdots, (Ax_1)''(0), (Ax_1)'(0))^\top,$$
$$B = (b_1, b_2, \cdots, b_{n-2}, b_{n-1})^\top,$$
$$b_i = -\frac{1}{i!T}\int_0^T (T-s)^i y_1(s)ds, \qquad i = 1, 2, \cdots, n-1,$$

and

$$c_j = \frac{T^j}{(j+1)!} \qquad j = 1, 2, \cdots, n-2,$$

$x_2^i(0)$, $i = 1, 2, \cdots, m-1$, are determined by the equation $E_2 W = F$, where

$$E_2 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ c_1 & 1 & 0 & \cdots & 0 & 0 \\ c_2 & c_1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{m-3} & c_{m-4} & c_{m-5} & \cdots & 1 & 0 \\ c_{m-2} & c_{m-3} & c_{m-4} & \cdots & c_1 & 1 \end{pmatrix}_{(m-1)\times(m-1)}$$
$$W = (x_2^{(m-1)}(0), x_2^{(m-2)}(0), \cdots, x_2''(0), x_2'(0))^\top,$$
$$F = (d_1, d_2, \cdots, d_{m-2}, d_{m-1})^\top,$$
$$d_i = -\frac{1}{i!T}\int_0^T (T-s)^i y_2(s)ds, \qquad i = 1, 2, \cdots, m-1,$$

and $c_j = \dfrac{T^j}{(j+1)!}$, $j = 1, 2, \cdots, m-2$.

From (2.4)and (2.5), it isn't hard to find that N is L-compact on $\overline{\Omega}$, where Ω is an arbitrary open bounded subset of X.

For the sake of convenience, we list the following assumptions which will be used by us in studing the existence of $T-$ periodic solution to Eq (1.1).

(H_1) There is a constant $d > 0$ such that:
  (1) $g(t, u_0, u_1, ..., u_k) > |e|_0, \forall (t, u_0, u_1, ..., u_k) \in [0, T] \times \mathbb{R}^{k+1}$ with $u_i > d$ $(i = 0, 1, ..., k)$.

(2) $g(t, u_0, u_1, ..., u_k) < -|e|_0, \forall (t, u_0, u_1, ..., u_k) \in [0, T] \times \mathbb{R}^{k+1}$ with $u_i < -d$ $(i = 0, 1, ..., k)$.

(H_2) $|g(t, u_0, u_1, ..., u_k)| \leq \sum_{i=0}^{k} \alpha_i |u_i|^{p-1} + \beta$, where $\alpha_i (i = 0, ..., k), \beta$ are positive constants.

(H_3) There exist positive constants l, b $|f(x)| \leq l|x|^{p-2} + b$.

3. Main results

Lemma 3.1. *Suppose that* $[H_1]$ *holds, if* $x \in D(L)$ *is an arbitrary solution of the equation* $Lx = \lambda Nx, \lambda \in]0, 1[$*,where* L *and* N *are defined by (2.3) and (2.4), respectively, then there must be a point* $t^* \in [0, T]$ *such that*

$$|x_1(t^*)| \leq d. \tag{3.6}$$

Proof. Suppose $x \in D(L)$ is an arbitrary solution of the equation $Lx = \lambda Nx$, for some $\lambda \in]0, 1[$, then

$$\begin{cases} (Ax_1)^{(n)}(t) = \lambda \varphi_q(x_2)(t) = \lambda |x_2(t)|^{q-2} x_2(t), \\ x_2^{(m)}(t) = \lambda f(x_1(t)) x_1'(t) + \lambda g(t, x_1(t), x_1(t - \tau_1(t)), ..., x_1(t - \tau_k(t))) + \lambda e(t). \end{cases} \tag{3.7}$$

From the first equation of (3.7), we have $x_2(t) = \varphi_p(\frac{1}{\lambda}(Ax_1)^{(n)})(t)$, and then by substituting it into the second equation of (3.7),we have

$$(\varphi_p(Ax_1)^{(n)}(t))^{(m)} = \lambda^p f(x_1(t)) x_1'(t) + \lambda^p g(t, x_1(t), x_1(t-\tau_1(t)), ..., x_1(t-\tau_k(t))) + \lambda^p e(t). \tag{3.8}$$

Integrating both sides of Eq. (3.8) on the interval $[0, T]$, we have

$$\int_0^T g(t, x_1(t), x_1(t - \tau_1(t)), ..., x_1(t - \tau_k(t))) + \int_0^T e(t) = 0.$$

By the integral mean value theorem, there is a constant $t_0 \in [0, T]$ such that

$$g(t, x_1(t_0), x_1(t_0 - \tau_1(t_0)), ..., x_1(t_0 - \tau_k(t_0))) = -\frac{1}{T} \int_0^T e(t) dt. \tag{3.9}$$

Case 1. If $|x_1(t_0)| \leq d$, then taking $t^* = t_0$ such that $|x_1(t^*)| \leq d$.

Case 2. If $|x_1(t_0)| > d$, in this case we need to prove that there exist $\xi \in \mathbb{R}$ such that $|x_1(\xi)| \leq d$. By (3.9), we can get

$$g(t, x_1(t_0), x_1(t_0 - \tau_1(t_0)), ..., x_1(t_0 - \tau_k(t_0))) = -\frac{1}{T} \int_0^T e(t) dt \leq |e|_0.$$

From assumption $(H_1)(1)$, we see that there exist $r \in \{1, 2, ..., k\}$ such that $x_1(t_0 - \tau_r(t_0)) \leq d$. On the other hand we have

$$g(t, x_1(t_0), x_1(t_0 - \tau_1(t_0)), ..., x_1(t_0 - \tau_k(t_0))) = -\frac{1}{T} \int_0^T e(t) dt \geq -|e|_0.$$

From $(H_1)(2)$ there exist $l \in \{1, 2, ..., k\}$ such that $x_1(t_0 - \tau_l(t_0)) \geq -d$. In this case we consider the following two other cases

- If $l = r$,we get $|x_1(t_0 - \tau_l(t_0))| \leq d$, then taking $\xi = x_1(t_0 - \tau_l(t_0))$ such that $|x_1(\xi)| \leq d$.
- If $l \neq r$ we consider three other cases:
  - If $x_1(t_0 - \tau_l(t_0)) \leq x_1(t_0 - \tau_r(t_0))$, which yields $|x_1(t_0 - \tau_l(t_0))| \leq d$ and $|x_1(t_0 - \tau_r(t_0))| \leq d$, let $\xi = x_1(t_0 - \tau_l(t_0))$ or $\xi = x_1(t_0 - \tau_r(t_0))$ obviously $|x_1(\xi)| \leq d$.
  - If $x_1(t_0 - \tau_r(t_0)) \leq x_1(t_0 - \tau_l(t_0))$ and one of the following assumptions holds: $x_1(t_0 - \tau_r(t_0)) \geq -d$ or $x_1(t_0 - \tau_l(t_0)) \leq d$, we assume $\xi = x_1(t_0 - \tau_l(t_0))$ or $\xi = x_1(t_0 - \tau_r(t_0))$, we can obtain $|x_1(\xi)| \leq d$.

– If $x_1(t_0-\tau_r(t_0)) \le x_1(t_0-\tau_l(t_0))$, $x_1(t_0-\tau_r(t_0)) < -d$ and $x_1(t_0-\tau_l(t_0)) > d$. By the intermediate value theorem there exist t_1 such that $x_1(t_1) = 0$, then taking $\xi = t_1$, we have $|x_1(\xi)| \le d$.

Let $k' = \left[\frac{\xi}{T}\right]$, where $\left[\frac{\xi}{T}\right]$ is integer part of the number $\frac{\xi}{T}$, then taking $t^* = \xi - k'T$. Furthermore $|x_1(t^*)| \le d$ with $t^* \in [0, T]$. □

Theorem 3.2. *Assume that* $|\tau_i'|_0 < 1,\ (i = 1, \cdots, k)$ *and assumption* $[H_1] - [H_3]$ *hold. Suppose one of the following conditions is satisfied:*

(1) $p > 2$ *and* $\left[\frac{2l + \left(\alpha_0 + \sum_{i=1}^k \alpha_i\delta_i\right) T}{2^{p+1}|1-|c||^{p-1}}\right] T^p \left(\frac{T}{2\pi}\right)^{(n-1)(p-1)+(m-2)} < 1$

(2) $p = 2$ *and*

$$\left[\frac{2l + \left(\alpha_0 + \sum_{i=1}^k \alpha_i\delta_i\right) T}{2^{p+1}|1-|c||^{p-1}}\right] T^p \left(\frac{T}{2\pi}\right)^{(n-1)(p-1)+(m-2)} + \frac{bT^2(\frac{T}{2\pi})^{n+m-3}}{4|1-|c||} < 1,$$

where δ_i *is defined in Lemma 2.2. Then Eq (1.1) has at one least one* $T-$*periodic solution.*

Proof. Let $\Omega_1 = \{x \in X : Lx = \lambda Nx, \lambda \in]0, 1[\}$ if $x(t) = (x_1(t), x_2(t))^\top \in \Omega_1$, then from (2.3) and (2.4), we have

$$\begin{cases} (Ax_1)^{(n)}(t) = \lambda\varphi_q(x_2)(t) = \lambda|x_2(t)|^{q-2}x_2(t), \\ x_2^{(m)}(t) = \lambda f(x_1(t))x_1'(t) + \lambda g(t, x_1(t), x_1(t-\tau_1(t)), ..., x_1(t-\tau_k(t))) + \lambda e(t). \end{cases} \tag{3.10}$$

From Lemma 3.1 we have

$$|x_1(t)| = |x_1(t^*) + \int_{t^*}^t x_1'(s)ds| \le d + \int_{t^*}^t |x_1'(s)|ds, \quad t \in [t^*, t^* + T],$$

and

$$|x_1(t)| = |x_1(t-T)| = |x_1(t^*) - \int_{t-T}^{t^*} x_1'(s)ds| \le d + \int_{t^*-T}^{t^*} |x_1'(s)|ds, \quad t \in [t^*, t^* + T].$$

Combining the above two inequalities, we obtain

$$\begin{aligned} |x_1|_0 &= \max_{t\in[0,T]} |x_1(t)| = \max_{t\in[t^*,t^*+T]} |x_1(t)| \\ &\le \max_{t\in[t^*,t^*+T]} \left\{ d + \frac{1}{2}\left(\int_{t^*}^t |x_1'(s)|ds + \int_{t-T}^{t^*} |x_1'(s)|ds\right)\right\} \\ &\le d + \frac{1}{2}\int_0^T |x_1'(s)|ds. \end{aligned} \tag{3.11}$$

From Lemma 2.1 and the first equation of 3.10, we have

$$\begin{aligned} |x_1^{(n)}|_0 &= \max_{t\in[0,T]} |A^{-1}Ax_1^{(n)}(t)| \\ &\le \frac{\max_{t\in[0,T]} |(Ax_1)^{(n)}(t)|}{|1-|c||} \\ &\le \frac{\varphi_q(|x_2|_0)}{|1-|c||}. \end{aligned} \tag{3.12}$$

On the other hand, from $x_2^{(m-2)}(0) = x_2^{(m-2)}(T)$, there exists a point $t_1 \in [0, T]$ such that $x_2^{(m-1)}(t_1) = 0$, which together with the integration of the second equation of 3.10 on the interval $[0, T]$ yields

$$\begin{aligned}2|x_2^{(m-1)}(t)| &\leq 2\left(x_2^{(m-1)}(t_1) + \frac{1}{2}\int_0^T |x_2^{(m)}(t)|dt\right) \\ &\leq \lambda \int_0^T |f(x_1(t))x_1'(t) + g(t, x_1(t), x_1(t-\tau_1(t)), ..., x_1(t-\tau_k(t))) + e(t)|dt \\ &\leq \int_0^T |f(x_1(t))||x_1'(t)|dt + \int_0^T |g(t, x_1(t), x_1(t-\tau_1(t)), ..., x_1(t-\tau_k(t)))|dt \\ &\quad + \int_0^T |e(t)|dt \\ &\leq l\int_0^T |x_1(t))|^{p-2}|x_1'(t)|dt + b\int_0^T |x_1'(t)|dt + \int_0^T \alpha_0|x_1(t)|^{p-1} \\ &\quad + \sum_{i=1}^k \alpha_i|x_1(t-\tau_i(t))|^{p-1}dt + T(|e|_0 + \beta).\end{aligned} \tag{3.13}$$

By, Lemma 2.2 and (3.13) we obtain

$$\begin{aligned}2|x_2^{(m-1)}(t)| \leq l|x_1|_0^{p-2}\int_0^T |x_1'(t)|dt + b\int_0^T |x_1'(t)|dt \\ + (\alpha_0 + \sum_{i=1}^k \alpha_i\delta_i)T|x_1|_0^{p-1} + T(|e|_0 + \beta).\end{aligned} \tag{3.14}$$

Substituting (3.11) into (3.14), we have

$$\begin{aligned}2|x_2^{(m-1)}(t)| &\leq l\left(d + \frac{1}{2}\int_0^T |x_1'(t)|dt\right)^{p-2}\int_0^T |x_1'(t)|dt + b\int_0^T |x_1'(t)|dt \\ &\quad + (\alpha_0 + \sum_{i=1}^k \alpha_i\delta_i)T\left(d + \frac{1}{2}\int_0^T |x_1'(t)|dt\right)^{p-1} + T(|e|_0 + \beta).\end{aligned} \tag{3.15}$$

Then we can get

$$\begin{aligned}2|x_2^{(m-1)}(t)| &\leq 2^{2-p}l\left(1 + \frac{2d}{\int_0^T |x_1'(t)|dt}\right)^{p-2}\left(\int_0^T |x_1'(t)|dt\right)^{p-1} + b\int_0^T |x_1'(t)|dt \\ &\quad + (\alpha_0 + \sum_{i=1}^k \alpha_i\delta_i)T2^{1-p}\left(1 + \frac{2d}{\int_0^T |x_1'(t)|dt}\right)^{p-1}\left(\int_0^T |x_1'(t)|dt\right)^{p-1} \\ &\quad + T(|e|_0 + \beta).\end{aligned} \tag{3.16}$$

By classical elementary inequalities, we see that there exist a $\theta > 0$ which is dependent on p, such that

$$(1 + x)^p \leq 1 + (1 + p)x. \qquad x \in [0, \theta]. \tag{3.17}$$

If $\int_0^T |x_1'(t)|dt \geq \frac{2d}{\theta}$, then it follows from (3.16) and (3.17) that

$$\begin{aligned} 2|x_2^{(m-1)}(t)| \le 2^{2-p} l \left(1+\frac{2d(p-1)}{\int_0^T |x_1'(t)|dt}\right)\left(\int_0^T |x_1'(t)|dt\right)^{p-1} + b\int_0^T |x_1'(t)|dt \\ + (\alpha_0+\sum_{i=1}^k \alpha_i\delta_i)T2^{1-p}\left(1+\frac{2dp}{\int_0^T |x_1'(t)|dt}\right)\left(\int_0^T |x_1'(t)|dt\right)^{p-1} \\ + T(|e|_0+\beta). \end{aligned} \tag{3.18}$$

From the Wirtinger inequality (see [18], Lemma 2.4), we have

$$\begin{aligned} \int_0^T |x_1'(t)|dt &\le T^{\frac{1}{2}}\left(\int_0^T |x_1'(t)|^2 dt\right)^{\frac{1}{2}} \\ &\le T^{\frac{1}{2}}\left(\frac{T}{2\pi}\right)^{n-1}\left(\int_0^T |x_1^{(n)}(t)|^2 dt\right)^{\frac{1}{2}} \\ &\le T\left(\frac{T}{2\pi}\right)^{n-1} |x_1^{(n)}|_0. \end{aligned} \tag{3.19}$$

Substituting (3.19) into (3.18), we can see that

$$\begin{aligned} 2|x_2^{(m-1)}(t)| \le \left[2^{2-p}l + 2^{1-p}(\alpha_0+\sum_{i=1}^k \alpha_i\delta_i)T\right] T^{p-1}\left(\frac{T}{2\pi}\right)^{(n-1)(p-1)} |x_1^{(n)}|_0^{p-1} \\ + \left[2^{3-p}ld(p-1) + 2^{2-p}(\alpha_0+\sum_{i=1}^k \alpha_i\delta_i)Tdp\right] T^{p-2}\left(\frac{T}{2\pi}\right)^{(n-1)(p-2)} |x_1^{(n)}|_0^{p-2} \\ + bT\left(\frac{T}{2\pi}\right)^{n-1} |x_1^{(n)}|_0 + T(|e|_0+\beta). \end{aligned} \tag{3.20}$$

Combination of (3.20) and (3.12) implies

$$\begin{aligned} 2|x_2^{(m-1)}(t)| \le \left[2^{2-p}l + 2^{1-p}(\alpha_0+\sum_{i=1}^k \alpha_i\delta_i)T\right] T^{p-1}\left(\frac{T}{2\pi}\right)^{(n-1)(p-1)} \left(\frac{\varphi_q|x_2|_0}{|1-|c||}\right)^{p-1} \\ + \left[2^{3-p}ld(p-1) + 2^{2-p}(\alpha_0+\sum_{i=1}^k \alpha_i\delta_i)Tdp\right] T^{p-2}\left(\frac{T}{2\pi}\right)^{(n-1)(p-2)} \left(\frac{\varphi_q|x_2|_0}{|1-|c||}\right)^{p-2} \\ + bT\left(\frac{T}{2\pi}\right)^{n-1} \left(\frac{\varphi_q|x_2|_0}{|1-|c||}\right) + T(|e|_0+\beta). \end{aligned} \tag{3.21}$$

So, we have

$$\begin{aligned} 2|x_2^{(m-1)}(t)| \le \left[2^{2-p}l + 2^{1-p}(\alpha_0+\sum_{i=1}^k \alpha_i\delta_i)T\right] T^{p-1}\left(\frac{T}{2\pi}\right)^{(n-1)(p-1)} \left(\frac{|x_2|_0}{|1-|c||^{p-1}}\right) \\ + \left[2^{3-p}ld(p-1) + 2^{2-p}(\alpha_0+\sum_{i=1}^k \alpha_i\delta_i)Tdp\right] T^{p-2}\left(\frac{T}{2\pi}\right)^{(n-1)(p-2)} \left(\frac{|x_2|_0^{2-q}}{|1-|c||^{p-2}}\right) \\ + bT\left(\frac{T}{2\pi}\right)^{n-1} \left(\frac{|x_2|_0^{q-1}}{|1-|c||}\right) + T(|e|_0+\beta). \end{aligned} \tag{3.22}$$

Since $\int_0^T (\varphi_q(x_2(t)))dt = \int_0^T (A(x_1(t)))^{(n)}dt = 0$, there exists a point $t_2 \in [0,T]$ such that $x_2(t_2) = 0$, hence we can get

$$|x_2(t)| = |x_2(t_2) + \int_{t_2}^{t} x_2'(s)ds| \le \int_{t_2}^{t} |x_2'(s)|ds, \quad t \in [t_2, t_2+T],$$

and

$$|x_2(t)| = |x_2(t-T)| = |x_2(t_2) - \int_{t-T}^{t_2} x_2'(s)ds| \le \int_{t-T}^{t_2} |x_2'(s)|ds, \quad t \in [t_2, t_2+T].$$

Combining the above two inequalities, we obtain

$$\begin{aligned}|x_2|_0 = \max_{t\in[0,T]} |x_2(t)| = \max_{t\in[t_2,t_2+T]} |x_2(t)| &\le \max_{t\in[t_2,t_2+T]} \left\{ \frac{1}{2} \left(\int_{t_2}^{t} |x_2'(s)|ds + \int_{t-T}^{t_2} |x_2'(s)|ds \right) \right\} \\ &\le \frac{1}{2} \int_0^T |x_2'(s)|ds.\end{aligned}$$

From the Wiritinger inequality, we have

$$\begin{aligned}|x_2|_0 &\le \frac{1}{2} \int_0^T |x_2'(t)|dt \\ &\le T^{\frac{1}{2}} \left(\int_0^T |x_2'(t)|^2 dt \right)^{\frac{1}{2}} \\ &\le T^{\frac{1}{2}} \left(\frac{T}{2\pi} \right)^{m-2} \left(\int_0^T |x_2^{(m-1)}(t)|^2 dt \right)^{\frac{1}{2}} \\ &\le \frac{T}{2} \left(\frac{T}{2\pi} \right)^{m-2} |x_2^{(m-1)}|_0.\end{aligned} \tag{3.23}$$

Substituting (3.23) into (3.22), we can see that

$$\begin{aligned}|x_2|_0 &\le \frac{T}{2} \left(\frac{T}{2\pi} \right)^{m-2} |x_2^{(m-1)}|_0 \\ &\le \left[\frac{2l + \left(\alpha_0 + \sum_{i=1}^k \alpha_i \delta_i \right) T}{2^{p+1} |1-|c||^{p-1}} \right] T^p \left(\frac{T}{2\pi} \right)^{(n-1)(p-1)+(m-2)} |x_2|_0 \\ &+ \left[\frac{2ld(p-1) + \left(\alpha_0 + \sum_{i=1}^k \alpha_i \delta_i \right) Tdp}{2^p |1-|c||^{p-2}} \right] T^{p-1} \left(\frac{T}{2\pi} \right)^{(n-1)(p-2)+(m-2)} |x_2|_0^{2-q} \\ &+ b\frac{T^2}{4} \left(\frac{T}{2\pi} \right)^{(n+m-3)} \left(\frac{|x_2|_0^{q-1}}{|1-|c||} \right) + \frac{T^2}{4} \left(\frac{T}{2\pi} \right)^{m-2} (|e|_0 + \beta).\end{aligned} \tag{3.24}$$

Case 1: If $p > 2$, we can get $1 < q < 2$. It follows from

$$\left[\frac{2l + \left(\alpha_0 + \sum_{i=1}^k \alpha_i \delta_i \right) T}{2^{p+1} |1-|c||^{p-1}} \right] T^p \left(\frac{T}{2\pi} \right)^{(n-1)(p+1)+(m-2)} < 1$$

that there exists a positive constant M_{21} such that

$$|x_2|_0 \le M_{21}. \tag{3.25}$$

Case 2: If $p = 2$, we can get $q = 2$. It follows from

$$\left[\frac{2l + \left(\alpha_0 + \sum_{i=1}^{k} \alpha_i \delta_i\right) T}{2^{p+1}|1-|c||^{p-1}}\right] T^p \left(\frac{T}{2\pi}\right)^{(n-1)(p+1)+(m-2)} + \frac{bT^2(\frac{T}{2\pi})^{n+m-3}}{4|1-|c||} < 1$$

that there exists a positive constant M_{21} such that

$$|x_2|_0 \leq M_{21}.$$

On the other hand From(3.12), we have

$$|x_1^{(n)}|_0 \leq \frac{\varphi_q(|x_2|_0)}{|1-|c||} \leq \frac{M_{21}^{q-1}}{|1-|c||} := M_1'. \tag{3.26}$$

Since $x_1(0) = x_1(T)$, there exists a point $t_3 \in [0, T]$ such that $x_1'(t_3) = 0$. From the Wirtinger inequality, we can get

$$\begin{aligned} |x_1'|_0 &\leq \frac{1}{2}\int_0^T |x_1''(t)|dt \\ &\leq T^{\frac{1}{2}} \left(\int_0^T |x_1''(t)|^2 dt\right)^{\frac{1}{2}} \\ &\leq \frac{T}{2}\left(\frac{T}{2\pi}\right)^{n-2} |x_1^{(n)}|_0 \\ &\leq \frac{T}{2}\left(\frac{T}{2\pi}\right)^{n-2} M_1' := M_{11}. \end{aligned} \tag{3.27}$$

Which together with (3.11) yields

$$|x_1|_0 \leq d + \frac{1}{2}\int_0^T |x_1'(s)|ds \leq T + \frac{TM_{11}}{2} := M_{12}. \tag{3.28}$$

Let $M_f = \max_{|u|\leq M_{12}} |f(u)|$, $M_g = \max_{t\in[0,T],|u_0|\leq M_{12},...,|u_k|\leq M_{12}} |g(t, u_0, ..., u_k)|$ Hence, from (3.27) and (3.25)

$$\begin{aligned} |x_2^{(m-1)}|_0 &\leq \frac{1}{2}\max\left|\int_0^T x_2^{(m)}(t)dt\right| \\ &\leq \frac{\lambda}{2}\int_0^T |f(x_1(t))x_1'(t) + g(t, x_1(t), x_1(t-\tau_1(t)), ..., x_1(t-\tau_k(t))) + e(t)|dt \\ &\leq \int_0^T |f(x_1(t))||x_1'(t)|dt + \int_0^T |g(t, x_1(t), x_1(t-\tau_1(t)), ..., x_1(t-\tau_k(t)))|dt \\ &\quad + \int_0^T |e(t)|dt \\ &\leq M_f T|x_1'|_0 + T(M_g + |e|_0) \\ &\leq M_f T M_{11} + T(M_g + |e|_0) := \overline{M_0}. \end{aligned} \tag{3.29}$$

Since $x_2(0) = x_2(T)$, there exists a point $t_4 \in [0, T]$ such that $x_2'(t_4) = 0$. From the Wirtinger inequality, we can get

$$\begin{aligned} |x_2'|_0 &\le \frac{1}{2}\int_0^T |x_2''(t)|dt \\ &\le T^{\frac{1}{2}}\left(\int_0^T |x_2''(t)|^2 dt\right)^{\frac{1}{2}} \\ &\le \frac{T}{2}\left(\frac{T}{2\pi}\right)^{m-3} |x_1^{(m-1)}|_0 \\ &\le \frac{T}{2}\left(\frac{T}{2\pi}\right)^{m-3} \overline{M_0} := M_{22}. \end{aligned} \tag{3.30}$$

If $\int_0^T |x_1'(t)|dt \le \frac{2d}{\theta}$, then it follows from (3.11)that $|x_1|_0 \le (1+\dfrac{1}{\theta})d$, which together with (3.14) and (3.23) implies that there exists positive constant such that $|x_2|_0 \le M_{21}$. This case can be treated similar.

Let $\Omega_2 = \{x | x \in KerL, QNx = 0\}$. If $x \in \Omega_2$ then $x \in \mathbb{R}^2$ is a constant vector with

$$\begin{cases} |x_2|^{q-2}x_2 = 0, \\ \dfrac{1}{T}\displaystyle\int_0^T [f(x_1(t))x_1'(t) + g(t, x_1(t), x_1(t-\tau_1(t)), ..., x_1(t-\tau_k(t))) + e(t)]dt = 0. \end{cases} \tag{3.31}$$

By the first formula of (3.31) , we have $x_2 = 0$. This together with the second equation of (3.31) yields

$$\frac{1}{T}\int_0^T [g(t, x_1, x_1, ..., x_1) + e(t)]dt = 0.$$

In view of (H_1), we see that $|x_1| \le d$. Now, Let

$$M_1 = \max\{M_{11}, M_{12}\}, \qquad M_2 = \max\{M_{21}, M_{22}\}.$$

Then $\|x_1\| \le M_1, \|x_2\| \le M_2$. Taking $\Omega = \{x | x = (x_1, x_2)^\top \in X, \|x_1\| < M_1 + d, \|x_2\| < M_2 + d\}$, we get $\Omega_1 \cup \Omega_2 \subset \Omega$. So from (3.25) and (3.28), it is easy to see that conditions (1) and (2) of Lemma 2.4 are satisfied.

Next, we verify the condition (3) of Lemma 2.4 . To do this, we define the isomorphism $J : ImQ \to KerL$, $J(x_1, x_2)^T = (x_1, x_2)^\top$. Then

$$JQN(x) = \begin{pmatrix} \varphi_q(x_2) \\ \dfrac{1}{T}\displaystyle\int_0^T [g(t, x_1, x_1, ..., x_1) + e(t)]dt. \end{pmatrix}, \qquad x \in \overline{KerL \cap \Omega}.$$

By Lemma 2.3, we need to prove that

$$JQN(x) \ne \mu(JQN(-x)),\ \forall x \in \partial\Omega \bigcap KerL\ ,\ \mu \in [0, 1]$$

Case 1. If $x = (x_1, x_2)^\top \in \partial\Omega \cap KerL \backslash \{(M_1 + d, 0)^\top, (-M_1 - d, 0)^\top\}$, then $x_2 \ne 0$ which, gives us $\varphi_q(x_2) \ne 0$ and

$$\varphi_q(x_2)\varphi_q(-x_2) < 0,$$

hence, $\forall \mu \in [0, 1]$ we have $JQN(x) \ne \mu(JQN(-x))$.

Case 2. If $x = (M_1 + d, 0)^\top$ or $x = (-M_1 - d, 0)^\top$ then

$$JQN(x) = \begin{pmatrix} 0 \\ \dfrac{1}{T}\displaystyle\int_0^T [g(t, x_1, x_1, ..., x_1) + e(t)]dt \end{pmatrix},$$

which, together with (H_1), yields $\forall\mu \in [0,1],\ JQN(x) \neq \mu(JQN(-x))$.

Thus, the condition (3) of Lemma 2.4 is also satisfied. Therefore, by applying Lemma 2.4, we conclude that the equation $Lx = Nx$ has a solution $x = (x_1, x_2)^\top$ on $\overline{\Omega} \cap D(L)$, so (1.1) has an T-periodic solution $x_1(t)$. □

Example 3.3. In this section, we provide an example to illustrate effectiveness of Theorem 3.2.

Let us consider the following equation

$$(\varphi_4(x(t) - 3(x - \frac{\pi}{6}))^{(2)}(t))^{(3)}$$
$$= f(x(t))x'(t) + g(t, x(t), x(t - \frac{\cos 20\pi t}{90}), x(t - \frac{\sin 20\pi t}{100})) + e(t), \quad (3.32)$$

where $p = 4,\ m = 3,\ n = 2,\ T = \frac{1}{10},\ c = 3,\ f(u) = \frac{1}{5}u^2 + \frac{1}{176},\ l = \frac{1}{5},\ b = \frac{1}{176}$, $\tau_1(t) = \frac{\cos 20\pi t}{90}$, $\tau_2(t) = \frac{\sin 20\pi t}{100}$, $e(t) = \frac{6}{225}\cos 20\pi t + \frac{1}{2}$, $g(t, u, v, w) = sgn(u)u^2(2 + \sin 20\pi t) + \frac{3}{225}\left(sgn(v)v^2 + sgn(w)w^2\right)|\cos 20\pi t|$.

Therefore we can choose $d = 1,\ \alpha_1 = \alpha_2 = 0,014$.

We can easily check the condition $(H_1), (H_2)$ of Theorem 3.2 hold. We can compute that

$$\left[\frac{2l + \left(\alpha_0 + \sum_{i=1}^{k} \alpha_i\delta_i\right) T}{2^{p+1}|1 - |c||^{p-1}}\right] T^p \left(\frac{T}{2\pi}\right)^{(n-1)(p-1)+(m-2)} < 1.$$

By Theorem 3.2, equation (3.32) has at least one $\dfrac{1}{10}$ -periodic solution.

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