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ASYMPTOTICALLY STABLE SOLUTIONS OF A NONLINEAR INTEGRAL EQUATION

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Abstract. The purpose of this paper is to study the existence and asymptotical stability of solutions of some functional integral equations which include a number of classical nonlinear integral equations as special cases. Our investigations will be carried out in the space of continuous and bounded functions on an unbounded interval. We use the technique associated with the measure of noncompactness and a suitable fixed point theorem of Darbo type. The applicability of the results is illustrated by examples showing the difference between our main result and some previous results.

Метою цієї статті є дослідження існування та асимптотичної стійкості розв'язків інтегрально функціональних рівнянь, спеціальними випадками яких є низка класичних нелінійних інтегральних рівнянь. Наші дослідження ведуться в просторах обмежених неперервних функцій на нескінченному інтервалі. Використовується техніка мір некомпактності та теореми про нерухому точку типу Дарбо. Результати ілюструються прикладами, що вказують на відмінності з деякими попередніми результатами.

1. Introduction

The theory of integral equations is an important part of nonlinear analysis and it is frequently applicable in engineering, mechanics, physics, economics, optimization, queing theory and so on. The theory of nonlinear integral equations is rapidly developing with the help of tools of nonlinear functional analysis, topology and fixed-point theory (see [1, 4–8, 10, 13–15, 17, 20, 21, 23, 24, 26–33]).

In this paper, we will investigate the existence of asymptotically stable solutions of the following nonlinear functional integral equation

$$x(t) = f\left(t, (T_1x)(t), (T_2x)(t)\int_0^\infty u\left(t, s, (T_3x)(s)\right) ds\right),\ t \in \mathbb{R}_+, \tag{1.1}$$

where the functions f, u and the operators T_i, $(i = 1, 2, 3)$ are known, while $x = x(t)$ is an unknown function. It is clear that the equation (1.1) includes the equations (1.2)-(1.14) given the following as special cases.

El-Abd [17] proved an existence theorem on monotonic solutions for the nonlinear functional integral equation of convolution type:

$$x(t) = f_1\left(t, \int_0^\infty k(t-s) f_2\left(s, x\left(\phi(s)\right)\right) ds\right),\ t \in \mathbb{R}_+. \tag{1.2}$$

Khosravi et al. [23] studied the existence of solutions for the following class of nonlinear functional integral equations of convolution type:

$$x(t) = f(t, x(t)) + \int_0^\infty k(t-s)(Qx)(s)ds,\ t \in \mathbb{R}_+. \tag{1.3}$$

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Agarwal and O'Regan [1] give the conditions on the existence of the solutions for the nonlinear integral equation:

$$x(t) = \int_0^\infty k(t,s) f(s, x(s))\, ds,\ t \in \mathbb{R}_+. \tag{1.4}$$

Meehan and O'Regan [26–28] discussed the existence of the solutions for the nonlinear integral equations:

$$x(t) = h(t) + \int_0^\infty k(t,s) f(s, x(s))\, ds, \qquad t \in \mathbb{R}_+, \tag{1.5}$$

$$x(t) = h(t) + \mu \int_0^\infty k(t,s) f(s, x(s))\, ds, \qquad t \in \mathbb{R}_+, \tag{1.6}$$

$$x(t) = h(t) + \int_0^\infty k(t,s) [f(x(s)) + g(x(s))] ds, \qquad t \in \mathbb{R}_+. \tag{1.7}$$

Salem [33] examined the existence of solutions of the quadratic integral equation:

$$x(t) = H(t, x(t)) + x(t) \int_0^\infty k(t,s) \varphi(s) (f(x(s)) + g(x(s))) ds, \qquad t \in \mathbb{R}_+. \tag{1.8}$$

Karoui et al. [21] researched the existence of the solution of nonlinear quadratic integral equation:

$$x(t) = a(t) + x(t) \int_0^\infty k(t,s) h(s, x(s))\, ds, \qquad t \in \mathbb{R}_+. \tag{1.9}$$

Banaś et al. [5] studied the existence and asymptotic stability of the solutions for the nonlinear integral equation:

$$x(t) = a(t) + g(t, x(t)) \int_0^\infty K(t,s) h(s, x(s))\, ds, \qquad t \in \mathbb{R}_+. \tag{1.10}$$

Banaś and Poludniak [4] investigated the monotonic solutions for the nonlinear integral equation:

$$x(t) = g(t) + \int_0^\infty u(t, s, x(s))\, ds, \qquad t \in \mathbb{R}_+. \tag{1.11}$$

Banaś and Olszowy [6], Cabellaro et al. [10], Darwish et al. [15], Karoui et al. [21] and Olszowy [29–31] studied the existence of the solutions for the Urysohn integral equation defined on unbounded interval:

$$x(t) = a(t) + f(t, x(t)) \int_0^\infty u(t, s, x(s))\, ds, \qquad t \in \mathbb{R}_+. \tag{1.12}$$

Olszowy [29] investigated the conditions on the existence of solutions of the equation:

$$x(t) = F\left(t, x(t), \int_0^\infty u(t, s, x(s))\, ds\right), \qquad t \in \mathbb{R}_+. \tag{1.13}$$

İlhan and Özdemir [20, 32] examined the nonlinear integral equation of the form:

$$x(t) = (T_1 x)(t) + (T_2 x)(t) \int_0^\infty u(t, s, x(s))\, ds, \qquad t \in \mathbb{R}_+. \tag{1.14}$$

It is worthwhile mentioning that equation (1.1) appears very often in a lot of applications to real world problems. For example, if

$$f(t,x,y) = f_1(t,y), \quad u(t,s,x) = k(t-s) f_2(s,x), \quad (T_1 x)(t) \equiv 0, \quad (T_2 x)(t) \equiv 1$$

and

$$(T_3 x)(t) = x(\phi(t)),$$

then (1.1) becomes the nonlinear integral equation (1.2). The equation (1.2) arises very often in applications of integral equations in many branches of mathematical physics (see [3, 17, 18, 25]). If

$$f(t,x,y) = a(t) + y, \quad u(t,s,x) = K(t,s)h(s,x), \quad (T_1x)(t) \equiv 0, \quad (T_2x)(t) = g(t,x(t))$$

and $(T_3x)(t) = x(t)$, then (1.1) reduces to the nonlinear quadratic integral equation (1.10). The integral equation (1.10) is applied in the theory of radiative transfer and the theory of neutron transport as well in the kinetic theory of gases (see [5, 9, 11, 12, 19, 22]). In the cace where

$$f(t,x,y) = g(t) + y, \quad (T_1x)(t) \equiv 0, \quad (T_2x)(t) \equiv 1, \quad (T_3x)(t) = x(t),$$

(1.1) has the form (1.11) which is a well-known Urysohn integral equation (see [4]).

Using the technique of a suitable measure of noncompactness, we prove a theorem on the existence and asymptotically stable of the solutions of the equation (1.1). We give some examples satisfying the conditions given in this paper. The approach applied in this paper depends on extending and generalizing of the methods and tools used in the study of some nonlinear integral equations which are presented above. It is worthwhile mentioning that the class of integral equations considered in this paper are more general then those investigated up to now.

2. Notations and preliminaries

In this section, we give a collection of auxiliary facts which will be needed in the sequel. Assume that $(E, \|.\|)$ is a real Banach space with zero element θ . Let $B(x,r)$ denote the closed ball centered at x and with radius r. The symbol B_r stands for the ball $B(\theta, r)$. If X is a subset of E, then $\overline{X}$ and $\mathrm{Conv}X$ denote the closure and convex closure of X, respectively. With the symbols λX and $X+Y$, we denote the standard algebraic operations on sets. Moreover, we denote by $\mathfrak{M}_E$ the family of all nonempty and bounded subsets of E and $\mathfrak{N}_E$ its subfamily consisting of all relatively compact subsets. The definition of the concept of a measure of noncompactness is presented below (see [2]).

Definition 2.1. A function $\mu : \mathfrak{M}_E \to \mathbb{R}_+ = [0, \infty)$ is said to be a measure of noncompactness in E if it satisfies following conditions:

(1) The family $\ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker\mu \subset \mathfrak{N}_E$.
(2) $X \subset Y \Rightarrow \mu(X) \le \mu(Y)$.
(3) $\mu(\overline{X}) = \mu(\mathrm{Conv}X) = \mu(X)$.
(4) $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$, for $\lambda \in [0,1]$.
(5) If $\{X_n\}$ is a sequence of nonempty, bounded, closed subsets of the set E such that $X_{n+1} \subset X_n, (n = 1, 2, ...)$ and $\lim_{n\to\infty} \mu(X_n) = 0$, then the set $X_\infty = \cap_{n=1}^{\infty} X_n$ is nonempty.

In the sequel, we will work in the Banach space $BC(\mathbb{R}_+, \mathbb{R})$ consisting of real functions defined, continuous and bounded on $\mathbb{R}_+$. This space is endowed with the standard norm $\|x\| = \sup\{|x(t)| : t \in \mathbb{R}_+\}$.

We will use a measure of noncompactness in the space $BC(\mathbb{R}_+, \mathbb{R})$. In order to define this measure let us fix a nonempty and bounded subset X of $BC(\mathbb{R}_+, \mathbb{R})$. For $x \in X$, $\varepsilon \ge 0$ and $L > 0$ denoted by $\omega^L(x, \varepsilon)$ the modulus of continuity of function x, i.e.,

$$\omega^L(x, \varepsilon) = \sup\{|x(s) - x(t)| : t, s \in [0, L] \text{ and } |t - s| \le \varepsilon\}.$$

Further let us put

$$\omega^L(X,\varepsilon) = \sup\{\omega^L(x,\varepsilon) : x \in X\},$$
$$\omega_0^L(X) = \lim_{\varepsilon\to 0} \omega^L(X,\varepsilon),$$
$$\omega_0(X) = \lim_{L\to\infty} \omega_0^L(X). \tag{2.15}$$

Moreover, let us denote

$$X(t) = \{x(t) : x \in X\}$$

and

$$\operatorname{diam} X(t) = \sup\{|x(t) - y(t)| : x, y \in X\}$$

for any fixed $t \in \mathbb{R}_+$. With help of the above mappings we define the following measure of noncompactness in $BC(\mathbb{R}_+, \mathbb{R})$ [2]:

$$\mu(X) = \omega_0(X) + \limsup_{t\to\infty} \operatorname{diam} X(t). \tag{2.16}$$

The kernel of μ consists of all nonempty and bounded subsets X of $BC(\mathbb{R}_+, \mathbb{R})$ such that functions from X are locally equicontinuous on $\mathbb{R}_+$ and the thickness of the bundle formed by functions from X tends to zero at infinity.

Now we recall definitions of the concepts of local attractivity and asymptotic stability of the solutions of operator equations. Let us assume that Ω is a nonempty subset of the space $BC(\mathbb{R}_+, \mathbb{R})$ and F be an operator defined on Ω with values in $BC(\mathbb{R}_+, \mathbb{R})$. Let us consider the operator equation of the form:

$$x(t) = (Fx)(t),\ t \in \mathbb{R}_+. \tag{2.17}$$

Definition 2.2. We say that solutions of (2.17) are locally attractive if there exist an $x_0 \in BC(\mathbb{R}_+, \mathbb{R})$ and an $r > 0$ such that for all solutions $x = x(t)$ and $y = y(t)$ of (2.17) belonging to $B(x_0, r) \cap \Omega$ we have that

$$\lim_{t\to\infty} (x(t) - y(t)) = 0.$$

In the case when limit is uniform with respect to the set $B(x_0, r) \cap \Omega$, that is, when for each $\varepsilon \geq 0$ there exists $L > 0$ such that

$$|x(t) - y(t)| \leq \varepsilon$$

for all $x, y \in B(x_0, r) \cap \Omega$ being solutions of (2.17) and for all $t \geq L$, we will say that solutions of (2.17) are uniformly locally attractive (or equivalently asymptotically stable) on $\mathbb{R}_+$, [16].

Finally, we present the following fixed-point theorem which we will need later, [2].

Theorem 2.3. *Let Q be a nonempty, bounded, closed and convex subset of the Banach space E and let*

$$T : Q \to Q$$

be a continuous transformation such that $\mu(TX) \leq c\mu(X)$ for any nonempty subset X of Q, where μ is a measure of noncompactness in E and $c \in [0, 1)$ is a constant. Then T has a fixed point in set Q.

Remark 2.4. Denote by $\operatorname{Fix} T$ the set of all fixed points of the operator T belonging to Q. It can be readily seen that the set $\operatorname{Fix} T$ belongs to the family $\ker \mu$, [2].

3. The Main Result

We will assume that the functions and operators involved in the equation (1.1) satisfy the following conditions:

(i) $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and the function $t \to f(t, 0, 0)$ is a member of the space $BC(\mathbb{R}_+, \mathbb{R})$.

(ii) The function f satisfies the Lipschitz condition with the nonnegative constants l_1 and l_2 with respect to second and third variables, i.e.

$$|f(t, x_1, y) - f(t, x_2, y)| \leq l_1|x_1 - x_2|$$

for all $t \in \mathbb{R}_+$ and $x_1, x_2, y \in \mathbb{R}$ and

$$|f(t, x, y_1) - f(t, x, y_2)| \leq l_2|y_1 - y_2|$$

for all $t \in \mathbb{R}_+$ and $x, y_1, y_2 \in \mathbb{R}$.

(iii) The operators $T_i : BC(\mathbb{R}_+, \mathbb{R}) \to BC(\mathbb{R}_+, \mathbb{R})$ are continuous and there exist nondecreasing functions $d_i : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|T_i x\| \leq d_i(\|x\|) \ (i = 1, 2, 3)$$

for all $x \in BC(\mathbb{R}_+, \mathbb{R})$.

(iv) $u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a continuous function and there exist a continuous function $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ and a nondecreasing function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|u(t, s, x)| \leq g(t, s)h(|x|)$$

for all $t, s \in \mathbb{R}_+$ and $x \in \mathbb{R}$. Besides, the function $s \to g(t, s)$ defining for each $t \in \mathbb{R}_+$ is integrable on $\mathbb{R}_+$, the function $q : \mathbb{R}_+ \to \mathbb{R}_+$ given by $q(t) = \int_0^\infty g(t, s)ds$ is bounded on $\mathbb{R}_+$ and $\lim_{t\to\infty} q(t) = 0$.

Remark 3.1. It can be shown that the requirements on the function g given in the assumption (iv) are independent.

For example; if $g(t, s) = te^{-ts}$, then the function q defined as

$$q(t) = \int_0^\infty g(t, s)ds = \begin{cases} 0, & t = 0 \\ 1, & t > 0 \end{cases}$$

is bounded on $\mathbb{R}_+$, but $\lim_{t\to\infty} q(t) = 1 \neq 0$.

If $g(t, s) = te^{-t^2 s}$, then

$$q(t) = \int_0^\infty g(t, s)ds = \begin{cases} 0, & t = 0 \\ \frac{1}{t}, & t > 0 \end{cases}$$

and $\lim_{t\to\infty} q(t) = 0$, but the function q is unbounded on $\mathbb{R}_+$.

(v) There exists the positive real number r_0 satisfying the inequality

$$l_1 d_1(r) + l_2 d_2(r)h(d_3(r))G + F \leq r,$$

where $F = \sup\{|f(t, 0, 0)| : t \geq 0\}$ and $G = \sup\{|q(t)| : t \geq 0\}$.

Remark 3.2. F and G given in (v) are the finite constants from the assumptions (i) and (iv).

(vi) There exist the nonnegative constants m_{r_0} and ϑ_{r_0} for r_0 such that the inequalities

$$\mu(T_1 X) \leq m_{r_0}\mu(X)$$

and

$$\omega_0(T_2 X) \leq \vartheta_{r_0}\omega_0(X)$$

hold for all nonempty and bounded subsets X of B_{r_0}, where ω_0 and μ are defined by (2.15) and (2.16).

(vii) We assume that

$$l_1 m_{r_0} + l_2 \vartheta_{r_0} h(d_3(r_0))G < 1.$$

(viii) There exists a continuous nondecreasing function $\phi_{r_0} : \mathbb{R}_+ \to \mathbb{R}_+$ which holds $\phi_{r_0}(0) = 0$ and

$$|u(t_2, s, x) - u(t_1, s, x)| \leq \phi_{r_0}(|t_2 - t_1|)p(s)$$

for all $t_1, t_2, s \in \mathbb{R}_+$ and $x \in \mathbb{R}$ with $|x| \leq r_0$, where $p \in BC(\mathbb{R}_+, \mathbb{R}_+)$ such that $\|p\|_1 = \int_0^\infty |p(s)|ds < \infty$.

(ix) There exists a continuous nondecreasing function $\eta_{r_0} : \mathbb{R}_+ \to \mathbb{R}_+$ which holds $\eta_{r_0}(0) = 0$ and

$$|u(t, s, x) - u(t, s, y)| \leq \eta_{r_0}(|x - y|)k(s)$$

for all $t, s \in \mathbb{R}_+$ and $x, y \in \mathbb{R}$ with $|x| \leq r_0, |y| \leq r_0$, where $k \in BC(\mathbb{R}_+, \mathbb{R}_+)$ such that $\|k\|_1 = \int_0^\infty |k(s)|ds < \infty$.

Now we can give an existence theorem on the functional integral equation (1.1).

Theorem 3.3. *Under the assumptions $(i)-(ix)$, there exists a positive real number r_0 such that the equation (1.1) has at least one solution $x = x(t)$ belonging to $B_{r_0} \subset BC(\mathbb{R}_+, \mathbb{R})$. Moreover, all the solutions of the equation (1.1) belonging to the ball B_{r_0} are asymptotically stable on $\mathbb{R}_+$.*

Proof. We define operator T on B_{r_0} in the following way:

$$(Tx)(t) = f\left(t, (T_1x)(t), (T_2x)(t)\int_0^\infty u\left(t, s, (T_3x)(s)\right) ds\right).$$

Notice that in view of assumptions (i), (iii) and (iv), the function $t \to (Tx)(t)$ is well defined on the interval $\mathbb{R}_+$. At first we show that the function Tx is continuous on $\mathbb{R}_+$ for an arbitrary fixed $x \in B_{r_0}$. Since the functions f and T_ix $(i = 1, 2)$ are continuous, it is sufficient to prove that the function $t \to \int_0^\infty u\left(t, s, (T_3x)(s)\right) ds$ is continuous on $\mathbb{R}_+$. Let us fix arbitrarily $\varepsilon \geq 0$ and take arbitrary numbers $t_1, t_2 \in \mathbb{R}_+$ with $|t_1 - t_2| \leq \varepsilon$. Then in view of assumption $(viii)$ we obtain the estimate

$$\begin{aligned}
&\left|\int_0^\infty \left[u\left(t_1, s, (T_3x)(s)\right) - u\left(t_2, s, (T_3x)(s)\right)\right] ds\right| \\
&\leq \int_0^\infty \left|u\left(t_1, s, (T_3x)(s)\right) - u\left(t_2, s, (T_3x)(s)\right)\right| ds \\
&\leq \int_0^\infty \phi_{r_0}(|t_1 - t_2|)p(s)ds \leq \phi_{r_0}(\varepsilon)\|p\|_1
\end{aligned}$$

which implies that the function $t \to \int_0^\infty u\left(t, s, (T_3x)(s)\right) ds$ is continuous on $\mathbb{R}_+$. Therefore, Tx is continuous on $\mathbb{R}_+$.

Next we show that Tx is bounded on $\mathbb{R}_+$. By our assumptions, we derive that

$$\begin{aligned} |(Tx)(t)| &= \left| f\left(t, (T_1x)(t), (T_2x)(t)\int_0^\infty u\left(t,s,(T_3x)(s)\right)ds\right)\right| \\ &\leq \left| f\left(t, (T_1x)(t), (T_2x)(t)\int_0^\infty u\left(t,s,(T_3x)(s)\right)ds\right)\right. \\ &\quad \left. - f\left(t, 0, (T_2x)(t)\int_0^\infty u\left(t,s,(T_3x)(s)\right)ds\right)\right| \\ &\quad + \left| f\left(t, 0, (T_2x)(t)\int_0^\infty u\left(t,s,(T_3x)(s)\right)ds\right) - f\left(t,0,0\right)\right| \\ &\quad + |f\left(t,0,0\right)| \end{aligned} \tag{3.18}$$

for arbitrarily fixed $t \in \mathbb{R}_+$. From (3.18), we have that

$$\begin{aligned} |(Tx)(t)| &\leq l_1\left|(T_1x)(t)\right| + \left|f\left(t,0,0\right)\right| \\ &\quad + l_2\left|(T_2x)(t)\right|\int_0^\infty \left|u\left(t,s,(T_3x)(s)\right)\right|ds \\ &\leq l_1d_1(\|x\|) + l_2d_2(\|x\|)\int_0^\infty g(t,s)h\left(|(T_3x)(s)|\right)ds + F \\ &\leq l_1d_1(\|x\|) + l_2d_2(\|x\|)h(\|T_3x\|)\int_0^\infty g(t,s)ds + F \\ &\leq l_1d_1(\|x\|) + l_2d_2(\|x\|)h(d_3(\|x\|))G + F \end{aligned} \tag{3.19}$$

for all $t \in \mathbb{R}_+$. Hence, from (3.19), we obtain the following evaluation:

$$\|Tx\| \leq l_1d_1(\|x\|) + l_2d_2(\|x\|)h(d_3(\|x\|))G + F. \tag{3.20}$$

The estimate (3.20) implies that the function Tx is bounded on $\mathbb{R}_+$. Combining this fact with the continuity of the function Tx on $\mathbb{R}_+$, we conclude that the operator T transforms the ball B_{r_0} into the space $BC(\mathbb{R}_+, \mathbb{R})$. Moreover linking (3.20) and the assumption (v) we deduce that the operator T maps the ball B_{r_0} into itself, where r_0 is a number indicated in assumption (v).

Now, we shall prove that the operator T is continuous on B_{r_0}. To do this, consider any $\varepsilon > 0$ and fixed $y \in B_{r_0}$. Then,

$$\begin{aligned} |(Tx)(t) - (Ty)(t)| &= \left| f\left(t, (T_1x)(t), (T_2x)(t)\int_0^\infty u\left(t,s,(T_3x)(s)\right)ds\right)\right. \\ &\quad \left. - f\left(t, (T_1y)(t), (T_2y)(t)\int_0^\infty u\left(t,s,(T_3y)(s)\right)ds\right)\right| \\ &\leq \left| f\left(t, (T_1x)(t), (T_2x)(t)\int_0^\infty u\left(t,s,(T_3x)(s)\right)ds\right)\right. \\ &\quad \left. - f\left(t, (T_1y)(t), (T_2x)(t)\int_0^\infty u\left(t,s,(T_3x)(s)\right)ds\right)\right| \\ &\quad + \left| f\left(t, (T_1y)(t), (T_2x)(t)\int_0^\infty u\left(t,s,(T_3x)(s)\right)ds\right)\right. \\ &\quad \left. - f\left(t, (T_1y)(t), (T_2y)(t)\int_0^\infty u\left(t,s,(T_3y)(s)\right)ds\right)\right|. \end{aligned} \tag{3.21}$$

From the hypothesis (ii) and the inequality (3.21) we derive that

$$\begin{aligned} |(Tx)(t)-(Ty)(t)| \leq\; & l_1\,|(T_1x)(t)-(T_1y)(t)| \\ & +l_2\left|(T_2x)(t)\int_0^\infty u\,(t,s,(T_3x)(s))\,ds\right. \\ & \left.-(T_2y)(t)\int_0^\infty u\,(t,s,(T_3y)(s))\,ds\right|. \end{aligned} \tag{3.22}$$

Since

$$\begin{aligned} & (T_2x)(t)\int_0^\infty u\,(t,s,(T_3x)(s))\,ds-(T_2y)(t)\int_0^\infty u\,(t,s,(T_3y)(s))\,ds \\ & =[(T_2x)(t)-(T_2y)(t)]\int_0^\infty u\,(t,s,(T_3x)(s))\,ds \\ & \quad +(T_2y)(t)\int_0^\infty [u\,(t,s,(T_3x)(s))-u\,(t,s,(T_3y)(s))]ds, \end{aligned}$$

we get by (3.22) that

$$\begin{aligned} |(Tx)(t)-(Ty)(t)| \leq\; & l_1\,|(T_1x)(t)-(T_1y)(t)| \\ & +l_2\,|(T_2x)(t)-(T_2y)(t)|\int_0^\infty |u\,(t,s,(T_3x)(s))|\,ds \\ & +l_2\,|(T_2y)(t)|\int_0^\infty |u\,(t,s,(T_3x)(s))-u\,(t,s,(T_3y)(s))|\,ds. \end{aligned} \tag{3.23}$$

By using of the assumptions, from the estimate (3.23) we get that

$$\begin{aligned} |(Tx)(t)-(Ty)(t)| \leq\; & l_1\,\|T_1x-T_1y\|+l_2\,\|T_2x-T_2y\|\,h(\|T_3x\|)\int_0^\infty g(t,s)ds \\ & +l_2d_2(\|y\|)\int_0^\infty k(s)\eta_{r_0}(|(T_3x)(s)-(T_3y)(s)|)ds \\ \leq\; & l_1\,\|T_1x-T_1y\|+l_2\,\|T_2x-T_2y\|\,h(d_3(\|x\|))G \\ & +l_2d_2(\|y\|)\eta_{r_0}\,(\|(T_3x-T_3y\|)\int_0^\infty k(s)ds \\ \leq\; & l_1\,\|T_1x-T_1y\|+l_2\,\|T_2x-T_2y\|\,h(d_3(r_0))G \\ & +l_2d_2(r_0)\eta_{r_0}\,(\|(T_3x-T_3y\|)\,\|k\|_1 \end{aligned} \tag{3.24}$$

for all $t\in\mathbb{R}_+$. Since the operators T_i $(i=1,2,3)$ are continuous for any fixed $y\in B_{r_0}$, there exists the number $\delta>0$ such that we have $\|T_ix-T_iy\|\leq\varepsilon$ for all $x\in B_{r_0}$ with $\|x-y\|<\delta$. In this case, (3.24) yields that

$$\|Tx-Ty\|\leq l_1\varepsilon+l_2h(d_3(r_0))G\varepsilon+l_2d_2(r_0)\|k\|_1\eta_{r_0}(\varepsilon) \tag{3.25}$$

for all $x\in B_{r_0}$ with $\|x-y\|<\delta$. By (3.25) and assumption (ix), we have that T is continuous at the fixed arbitrary point $y\in B_{r_0}$ and thus T is continuous on the ball B_{r_0}.

Further, we shall show that the operator F satisfies the Darbo condition on the ball B_{r_0}. In order to do this, let us take a nonempty subset X of the ball B_{r_0}. Fix $\varepsilon\geq 0$,

$L > 0$ and choose $x \in X$ and $t_1, t_2 \in [0, L]$ with $|t_1 - t_2| \leq \varepsilon$. Then,

$$
\begin{aligned}
|(Tx)(t_1) - (Tx)(t_2)| = & \left| f\left(t_1, (T_1x)(t_1), (T_2x)(t_1) \int_0^\infty u\left(t_1, s, (T_3x)(s)\right) ds\right)\right. \\
& \left. - f\left(t_2, (T_1x)(t_2), (T_2x)(t_2) \int_0^\infty u\left(t_2, s, (T_3x)(s)\right) ds\right)\right| \\
\leq & \left| f\left(t_1, (T_1x)(t_1), (T_2x)(t_1) \int_0^\infty u\left(t_1, s, (T_3x)(s)\right) ds\right)\right. \\
& \left. - f\left(t_2, (T_1x)(t_1), (T_2x)(t_1) \int_0^\infty u\left(t_1, s, (T_3x)(s)\right) ds\right)\right| \\
& + \left| f\left(t_2, (T_1x)(t_1), (T_2x)(t_1) \int_0^\infty u\left(t_1, s, (T_3x)(s)\right) ds\right)\right. \\
& \left. - f\left(t_2, (T_1x)(t_2), (T_2x)(t_1) \int_0^\infty u\left(t_1, s, (T_3x)(s)\right) ds\right)\right| \\
& + \left| f\left(t_2, (T_1x)(t_2), (T_2x)(t_1) \int_0^\infty u\left(t_1, s, (T_3x)(s)\right) ds\right)\right. \\
& \left. - f\left(t_2, (T_1x)(t_2), (T_2x)(t_2) \int_0^\infty u\left(t_2, s, (T_3x)(s)\right) ds\right)\right|. \qquad (3.26)
\end{aligned}
$$

Taking into account assumptions, we have by (3.26) that

$$
\begin{aligned}
|(Tx)(t_1) - (Tx)(t_2)| \leq & \, \omega_{r_0}^L(f, \varepsilon) + l_1 \left|(T_1x)(t_1) - (T_1x)(t_2)\right| \\
& + l_2 \left|(T_2x)(t_1) \int_0^\infty u\left(t_1, s, (T_3x)(s)\right) ds \right. \\
& \left. -(T_2x)(t_2) \int_0^\infty u\left(t_2, s, (T_3x)(s)\right) ds\right| \\
= & \, \omega_{r_0}^L(f, \varepsilon) + l_1 \left|(T_1x)(t_1) - (T_1x)(t_2)\right| \\
& + l_2 \left| \left[(T_2x)(t_1) - (T_2x)(t_2)\right] \int_0^\infty u\left(t_1, s, (T_3x)(s)\right) ds \right. \\
& \left. +(T_2x)(t_2) \int_0^\infty \left[u\left(t_1, s, (T_3x)(s)\right) - u\left(t_2, s, (T_3x)(s)\right)\right] ds\right| \\
\leq & \, \omega_{r_0}^L(f, \varepsilon) + l_1 \left|(T_1x)(t_1) - (T_1x)(t_2)\right| \\
& + l_2 \left|(T_2x)(t_1) - (T_2x)(t_2)\right| \int_0^\infty \left|u\left(t_1, s, (T_3x)(s)\right)\right| ds \\
& + l_2 \left|(T_2x)(t_2)\right| \int_0^\infty \left|u\left(t_1, s, (T_3x)(s)\right) - u\left(t_2, s, (T_3x)(s)\right)\right| ds \\
\leq & \, \omega_{r_0}^L(f, \varepsilon) + l_1 \omega^L(T_1x, \varepsilon) + l_2 \omega^L(T_2x, \varepsilon) h(\|T_3x\|) \int_0^\infty g(t_1, s) ds \\
& + l_2 d_2(\|x\|) \int_0^\infty \phi_{r_0}(|t_1 - t_2|) p(s) ds \\
\leq & \, \omega_{r_0}^L(f, \varepsilon) + l_1 \omega^L(T_1x, \varepsilon) + l_2 \omega^L(T_2x, \varepsilon) h(d_3(\|x\|)) G \\
& + l_2 d_2(\|x\|) \phi_{r_0}(|t_1 - t_2|) \int_0^\infty p(s) ds \\
\leq & \, \omega_{r_0}^L(f, \varepsilon) + l_1 \omega^L(T_1x, \varepsilon) + l_2 h(d_3(r_0)) G \omega^L(T_2x, \varepsilon) \\
& + l_2 d_2(r_0) \|p\|_1 \phi_{r_0}(\varepsilon)
\end{aligned}
$$

which implies that

$$\begin{aligned}\omega^L(Tx,\varepsilon) &\leq \omega^L_{r_0}(f,\varepsilon)+l_1\omega^L(T_1x,\varepsilon)+l_2h(d_3(r_0))G\omega^L(T_2x,\varepsilon)\\ &\quad +l_2d_2(r_0)\|p\|_1\phi_{r_0}(\varepsilon),\end{aligned} \tag{3.27}$$

where,

$$\begin{aligned}\omega^L_{r_0}(f,\varepsilon) &= \sup\{|f(t_1,x_1,y)-f(t_2,x_1,y)| : t_1,t_2\in[0,L], x_1\in[-d_1(r_0),d_1(r_0)],\\ &\quad y\in[-d_2(r_0)h(d_3(r_0))G, d_2(r_0)h(d_3(r_0))G] \text{ and } |t_1-t_2|\leq\varepsilon\},\\ \omega^L(T_ix,\varepsilon) &= \sup\{|(T_ix)(t_1)-(T_ix)(t_2)| : t_1,t_2\in[0,L] \text{ and } |t_1-t_2|\leq\varepsilon\}\end{aligned}$$

for $i=1,2$. By (3.27) we get that

$$\begin{aligned}\omega^L(TX,\varepsilon) &\leq \omega^L_{r_0}(f,\varepsilon)+l_1\omega^L(T_1X,\varepsilon)+l_2h(d_3(r_0))G\omega^L(T_2X,\varepsilon)\\ &\quad +l_2d_2(r_0)\|p\|_1\phi_{r_0}(\varepsilon).\end{aligned} \tag{3.28}$$

By uniform continuity of the function f on the set $[0,L]\times[-d_1(r_0),d_1(r_0)]\times[-d_2(r_0)h(d_3(r_0))G, d_2(r_0)h(d_3(r_0))G]$, it is easily deduced that $\omega^L_{r_0}(f,\varepsilon)\to 0$ as $\varepsilon\to 0$. Also, by $(viii)$, $\phi_{r_0}(\varepsilon)\to 0$ as $\varepsilon\to 0$. Thus, (3.28) yields that

$$\omega^L_0(TX)\leq l_1\omega^L_0(T_1X)+l_2h(d_3(r_0))G\omega^L_0(T_2X). \tag{3.29}$$

If we take limit as $L\to\infty$, we have by (3.29) that

$$\omega_0(TX)\leq l_1\omega_0(T_1X)+l_2h(d_3(r_0))G\omega_0(T_2X). \tag{3.30}$$

Further let us take a nonempty subset X of the ball B_{r_0}. From estimate (3.23) and the conditions (iii) and (iv), it is shown that

$$\begin{aligned}\operatorname{diam}(TX)(t) &\leq l_1\operatorname{diam}(T_1X)(t)+l_2\operatorname{diam}(T_2X)(t)h(\|T_3x\|)\int_0^\infty g(t,s)ds\\ &\quad + l_2d_2(\|y\|)\left[h(\|T_3x\|)+h(\|T_3y\|)\right]\int_0^\infty g(t,s)ds\\ &\leq l_1\operatorname{diam}(T_1X)(t)+l_2\operatorname{diam}(T_2X)(t)h(d_3(\|x\|))q(t)\\ &\quad + l_2d_2(\|y\|)\left[h(d_3(\|x\|))+h(d_3(\|y\|))\right]q(t)\\ &\leq l_1\operatorname{diam}(T_1X)(t)+l_2\operatorname{diam}(T_2X)(t)h(d_3(r_0))q(t)\\ &\quad + 2l_2d_2(r_0)h(d_3(r_0))q(t)\end{aligned} \tag{3.31}$$

for all $t\in\mathbb{R}_+$. If we take limit superior as $t\to\infty$ in (3.31) by considering $\lim_{t\to\infty} q(t)=0$, we have the inequality:

$$\limsup_{t\to\infty}\operatorname{diam}(TX)(t)\leq l_1\limsup_{t\to\infty}\operatorname{diam}(T_1X)(t). \tag{3.32}$$

By linking (2.16), (3.30) and (3.32), we derive by assumption (vi) and the inequality $\omega_0(X)\leq\mu(X)$ that

$$\begin{aligned}\mu(TX) &\leq l_1\mu(T_1X)+l_2\vartheta_{r_0}h(d_3(r_0))G\omega_0(X)\\ &\leq l_1m_{r_0}\mu(X)+l_2\vartheta_{r_0}h(d_3(r_0))G\mu(X)\\ &= \left[l_1m_{r_0}+l_2\vartheta_{r_0}h(d_3(r_0))G\right]\mu(X).\end{aligned} \tag{3.33}$$

Now let us observe that by assumption (vii) and (3.33) we have that T is a contraction with respect to the measure of noncompactness μ. By Theorem 2.3 the operator T has at least one fixed point x in the ball B_{r_0}. Obviously, every function $x=x(t)$ being a fixed point of the operator T is a solution of (1.1). Further, keeping in mind Remark 2.4, we conclude that the set $\operatorname{Fix} T$ of all fixed points of the operator T belonging to the ball B_{r_0} is a member of the $\ker\mu$. Hence, in view of the description of the $\ker\mu$ we infer that all of solutions of (1.1) belonging to the ball B_{r_0} are asymptotically stable on $\mathbb{R}_+$. This step completes the proof of our theorem. □

4. Examples

Example 4.1. Consider the following integral equation:

$$x(t) = \frac{\sin\left(tx(t)\int_0^{\frac{1}{1+t}} \sin x(\tau)d\tau + \frac{\pi}{2}\right)}{7\left(1+\left|x(t)\int_0^{\frac{1}{1+t}} \sin x(\tau)d\tau\right|\right)(1+t)} + \frac{1}{8\left(\left|x^2(t)\int_0^{\infty} \frac{x^3(s)+1}{(t+1)(s^2+1)}ds\right|+1\right)}, \tag{4.34}$$

where $t \in \mathbb{R}_+$. Notice that (4.34) is a special case of (1.1) if we put

$$f(t,x,y) = \frac{\sin\left(tx+\frac{\pi}{2}\right)}{7(1+|x|)(1+t)} + \frac{1}{8(|y|+1)},$$

$$(T_1x)(t) = x(t)\int_0^{\frac{1}{1+t}} \sin x(\tau)d\tau,\ (T_2x)(t) = x^2(t),\ (T_3x)(t) = x^3(t)$$

and $u(t,s,x) = \frac{x+1}{(t+1)(s^2+1)}$.

It is verified that the assumptions of Theorem 3.3 are satisfied.

Indeed, f is continuous on $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$ and the function $t \to f(t,0,0) = \frac{1}{7(1+t)} + \frac{1}{8}$ is a member of the space $BC(\mathbb{R}_+, \mathbb{R})$.

Since

$$\begin{aligned}
\left|f(t,x_1,y) - f(t,x_2,y)\right| &= \frac{1}{7(1+t)}\left|\frac{\sin\left(tx_1+\frac{\pi}{2}\right)}{1+|x_1|} - \frac{\sin\left(tx_2+\frac{\pi}{2}\right)}{1+|x_2|}\right| \\
&\le \frac{1}{7(1+t)(1+|x_1|)(1+|x_2|)}\left(\left|1+|x_2|-1-|x_1|\right|\left|\sin\left(tx_1+\frac{\pi}{2}\right)\right|\right. \\
&\left. +(1+|x_1|)\left|\sin\left(tx_1+\frac{\pi}{2}\right) - \sin\left(tx_2+\frac{\pi}{2}\right)\right|\right) \\
&\le \frac{1}{7}|x_1-x_2| + \frac{1}{7(1+t)(1+|x_1|)(1+|x_2|)}(1+|x_1|)|x_1-x_2|t \\
&\le \frac{2}{7}|x_1-x_2|
\end{aligned}$$

and

$$|f(t,x,y_1) - f(t,x,y_2)| = \frac{1}{8}\left|\frac{1}{|y_1|+1} - \frac{1}{|y_2|+1}\right| \le \frac{1}{8}|y_1-y_2|$$

for all $t \in \mathbb{R}_+$ and $x, x_1, x_2, y, y_1, y_2 \in \mathbb{R}$, we can take the constants l_1 and l_2 satisfying (ii) as $l_1 = \frac{2}{7}$ and $l_2 = \frac{1}{8}$.

T_1, T_2 and T_3 are the continuous operators on the space $BC(\mathbb{R}_+, \mathbb{R})$. Further for all $x \in BC(\mathbb{R}_+, \mathbb{R})$ and $t \in \mathbb{R}_+$ the inequalities

$$|(T_1x)(t)| = \left|x(t)\int_0^{\frac{1}{1+t}} \sin x(\tau)d\tau\right| \le \frac{1}{1+t}\|x\| \le \|x\|,$$

$$|(T_2x)(t)| \le |x^2(t)| \le \|x\|^2$$

and

$$|(T_3x)(t)| \le |x^3(t)| \le \|x\|^3$$

hold. So, the assumption (iii) is satisfied with $d_1(t) = t$, $d_2(t) = t^2$ and $d_3(t) = t^3$.

Now notice that the function u is continuous on the set $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$. Moreover, we get that

$$|u(t,s,x)| = \left| \frac{x+1}{(t+1)(s^2+1)} \right| \le \frac{|x|+1}{(t+1)(s^2+1)}$$

for all $t, s \in \mathbb{R}_+$ and $x \in \mathbb{R}$. Thus, according to the assumption (iv) we may put $g(t,s) = \frac{1}{(t+1)(s^2+1)}$ and $h(t) = t+1$. Further we get

$$q(t) = \int_0^\infty g(t,s)ds = \int_0^\infty \frac{ds}{(t+1)(s^2+1)} = \frac{\pi}{2(t+1)}$$

and, obviously, we have that

$$G = \sup\{q(t) : t \ge 0\} = \frac{\pi}{2}$$

and $q(t) \to 0$ as $t \to \infty$.

Now if we consider the values $d_1(r) = r, d_2(r) = r^2$ and $d_3(r) = r^3$ together with $F = \sup\{|f(t,0,0)| : t \ge 0\} = \frac{15}{56}$, the inequality

$$l_1 d_1(r) + l_2 d_2(r) h(d_3(r)) G + F \le r$$

in the assumption (v) takes the following form:

$$\frac{2r}{7} + \frac{r^2}{8}\frac{\pi}{2}(r^3+1) + \frac{15}{56} \le r. \tag{4.35}$$

The number r_0 chosen as $0.429816 \le r_0 < 0.962102$ satisfies (4.35).

Apart from this, fixing a nonempty and bounded subset X of the ball B_{r_0}, let $x \in X$, $\varepsilon \ge 0$, $L > 0$ and $t, s \in [0, L]$ such that $|t-s| \le \varepsilon$.

Without loss of generality, we assume that $t \le s$. So,

$$\begin{aligned}
(T_1x)(t) - (T_1x)(s) &= x(t)\int_0^{\frac{1}{1+t}} \sin x(\tau)d\tau - x(s)\int_0^{\frac{1}{1+s}} \sin x(\tau)d\tau \\
&= (x(t)-x(s))\int_0^{\frac{1}{1+t}} \sin x(\tau)d\tau \\
&\quad + x(s)\left(\int_0^{\frac{1}{1+t}} \sin x(\tau)d\tau - \int_0^{\frac{1}{1+s}} \sin x(\tau)d\tau\right) \\
&= (x(t)-x(s))\int_0^{\frac{1}{1+t}} \sin x(\tau)d\tau + x(s)\left(\int_{\frac{1}{1+s}}^{\frac{1}{1+t}} \sin x(\tau)d\tau\right)
\end{aligned}$$

which yields that

$$\begin{aligned}
|(T_1x)(t) - (T_1x)(s)| &\le \frac{1}{1+t}|x(t)-x(s)| + |x(s)|\left|\frac{1}{1+t} - \frac{1}{1+s}\right| \\
&\le |x(t)-x(s)| + r_0|t-s|
\end{aligned} \tag{4.36}$$

for all $t, s \in [0, L]$ such that $|t-s| \le \varepsilon$. Besides, it is clear that

$$\begin{aligned}
|(T_2x)(t) - (T_2x)(s)| &= \left|x^2(t) - x^2(s)\right| \\
&= |x(t)+x(s)|\,|x(t)-x(s)| \\
&\le 2r_0|x(t)-x(s)|
\end{aligned} \tag{4.37}$$

for all $t, s \in [0, L]$ such that $|t-s| \le \varepsilon$.

From estimates (4.36), (4.37) and in the view of the (2.15), we get that

$$\omega_0(T_1X) \le \omega_0(X), \tag{4.38}$$

$$\omega_0(T_2X) \le 2r_0\omega_0(X). \tag{4.39}$$

Inequality (4.39) implies that the second inequality of assumption (vi) is satisfied with the constant $\vartheta_{r_0} = 2r_0$.

Furthermore, the estimate

$$\begin{aligned}
|(T_1x)(t) - (T_1y)(t)| &= \left| x(t) \int_0^{\frac{1}{1+t}} \sin x(\tau) d\tau - y(t) \int_0^{\frac{1}{1+t}} \sin y(\tau) d\tau \right| \\
&= \left| (x(t) - y(t)) \int_0^{\frac{1}{1+t}} \sin x(\tau) d\tau \right. \\
&\left. + y(t) \int_0^{\frac{1}{1+t}} (\sin x(\tau) - \sin y(\tau))\, d\tau \right| \\
&\leq |x(t) - y(t)| \int_0^{\frac{1}{1+t}} |\sin x(\tau)| d\tau \\
&+ |y(t)| \int_0^{\frac{1}{1+t}} |\sin x(\tau) - \sin y(\tau)| d\tau \\
&\leq \frac{1}{1+t} \|x - y\| + \frac{1}{1+t} \|y\|\, \|x - y\| \\
&\leq \frac{1}{1+t} \left(2r_0^2 + 2r_0\right) \qquad (4.40)
\end{aligned}$$

holds for all $x, y \in X$ and $t \in \mathbb{R}_+$. Using (4.40), we have the equality:

$$\limsup_{t\to\infty} \operatorname{diam} (T_1X)(t) = 0. \qquad (4.41)$$

From (2.16), (4.38) and (4.41), we get that:

$$\mu(T_1X) \leq \omega_0(X) \leq \mu(X). \qquad (4.42)$$

So, we derive by (4.42) that the first inequality of assumption (vi) is satisfied with $m_{r_0} = 1$.

The inequality of assumption (vii) is equivalent to:

$$\frac{2}{7} + \frac{2r_0}{8} \frac{\pi}{2} (r_0^3 + 1) < 1. \qquad (4.43)$$

The inequality (4.43) holds for $0.429816 \leq r_0 < 0.962102$.

Additionally, for all $t_1, t_2, s \in \mathbb{R}_+$ and $x \in \mathbb{R}$ with $|x| \leq r_0$ we have that:

$$\begin{aligned}
|u(t_1, s, x) - u(t_2, s, x)| &= \left| \frac{x+1}{(t_1+1)(s^2+1)} - \frac{x+1}{(t_2+1)(s^2+1)} \right| \\
&= \frac{|x+1|}{s^2+1} \left| \frac{1}{t_1+1} - \frac{1}{t_2+1} \right| \\
&\leq \frac{|x|+1}{s^2+1} \frac{|t_1 - t_2|}{(t_1+1)(t_2+1)} \\
&\leq \frac{r_0+1}{s^2+1} |t_1 - t_2|.
\end{aligned}$$

If we put $\phi_{r_0}(t) = t$ and $p(s) = \frac{r_0+1}{s^2+1}$, the assumption $(viii)$ is satisfied.

Finally, for all $t, s \in \mathbb{R}_+$ and $x, y \in \mathbb{R}$ with $|x| \leq r_0, |y| \leq r_0$, we get that:

$$|u(t, s, x) - u(t, s, y)| = \left| \frac{x+1-y-1}{(t+1)(s^2+1)} \right| \leq \frac{|x-y|}{1+s^2}.$$

If we take $\eta_{r_0}(t) = t$ and $k(s) = \frac{1}{s^2+1}$, the assumption (ix) is satisfied.

Since all of the assumptions of Theorem 3.3 are fullfilled, we deduce that the integral equation (4.34) has at least one solution belonging to the ball B_{r_0} of the space $BC(\mathbb{R}_+, \mathbb{R})$.

Taking into account Remark 2.4 and the measure of noncompactness μ given by (2.16), we infer easily that any solutions of (4.34) which belong to the ball B_{r_0} are asymptotically stable on $\mathbb{R}_+$ as defined in Definition 2.2.

Example 4.2. Let us consider the following integral equation:

$$\begin{aligned} x(t) &= \frac{t}{3(1+t)} + \frac{1}{14}\ln\left(e + \exp\left(-\left|x(t)\right|\right)\right) \\ &+\frac{1}{5}\cos\left(\sqrt{x^2(t)+2}\int_0^\infty \frac{\left(\int_0^{\sin s} x(\tau)d\tau\right)^3}{\exp(t+s+1)}ds\right), \end{aligned} \tag{4.44}$$

where $t \in \mathbb{R}_+$. Observe that

$$f(t,x,y) = \frac{t}{3(1+t)} + \frac{1}{14}\ln\left(e+|x|\right) + \frac{1}{5}\cos y,$$

$$(T_1x)(t) = \exp\left(-\left|x(t)\right|\right),\ (T_2x)(t) = \sqrt{x^2(t)+2},\ (T_3x)(t) = \int_0^{\sin t} x(\tau)d\tau$$

and $u(t,s,x) = \frac{x^3}{\exp(t+s+1)}$.

It is clear that the function $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and the function $t \to f(t,0,0) = \frac{t}{3(1+t)} + \frac{19}{70}$ is a member of the space $BC(\mathbb{R}_+,\mathbb{R})$.

Without loss of generality we can suppose that $|x_1| < |x_2|$. So, there exists a number $\xi \in (|x_1|,|x_2|)$ satisfying the inequality

$$\begin{aligned} |f(t,x_1,y) - f(t,x_2,y)| &= \frac{1}{14}\left|\ln\left(e+|x_1|\right) - \ln(e+|x_2|)\right| \\ &\leq \frac{1}{14(e+\xi)}|x_1 - x_2| \end{aligned} \tag{4.45}$$

for all $t \in \mathbb{R}_+$ and $y \in \mathbb{R}$. Taking into account (4.45), we have that

$$|f(t,x_1,y) - f(t,x_2,y)| \leq \frac{1}{14}|x_1 - x_2|$$

for all $t \in \mathbb{R}_+$ and $x_1, x_2, y \in \mathbb{R}$.

Besides, we can easily see that the inequality

$$|f(t,x,y_1) - f(t,x,y_2)| = \frac{1}{5}|\cos y_1 - \cos y_2| \leq \frac{1}{5}|y_1 - y_2|$$

holds for all $t \in \mathbb{R}_+$ and $x, y_1, y_2 \in \mathbb{R}$.

Therefore, we can chose the nonnegative constants l_1 and l_2 satisfying the condition (ii) of Theorem 3.3 as $l_1 = \frac{1}{14}$ and $l_2 = \frac{1}{5}$.

It is clear that T_1, T_2 and T_3 are the continuous operators on the space $BC(\mathbb{R}_+,\mathbb{R})$. Moreover for all $x \in BC(\mathbb{R}_+,\mathbb{R})$ and $t \in \mathbb{R}_+$, we get that:

$$|(T_1x)(t)| = \left|\exp\left(-\left|x(t)\right|\right)\right| \leq 1,$$

$$|(T_2x)(t)| = \left|\sqrt{x^2(t)+2}\right| \leq \sqrt{\|x\|^2+2}$$

and

$$|(T_3x)(t)| = \left|\int_0^{\sin t} x(\tau)d\tau\right| \leq |\sin t|\,\|x\| \leq \|x\|.$$

Hence the assumption (iii) is satisfied with $d_1(t) = 1$, $d_2(t) = \sqrt{t^2+2}$ and $d_3(t) = t$.

The function $u(t,s,x)$ is continuous on the set $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$. Further, it is clear that

$$|u(t,s,x)| = \left|\frac{x^3}{\exp(t+s+1)}\right| = \frac{|x|^3}{\exp(t+s+1)}$$

for all $t, s \in \mathbb{R}_+$ and $x \in \mathbb{R}$. Thus the functions appearing in the assumption (iv) have the form $g(t,s) = \exp(-t-1)\exp(-s)$ and $h(t) = t^3$. Clearly we have that:

$$q(t) = \int_0^\infty g(t,s)ds = \exp(-t-1)\int_0^\infty \exp(-s)ds = \exp(-t-1),$$

$$G = \sup\{|q(t)| : t \geq 0\} = \sup\{\exp(-t-1) : t \geq 0\} = \frac{1}{e}$$

and $q(t) \to 0$ as $t \to \infty$.

$F = \sup\{|f(t,0,0)| : t \geq 0\} = \frac{127}{210}$ and the assumption (v) which has the form:

$$\frac{1}{14} + \frac{\sqrt{r^2+2}\, r^3}{5e} + \frac{127}{210} \leq r. \tag{4.46}$$

By computation we see that the number $r_0 \in [0.719716, 1.92146]$ is the solution of the inequality (4.46).

Moreover the operators T_1 and T_2 satisfy the assumption (vi). Indeed for $\varepsilon \geq 0$, $L > 0$, $\|x\| \leq r_0$ and $t, s \in [0, L]$ such that $|t-s| \leq \varepsilon$.

Without loss of generality, assuming that $|x(t)| < |x(s)|$, we obtain that

$$\begin{aligned} |(T_1x)(t) - (T_1x)(s)| &= |\exp(-|x(t)|) - \exp(-|x(s)|)| \\ &= \frac{|\,|x(t)| - |x(s)|\,|}{\exp\rho} \\ &\leq |x(t) - x(s)|, \end{aligned} \tag{4.47}$$

where $\rho \in (|x(t)|, |x(s)|)$.

Besides, without loss of generality, assuming that
$x(t) < x(s)$, we get that

$$\begin{aligned} |(T_2x)(t) - (T_2x)(s)| &= \left|\sqrt{x^2(t)+2} - \sqrt{x^2(s)+2}\right| \\ &= \frac{|x(t)-x(s)|\, 2|\xi|}{2\sqrt{\xi^2+2}} \\ &\leq |x(t) - x(s)|, \end{aligned} \tag{4.48}$$

where $\xi \in (x(t), x(s))$.

In the view of (2.15), we have by (4.47) and (4.48) that:

$$\omega_0(T_1X) \leq \omega_0(X) \tag{4.49}$$

and

$$\omega_0(T_2X) \leq \omega_0(X). \tag{4.50}$$

Fixing a nonempty and bounded subset X of the ball B_{r_0}, for $x, y \in X$, by taking $y(t)$ and $(T_1y)(t)$ instead of $x(s)$ and $(T_1x)(s)$ in (4.47), respectively, we get that:

$$|(T_1x)(t) - (T_1y)(t)| \leq |x(t) - y(t)|. \tag{4.51}$$

Using (4.51), we have that:

$$\limsup_{t\to\infty} \operatorname{diam}(T_1X)(t) \leq \limsup_{t\to\infty} \operatorname{diam} X(t). \tag{4.52}$$

From (2.16), (4.49) and (4.52), we derive that:

$$\mu(T_1X) \leq \mu(X). \tag{4.53}$$

So, it is shown by (4.53) and (4.50) that the inequalities of assumption (vi) are satisfied with the constants $m_{r_0} = 1$ and $\vartheta_{r_0} = 1$.

Next we have that the inequality of assumption (vii) corresponds to

$$\frac{1}{14} + \frac{r_0^3}{5e} < 1 \tag{4.54}$$

and (4.54) holds for $r_0 \in [0.719716, 1.92146]$.

Further, without loss of generality, we assume that $t_1 < t_2$, for all $t_1, t_2, s \in \mathbb{R}_+$ and $x \in \mathbb{R}$ with $|x| \leq r_0$, we have that

$$\begin{aligned} |u(t_1,s,x)-u(t_2,s,x)| &= \left|\frac{x^3}{\exp(t_1+s+1)} - \frac{x^3}{\exp(t_2+s+1)}\right| \\ &\leq \frac{x^3}{\exp(s+1)}\left|\frac{1}{\exp(t_1)} - \frac{1}{\exp(t_2)}\right| \\ &\leq \frac{x^3}{\exp(s+1)}\frac{|\exp(t_2)-\exp(t_1)|}{\exp(t_1+t_2)} \\ &\leq \frac{x^3}{\exp(s+1)}\frac{|t_2-t_1|\exp(\xi)}{\exp(t_1+t_2)} \\ &\leq \frac{r_0^3}{\exp(s+1)}|t_2-t_1|, \end{aligned}$$

where $\xi \in (t_1, t_2)$. If we put $\phi_{r_0}(t) = r_0^3\, t$ and $p(s) = \frac{1}{\exp(s+1)}$, the assumption $(viii)$ is satisfied.

Finally, it is clear that the inequality

$$\begin{aligned} |u(t,s,x)-u(t,s,y)| &= \left|\frac{x^3}{\exp(t+s+1)} - \frac{y^3}{\exp(t+s+1)}\right| \\ &\leq \frac{|x-y|\left|x^2+xy+y^2\right|}{\exp(t+s+1)} \\ &\leq \frac{3r_0^2|x-y|}{\exp(s+1)} \end{aligned}$$

holds for all $t, s \in \mathbb{R}_+$ and $x, y \in \mathbb{R}$ with $|x| \leq r_0, |y| \leq r_0$. If we take $\eta_{r_0}(t) = 3r_0^2 t$ and $k(s) = \frac{1}{\exp(s+1)}$, the assumption (ix) is satisfied.

The result follows from Theorem 3.3.

Remark 4.3. The nonlinear integral equations (4.34) and (4.44) can't be derived from the integral equations (1.2)-(1.14) examined in [1, 4–6, 10, 15, 17, 20, 21, 23, 26–33].

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