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ABSOLUTELY SUMMING POLYNOMIALS

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Abstract. In this paper, we introduce an abstract approach to the notion of absolutely summing polynomials, and we explore several of its properties, among them that this class is a Banach ideal of homogeneous polynomials. As a consequence of the abstract approach introduced in this paper, we show that in addition to obtaining several previous results in different contexts as particular cases, it is possible to easily create new classes of homogeneous polynomials that are absolutely summing.

Розвинуто абстрактний підхід до поняття абсолютно підсумовуючих поліномів. Досліджуються їхні властивості, зокрема, показано, що цей клас є банаховим ідеалом однорідних поліномів. Наслідком абстактного підходу є не тільки результати, отримані раніше для спеціальних випадків, але й можливість побудови нових класів абсолютно підсумовуючих поліномів.

1. Introduction

There is a large number of classes of operators in the literature, see for example [1, 2, 6, 9, 11, 14, 16, 18]. Most of these previous works have followed a very similar script, trying to prove similar properties, of which we can highlight the following: characterize the elements of space by inequalities, build a suitable norm in the space, and then show that the normed space that has just been constructed is a Banach ideal of multilinear operators. Some works have also explored the concept of the n-homogeneous polynomials by seeking the same properties found for the space of multilinear applications.

Faced with so many coincidences, the concern arose to create an abstract class of operators that could generalize as many as possible of those already existing in the literature. Thinking in this direction, D. Serrano-Rodríguez in [19] introduced the abstract class of γ-summing multilinear operators. This work shows that this class is a Banach ideal of multilinear applications. However, it should be noted that the work of abstraction is not an easy task. For example, Serrano-Rodríguez's work [19] contained small gaps, which were filled by the work of G. Botelho and J. Campos in [3].

Thus, following the natural script, the proposal of this work is, in a first moment, to construct the abstract class of the absolutely γ-summing n -homogeneous polynomials. Once the abstract ideal of multilinear applications [19] and the abstract ideal of polynomials are now known, we will start to study the coherence and compatibility of the pair in the sense of Pellegrino and Ribeiro [17].

We will use the letters $E, E_1, \dots, E_n, F, G, H$ to represent Banach spaces over the same scalar-field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and E' for the topological dual of E. The closed unit ball of E is denoted by B_E. We use BAN to denote the class of all Banach spaces over $\mathbb{K}$. Given Banach spaces E and F, the symbol $E \overset{1}{\hookrightarrow} F$ means that E is a linear subspace of F and $\|x\|_F \le \|x\|_E$ for every $x \in E$. By $c_{00}(E)$ we denote the set of all E-valued finite sequences, which, as usual, can be regarded as infinite sequences by completing with zeros. For every $j \in \mathbb{N}$, $e_j = (0, \dots, 0, 1, 0, 0, \dots)$ where 1 appears at the j-th coordinate. The space of all continuous n-linear operators between $E_1, \dots, E_n$ and F is represented by

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$\mathcal{L}_n(E_1,\ldots,E_n;F)$. If $E_1=\cdots=E_n=E$, we write only $\mathcal{L}_n({}^nE;F)$ and when $n=1$, we write $\mathcal{L}(E;F)$.

A mapping $P\colon E\to F$ is said to be *n-homogeneous polynomials* if there is an $A\in\mathcal{L}_n({}^nE;F)$, such that $P(x)=A(x)^n$ for every $x\in E$, where $A(x)^n:=A\,(x,\overset{n}{\ldots},x)$. In this case, we write $P=\hat{A}$. By $\check{P}$ we denote the unique symmetric continuous n-linear operator associated to P. For each positive integer n, we denote by $\mathcal{P}_n$ the class of all continuous n-homogeneous polynomials between Banach spaces.

An ideal of homogeneous polynomials $\mathcal{Q}$ is a subclass of the class $\mathcal{P}=\bigcup\limits_{n=1}^{\infty}\mathcal{P}_n$ of all continuous homogeneous polynomials between Banach spaces, such that, for a positive integer n, Banach spaces E and F, the components

$$\mathcal{Q}_n(E;F):=\mathcal{P}_n({}^nE;F)\cap\mathcal{P}$$

satisfy:

(Ma) $\mathcal{Q}_n({}^nE;F)$ is a linear subspace of $\mathcal{P}_n({}^nE;F)$, which contains the n-homogeneous finite type polynomials, where an n-homogeneous polynomials is said finite type when

$$P(x)=\sum_{i=1}^{k}[\varphi_i(x)]^n b_i$$

with $k\in\mathbb{N}$ and $\varphi_i\in E'$, $b_i\in F$, $i=1,\ldots,k$.

(Mb) If $P\in\mathcal{Q}_n({}^nE;F)$, $u\in\mathcal{L}(G;E)$ and $v\in\mathcal{L}(F;H)$, then

$$v\circ P\circ u\in\mathcal{Q}_n({}^nG;H).$$

Moreover, $\mathcal{Q}$ is a (quasi-) normed multi-ideal if there is a function $\|\cdot\|_{\mathcal{Q}}\colon\mathcal{Q}\longrightarrow[0,\infty)$ satisfying

(M1) $\|\cdot\|_{\mathcal{Q}}$ restricted to $\mathcal{Q}_n({}^nE;F)$ is a (quasi-) norm, for all Banach spaces E and F.

(M2) $\|P_n\colon\mathbb{K}\longrightarrow\mathbb{K}:P_n(\lambda)=\lambda^n\|_{\mathcal{Q}}=1$ for all n,

(M3) If $P\in\mathcal{Q}_n({}^nE;F)$, $u\in\mathcal{L}(G;E)$ and $v\in\mathcal{L}(F;H)$, then

$$\|v\circ P\circ u\|_{\mathcal{Q}}\le\|v\|\|P\|_{\mathcal{Q}}\|u\|^n.$$

When all of the components $\mathcal{Q}_n({}^nE;F)$ are complete under this (quasi-) norm, $\mathcal{Q}$ is called the (quasi-) Banach homogeneous polynomials ideal. For a fixed homogeneous polynomials ideal $\mathcal{Q}$ and a positive integer n, the class

$$\mathcal{Q}_n:=\cup_{E,F}\mathcal{Q}_n\left({}^nE;F\right)$$

is called of n-homogeneous polynomials ideal. For more details, see [12].

Throughout this paper, we will also use the definitions of *finitely determined* and *linearly stable* sequence classes, which were recently introduced in the literature by Botelho and Campos in [3], as follows.

Definition 1.1.
- A class of vector-valued sequences γ_s, or simply a sequence class γ_s, is a rule that assigns to each $E\in BAN$ a Banach space $\gamma_s(E)$ of E-valued sequences; that is, $\gamma_s(E)$ is a vector subspace of $E^{\mathbb{N}}$ with the coordinate wise operations, such that:

$$c_{00}(E)\subseteq\gamma_s(E)\overset{1}{\hookrightarrow}\ell_\infty(E)\text{ and }\|e_j\|_{\gamma_s(\mathbb{K})}=1\text{ for every j.}$$

- A sequence class γ_s is *finitely determined* if for every sequence $(x_j)_{j=1}^{\infty}\in E^{\mathbb{N}}$, $(x_j)_{j=1}^{\infty}\in\gamma_s(E)$ if, and only if, $\sup_k\|(x_j)_{j=1}^{k}\|_{\gamma_s(E)}<+\infty$ and, in this case,

$$\|(x_j)_{j=1}^{\infty}\|_{\gamma_s(E)}=\sup_k\|(x_j)_{j=1}^{k}\|_{\gamma_s(E)}.$$

- A sequence class γ_s is said to be linearly stable if for every $u \in \mathcal{L}(E;F)$ it holds
$$(u(x_j))_{j=1}^{\infty} \in \gamma_s(F)$$
wherever $(x_j)_{j=1}^{\infty} \in \gamma_s(E)$ and $\|\hat{u} : \gamma_s(E) \to \gamma_s(F)\| = \|u\|$.
- Given sequence classes $\gamma_{s_1}, \ldots, \gamma_{s_n}, \gamma_s$, we say that $\gamma_{s_1}(\mathbb{K}) \cdots \gamma_{s_n}(\mathbb{K}) \overset{1}{\hookrightarrow} \gamma_s(\mathbb{K})$ if $\left(\lambda_j^{(1)} \cdots \lambda_j^{(n)}\right)_{j=1}^{\infty} \in \gamma_s(\mathbb{K})$ and
$$\left\|\left(\lambda_j^{(1)} \cdots \lambda_j^{(n)}\right)_{j=1}^{\infty}\right\|_{\gamma_s(\mathbb{K})} \leq \prod_{m=1}^{n} \left\|\left(\lambda_j^{(m)}\right)_{j=1}^{\infty}\right\|_{\gamma_{s_m}(\mathbb{K})}$$
whenever $\left(\lambda_j^{(m)}\right)_{j=1}^{\infty} \in \gamma_{s_m}(\mathbb{K})$, $m = 1, \ldots, n$.

Example 1.2. Let $1 \leq p < \infty$. The correspondences below are linearly stable sequence classes:

(a) The correspondence, $E \mapsto \ell_p(E)$, where
$$\ell_p(E) = \left\{(x_j)_{j=1}^{\infty} \in E^{\mathbb{N}};\ \sum_{j=1}^{\infty} \|x_j\|^p < \infty\right\},$$
endowed with the norm $\left\|(x_j)_{j=1}^{\infty}\right\|_p = \left(\sum_{j=1}^{\infty} \|x_j\|^p\right)^{\frac{1}{p}}$.

(b) The correspondence, $E \mapsto \ell_p^w(E)$, where
$$\ell_p^w(E) = \left\{(x_j)_{j=1}^{\infty} \in E^{\mathbb{N}};\ \sum_{j=1}^{\infty} |\varphi(x_j)|^p < \infty, \text{ for all } \varphi \in E'\right\},$$
endowed with the norm $\left\|(x_j)_{j=1}^{\infty}\right\|_{w,p} = \sup_{\varphi \in B_{E'}} \left\|(\varphi(x_j))_{j=1}^{\infty}\right\|_p$.

(c) The correspondence, $E \mapsto \ell_p\langle E\rangle$, where
$$\ell_p\langle E\rangle =$$
$$\left\{(x_j)_{j=1}^{\infty} \in E^{\mathbb{N}};\ \sum_{j=1}^{\infty} |\varphi_j(x_j)| < \infty, \text{ for all } (\varphi_j)_{j=1}^{\infty} \in \ell_{p*}^w(E') \text{ with } \frac{1}{p} + \frac{1}{p*} = 1\right\},$$
endowed with the norm $\left\|(x_j)_{j=1}^{\infty}\right\|_{C,p} = \sup_{(\varphi_j)_{j=1}^{\infty} \in B_{\ell_{p*}^w(E')}} \sum_{j=1}^{\infty} |\varphi_j(x_j)|$.

(d) The correspondence, $E \mapsto \ell_p^{mid}(E)$, where
$$\ell_p^{mid}(E) = \left\{(x_j)_{j=1}^{\infty} \in \ell_p^w(E);\ \sum_{j,n=1}^{\infty} |\varphi_n(x_j)|^p < \infty \text{ for all } (\varphi_n)_{n=1}^{\infty} \in \ell_p^w(E')\right\},$$
endowed with the norm $\|(x_j)_{j=1}^{\infty}\|_{mid,p} = \sup_{(\varphi_n)_{n=1}^{\infty} \in B_{\ell_p^w(E')}} \left(\sum_{n=1}^{\infty}\sum_{j=1}^{\infty} |\varphi_n(x_j)|^p\right)^{\frac{1}{p}}$.

(e) The correspondence, $E \mapsto \ell_p^u(E)$, where
$$\ell_p^u(E) = \left\{(x_j)_{j=1}^{\infty} \in \ell_p^w(E);\ \lim_{k\to\infty} \left\|(x_j)_{j=k}^{\infty}\right\|_{w,p} < \infty\right\},$$
endowed with the norm $\|(x_j)_{j=1}^{\infty}\|_{u,p} := \|(x_j)_{j=1}^{\infty}\|_{w,p}$.

(f) The correspondence, $E \mapsto Rad(E)$, where $Rad(E)$ is the set of all almost unconditionally summable E-valued sequences, in the sense of [10, Chapter 12], endowed with the norm $\|(x_j)_{j=1}^{\infty}\|_{Rad(E)} = \left(\int_0^1 \left\|\sum_{j=1}^{\infty} r_j(t)x_j\right\|^2 dt\right)^{\frac{1}{2}}$, where $(r_j)_{j=1}^{\infty}$ are the Rademacher functions.

In [3, 4], it has been shown that the items (a), (b), (c) and (d) of Example 1.2 are finitely determined sequence classes.

2. Absolutely γ - summing polynomials

For this study, we will consider sequence class $\gamma_s, \gamma_{s_1}, ..., \gamma_{s_m}$ to be finitely determined and linearly stable, as defined in [3].

Definition 2.1. Let E and F be Banach spaces. An m-homogeneous polynomial $P : E \longrightarrow F$ is said to be γ_{s,s_1} - summing at $a \in E$, if

$$(P(a + x_j) - P(a))_{j=1}^{\infty} \in \gamma_s(F)$$

whenever $(x_j)_{j=1}^{\infty} \in \gamma_{s_1}(E)$.

The space of all m-homogeneous polynomials γ_{s,s_1} - summing at a, as denoted by $\mathcal{P}^{(a)}_{\gamma_{s,s_1}}(^mE;F)$, is a linear subspace of the $\mathcal{P}(^mE;F)$. When $a = 0$, we write only $\mathcal{P}_{\gamma_{s,s_1}}(^mE;F)$. The space of all m-homogeneous polynomials γ_{s,s_1}-summing at every point will be denoted by $\mathcal{P}^{(ev)}_{\gamma_{s,s_1}}(^mE;F)$.

Example 2.2. (a) When we consider $\gamma_s = \ell_p$ and $\gamma_{s_1} = \ell_q^w$, we obtain the class of (p,q)-summing homogeneous polynomials, where an n-homogeneous polynomial is said (p,q)-summing at $a \in E$ when

$$(P(a + x_j) - P(a))_{j=1}^{\infty} \in \ell_p(F)$$

whenever $(x_j)_{j=1}^{\infty} \in \ell_q^w(E)$. For more details, see [14].

(b) When we consider $\gamma_s = \ell_p\langle\rangle$ and $\gamma_{s_1} = \ell_p$, we obtain the class of p-summing absolutely Cohen n-homogeneous polynomials, where an n-homogeneous polynomials is said p-summing absolutely Cohen at $a \in E$ when

$$(P(a + x_j) - P(a))_{j=1}^{\infty} \in \ell_p\langle F\rangle$$

whenever $(x_j)_{j=1}^{\infty} \in \ell_p(E)$. This class was explored in details in the thesis [7].

(c) When we consider $\gamma_s = Rad$ and $\gamma_{s_1} = \ell_p^u$, we obtain the class of almost (p)-summing n-homogeneous polynomials, where an n-homogeneous polynomials is said almost (p)-summing at $a \in E$ when

$$(P(a + x_j) - P(a))_{j=1}^{\infty} \in Rad(F)$$

whenever $(x_j)_{j=1}^{\infty} \in \ell_p^u(E)$. For more details about this class, see [16]. In the paper [3] was showed that the classes Rad and ℓ_p^u are not finitely determined. Thus, this definition can be applied in classes more general than those cited so far.

By using the Polarization Formula [10, Corollary 1.6] we can easily prove the following result:

Proposition 2.3. *$P \in \mathcal{P}^{(ev)}_{\gamma_{s,s_1}}(^mE;F)$ if, and only if, $\check{P}$ is γ_{s,s_1}-summing in every point $(a_1, \ldots, a_m) \in E \times \overset{m}{\cdots} \times E$, according to* [19, Definition 3].

The following lemma, whose proof can be obtained by using Proposition 2.3 and [2, Lemma 2], it is essential for the proof of the main result of this section.

Lemma 2.4. *If $P \in \mathcal{P}^{(ev)}_{\gamma_{s,s_1}}(^mE;F)$ and $a \in E$, then there is a constant $C_a > 0$, such that*

$$\left\|(P(a+x_j)-P(a))_{j=1}^{\infty}\right\|_{\gamma_s(F)} \leq C_a,$$

for all $(x_j)_{j=1}^{\infty} \in \gamma_{s_1}(E)$ and $\left\|(x_j)_{j=1}^{\infty}\right\|_{\gamma_{s_1}(E)} \leq 1$.

The next result is a characterization by inequality of the operators in $\mathcal{P}^{(ev)}_{\gamma_{s,s_1}}(^mE;F)$. The same is very important because from it we can extract a norm that makes $\mathcal{P}^{(ev)}_{\gamma_{s,s_1}}(^mE;F)$ a Banach space. The proof was inspired on [1] and [14].

Theorem 2.5. *Let $P \in \mathcal{P}(^mE;F)$. The following assertions are equivalents:*

(a) $P \in \mathcal{P}^{(ev)}_{\gamma_{s,s_1}}(^mE;F)$;

(b) *There is $C > 0$ satisfying*

$$\left\|(P(b+x_j)-P(b))_{j=1}^{\infty}\right\|_{\gamma_s(F)} \leq C\left(\|b\| + \left\|(x_j)_{j=1}^{\infty}\right\|_{\gamma_{s_1}(E)}\right)^m \tag{2.1}$$

for all $b \in E$ and $(x_j)_{j=1}^{\infty} \in \gamma_{s_1}(E)$.

(c) *There is $C > 0$ satisfying*

$$\left\|(P(b+x_j)-P(b))_{j=1}^{n}\right\|_{\gamma_s(F)} \leq C\left(\|b\| + \left\|(x_j)_{j=1}^{n}\right\|_{\gamma_{s_1}(E)}\right)^m$$

for all $n \in \mathbb{N}$ and $x_1, \cdots, x_n, b \in E$.

Proof. $(b) \Rightarrow (c)$ is immediate. Using the fact that the sequence classes considered are finitely determined, it follows that $(c) \Rightarrow (a)$.

Therefore, it remains to prove that $(a) \Rightarrow (b)$.

Let $G = E \times \gamma_{s_1}(E)$. For each $P \in \mathcal{P}^{(ev)}_{\gamma_{s,s_1}}(^mE;F)$, set the following application

$$\eta_{\gamma_{s,s_1}}(P): G \longrightarrow \gamma_s(F)$$

given by

$$\eta_{\gamma_{s,s_1}}(P)\left(\left(b,(x_j)_{j=1}^{\infty}\right)\right) = (P(b+x_j)-P(b))_{j=1}^{\infty}.$$

$\eta_{\gamma_{s,s_1}}(P)$ is an m-homogeneous polynomial. Indeed, for each

$$T \in \prod_{\gamma_{s,s_1,\ldots,s_m}}^{ev}(E_1,\ldots,E_n;F),$$

we can consider the continuous m-linear operator,

$$\Phi(T)\colon G_1 \times \cdots \times G_m \to \gamma_s(F)$$

where $G_i = E_i \times \gamma_{s_i}(E_i)$, $i = 1,\ldots,m$, given by

$$\Phi(T)\left(a_1,\left(x_j^{(1)}\right)_{j=1}^{\infty},\ldots,a_m,\left(x_j^{(m)}\right)_{j=1}^{\infty}\right) = \left(T\left(a_1+x_j^{(1)},\ldots,a_m+x_j^{(m)}\right)-T(a_1,\ldots,a_m)\right)_{j=1}^{\infty}.$$

For more details, see demonstration of [19, Theorem 2]. In this way,

$$\eta_{\gamma_{s,s_1}}(P)\left(\left(b,(x_j)_{j=1}^{\infty}\right)\right) = \left(\check{P}(b+x_j)^m - \check{P}(b)^m\right) = \Phi(\check{P})\left(\left(b,(x_j)_{j=1}^{\infty}\right)\right)^m.$$

To show that $\eta_{\gamma_{s,s_1}}(P)$ is continuous, we will consider, for all $k \in \mathbb{N}$ and $(x_j)_{j=1}^{\infty} \in \gamma_{s_1}(F)$, the set

$$F_{k,(x_j)_{j=1}^{\infty}} = \left\{b \in E : \left\|\eta_{\gamma_{s,s_1}}(P)\left(\left(b,(x_j)_{j=1}^{\infty}\right)\right)\right\|_{\gamma_s(F)} \leq k\right\}.$$

Note that the set $F_{k,(x_j)_{j=1}^{\infty}}$ is closed for all $b \in E$ and $(x_j)_{j=1}^{\infty} \in B_{\gamma_{s_1}(F)}$. Indeed, for each $n \in \mathbb{N}$, let

$$F_{k,(x_j)_{j=1}^{n}} = \left\{ b \in E : \left\| \eta_{\gamma_{s,s_1}}(P)\left(\left(b,(x_j)_{j=1}^{n}\right)\right) \right\|_{\gamma_s(F)} \leq k \right\}.$$

So,

$$F_{k,(x_j)_{j=1}^{\infty}} = \bigcap_{n\in\mathbb{N}} F_{k,(x_j)_{j=1}^{n}}. \tag{2.2}$$

For each $(x_j)_{j=1}^{\infty} \in B_{\gamma_{s_1}(E)}$, and fixed $k \in \mathbb{N}$, we can define

$$D_k : E \longrightarrow [0,\infty)$$

given by

$$D_k(b) = \left\| (P(b+x_j) - P(b))_{j=1}^{n} \right\|_{\gamma_s(F)}.$$

It following the same ideas as [19, Theorem 2], we can see that D_k is a continuous application. So, each $F_{k,(x_j)_{j=1}^{n}}$ is closed because

$$F_{k,(x_j)_{j=1}^{n}} = D_k^{-1}([0,k]).$$

Therefore, from (2.2) it follows that $F_{k,(x_j)_{j=1}^{\infty}}$ is closed because it is the intersection of closed sets.

Let

$$F_k = \bigcap_{(x_j)_{j=1}^{\infty} \in B_{\gamma_{s_1}^u(E)}} F_{k,(x_j)_{j=1}^{\infty}}.$$

By the Lemma (2.4) it follows that

$$E = \bigcup_{k\in\mathbb{N}} F_k.$$

Using the Baire Category Theorem, we know that there is a constant $k_0 \in \mathbb{N}$ such that F_{k_0} has an interior point. Let b be in the interior of F_{k_0}. Thus there is $0 < \epsilon < 1$ such that

$$\left\| \eta_{\gamma_{s,s_1}}(P)\left(c,(x_j)_{j=1}^{\infty}\right) \right\|_{\gamma_s(F)} \leq k_0 \tag{2.3}$$

whenever $\|c-b\| < \epsilon$ and $(x_j)_{j=1}^{\infty} \in B_{\gamma_{s_1}(E)}$. Note that, if

$$\left\| \left(v,(x_j)_{j=1}^{\infty}\right) \right\| < \epsilon$$

we have that

$$\|v\| < \epsilon \text{ and } \left\| (x_j)_{j=1}^{\infty} \right\|_{\gamma_{s_1}(E)} < \epsilon < 1.$$

So, by (2.3) it follows that

$$\left\| \eta_{\gamma_{s,s_1}}(P)\left(b+v,(x_j)_{j=1}^{\infty}\right) \right\|_{\gamma_s(F)} \leq k_0.$$

Therefore, $\eta_{\gamma_{s,s_1}}(P)$ is bounded in the open ball of radius ϵ centered at

$$\left(b,(0)_{j=1}^{\infty}\right) \in G,$$

and we conclude that $\eta_{\gamma_{s,s_1}}(P)$ is continuous. Therefore,

$$\begin{aligned} \left\| (P(b+x_j) - P(b))_{j=1}^{\infty} \right\|_{\gamma_s(F)} &= \left\| \eta_{\gamma_{s,s_1}}(P)\left(\left(b,(x_j)_{j=1}^{\infty}\right)\right) \right\|_{\gamma_s(F)} \\ &\leq \left\| \eta_{\gamma_{s,s_1}}(P) \right\| \left(\|b\| + \left\| (x_j)_{j=1}^{\infty} \right\|_{\gamma_{s_1}(E)} \right)^m. \end{aligned} \tag{2.4}$$

□

By straightforward computations, we can get the following result.

Corollary 2.6. *The infimum of the constants* $C > 0$ *that satisfy the inequality* (2.1) *defines a norm in* $\mathcal{P}^{(ev)}_{\gamma_{s,s_1}}({}^mE;F)$, *that will be denoted by* $\pi^{(ev)}(\cdot)$.

It is not difficult to see that

Remark 2.7. $\pi^{(ev)}(P) = \left\|\eta_{\gamma_{s,s_1}}(P)\right\|$.

An alternative way of constructing a normed space of the polynomials associated by $\prod^{(ev)}_{s,s_1}$ is to consider the set

$$\mathcal{P}_{\prod^{ev}_{\gamma_{s,s_1}}} := \left\{P \in \mathcal{P};\ \check{P} \text{ is } \gamma_{s,s_1} \text{ - summing in every point}\right\}.$$

and, in this set, to use the norm inherited from the ideal of multilinear applications $\prod^{ev}_{\gamma_{s,s_1}}$, that is,

$$\|P\|_{\mathcal{P}_{\prod^{ev}_{\gamma_{s,s_1}}}} := \|\check{P}\|_{\prod^{ev}_{\gamma_{s,s_1}}} = \pi^{ev}(\check{P}).$$

The advantage of this approach is that it is already established in the literature (see, for example, [2, page 46]) that this set, with this norm, is a Banach ideal of n-homogeneous polynomials.

But then, one question arises: What is the relationship between the norms $\pi^{(ev)}(P)$ and $\|P\|_{\mathcal{P}_{\prod^{ev}_{\gamma_{s,s_1}}}}$? The answer of this question is given in the next proposition.

Proposition 2.8. *The norm* $\pi^{(ev)}(\cdot)$, *defined in Corollary* 2.6, *satisfies the relation*

$$\pi^{(ev)}(P) \leq \pi^{ev}(\check{P}) \leq \frac{m^m}{m!}\pi^{(ev)}(P)$$

for any $P \in \mathcal{P}^{(ev)}_{\gamma_{s,s_1}}({}^mE;F)$.

Proof. If $P \in \mathcal{P}^{(ev)}_{\gamma_{s,s_1}}({}^mE;F)$, then, by Proposition 2.3, $\check{P}$ is γ_{s,s_1}-summing in every point. In this way, for any $(x_j)_{j=1}^{\infty} \in \gamma_{s_1}(E)$ and $a \in E$, we have

$$\begin{aligned}\left\|\left(P\left(a+x_j\right)-P(a)\right)_{j=1}^{\infty}\right\|_{\gamma_s(F)} &= \left\|\left(\check{P}\left(a+x_j\right)^m-\check{P}(a)^m\right)_{j=1}^{\infty}\right\|_{\gamma_s(F)} \\ &\leq \pi^{ev}(\check{P})\left(\|a\|+\left\|(x_j)_{j=1}^{\infty}\right\|_{\gamma_{s_1}(E)}\right)^m,\end{aligned}$$

from which it follows that $\pi^{(ev)}(P) \leq \pi^{ev}(\check{P})$.

For the other inequality, we will use the same tools that appear in the demonstration of [19, Theorem 2]. Let $G = E \times \gamma_{s_1}(E)$ be gifted with sum norm and $\Phi : \prod^{ev}_{\gamma_{s,s_1}}(E^m;F) \to \mathcal{L}(G, \overset{m}{\ldots}, G; \gamma_s(F))$ be defined by

$$\begin{aligned}\Phi(T)&\left(\left(a_1,\left(x_j^{(1)}\right)_{j=1}^{\infty}\right),\ldots,\left(a_m,\left(x_j^{(m)}\right)_{j=1}^{\infty}\right)\right) \\ &= \left(T\left(a_1+x_j^{(1)},\ldots,a_m+x_j^{(m)}\right)-T(a_1,\ldots,a_m)\right)_{j=1}^{\infty}.\end{aligned}$$

In [19] it was proved that $\pi^{ev}(\check{P}) = \|\Phi(\check{P})\|$. Using the Polarization Formula and that $\eta_{\gamma_{s,s_1}}(P)$ is an m-homogeneous polymonial, defined in the proof of Theorem 2.5, we obtain

$$\|\Phi(\check{P})\| \leq \frac{m^m}{m!}\|\eta_{\gamma_{s,s_1}}(P)\|$$

Therefore, it follows from Remark 2.7 that

$$\pi^{(ev)}(P) \leq \pi^{ev}(\check{P}) \leq \frac{m^m}{m!}\pi^{(ev)}(P).$$

□

The inequality established in this proposition was already expected, since there is in the literature a relationship between the norm of the an m-homogeneous polynomial P and the symmetric m-linear application associate to P, by.

$$\|P\| \leq \left\|\check{P}\right\| \leq \frac{m^m}{m!}\|P\|$$

which was shown in [15, Theorem 2.2]. This same shows that the constant $m^m/m!$ is the best possible solution. For more details, see [15, example 2I].

We also emphasize that the Proposition 2.3 is of great importance because through it we have that $\mathcal{P}^{(ev)}_{\gamma_{s,s_1}}$ is a homogeneous polynomials ideal. We now need proof that this is a normed homogeneous polynomials ideal and complete (Banach), with the norm $\pi^{(ev)}(\cdot)$.

Using standard computations, it is not difficult to show the following result.

Proposition 2.9. (a) *Let $id_{\mathbb{K}} : \mathbb{K} \longrightarrow \mathbb{K}$ be given by $id_{\mathbb{K}}(x) = x^m$ and suppose that*

$$\gamma_{s_1}(\mathbb{K})\overset{m}{\cdots}\gamma_{s_1}(\mathbb{K}) \overset{1}{\hookrightarrow} \gamma_s(\mathbb{K}).$$

Then, $id_{\mathbb{K}} \in \mathcal{P}^{(ev)}_{\gamma_{s,s_1}}({}^m\mathbb{K};\mathbb{K})$ and $\pi^{(ev)}(id_{\mathbb{K}}) = 1$.

(b) *The space $\mathcal{P}^{(ev)}_{\gamma_{s,s_1}}({}^mE;F)$ is complete under the norm $\pi^{(ev)}(\cdot)$.*

Theorem 2.10. $\left(\mathcal{P}^{(ev)}_{\gamma_{s,s_1}}, \pi^{(ev)}(\cdot)\right)$ *is a Banach ideal of homogeneous polynomial between Banach spaces.*

Proof. Let $u \in \mathcal{L}(G;E)$, $P \in \mathcal{P}({}^mE;F)$, $t \in \mathcal{L}(F;H)$ and $a \in G$. Given that $\mathcal{P}^{(ev)}_{\gamma_{s,s_1}}$ is a homogeneous polynomials ideal, then $t\circ P\circ u \in \mathcal{P}^{(ev)}_{\gamma_{s,s_1}}({}^mG;H)$. Now, if $(x_j)_{j=1}^\infty \in \gamma_{s_1}(G)$, then it follows from the linear stability of γ_s and γ_{s_1} that

$$\begin{aligned}
\left\|\left(t\circ P\circ u(a+x_j) - t\circ P\circ u(a)\right)_{j=1}^\infty\right\|_{\gamma_s(H)} &\leq \|t\| \left\|\left(P\left(u(a+x_j)\right) - P\left(u(a)\right)\right)_{j=1}^\infty\right\|_{\gamma_s(F)} \\
&= \|t\| \left\|\left(P\left(u(a)+u(x_j)\right) - P\left(u(a)\right)\right)_{j=1}^\infty\right\|_{\gamma_s(F)} \\
&\leq \|t\|\pi^{(ev)}(P)\left(\|u(a)\| + \left\|(u(x_j))_{j=1}^\infty\right\|_{\gamma_{s_1}(E)}\right)^m \\
&\leq \|t\|\pi^{(ev)}(P)\|u\|\left(\|a\| + \left\|(x_j)_{j=1}^\infty\right\|_{\gamma_{s_1}(G)}\right)^m.
\end{aligned}$$

So, $\mathcal{P}^{(ev)}_{\gamma_{s,s_1}}$ satisfies the ideal property and

$$\pi^{(ev)}(t\circ P\circ u) \leq \|t\|\pi^{(ev)}(P)\|u\|.$$

Therefore, it follows from Proposition 2.9 that $\left(\mathcal{P}^{(ev)}_{\gamma_{s,s_1}}, \pi^{(ev)}(\cdot)\right)$ is a homogeneous polynomial ideal Banach between Banach spaces. □

The results of this section show us several properties about the homogeneous polynomial classes shown in Example 2.2. Among other advantages, we obtain that this class is a Banach ideal of homogeneous polynomials. However, since *Rad* and ℓ_p^u are not linearly stable, we cannot claim that the class of item (c) of Example 2.2 satisfies the equivalence (a) and (b) of Theorem 2.5. For more detail, see [3].

3. Coherence and compatibility

In this section, we will study the coherence and the compatibility of the pairs formed by the ideals of γ-summing multilinear applications and γ-summing homogeneous polynomials. This concept that was introduced in the literature by Pellegrino and Ribeiro in [17], and their definitions are presented below.

We will consider the sequence $(\mathcal{U}_k, \mathcal{M}_k)_{k=1}^N$, where each $\mathcal{U}_k$ is a (quasi-) normed ideal of k-homogeneous polynomials and each $\mathcal{M}_k$ is a (quasi-) normed ideal of k-linear mappings. The parameter N can eventually be infinity.

Definition 3.1 (Compatible pair of ideals)**.** Let $\mathcal{U}$ be a normed operator ideal and $N \in (\mathbb{N} - \{1\}) \cup \{\infty\}$. A sequence $(\mathcal{U}_n, \mathcal{M}_n)_{n=1}^N$, with $\mathcal{U}_1 = \mathcal{M}_1 = \mathcal{U}$, is compatible with $\mathcal{U}$ if there exists positive constants $\alpha_1, \alpha_2, \alpha_3$ such that for all Banach spaces E and F, the following conditions hold for all $n \in \{2, \cdots, N\}$:

(CP1) If $k \in \{1, \ldots, n\}$, $T \in \mathcal{M}_n(E_1, \ldots, E_n; F)$ and $a_j \in E_j$ for all $j \in \{1, \ldots, n\} \setminus \{k\}$, then $T_{a_1,\ldots,a_{k-1},a_{k+1},\ldots,a_n} \in \mathcal{U}(E_k; F)$ and
$$\left\|T_{a_1,\ldots,a_{k-1},a_{k+1},\ldots,a_n}\right\| \leq \alpha_1 \|T\|_{\mathcal{M}_n} \|a_1\| \cdots \|a_{k-1}\| \, \|a_{k+1}\| \cdots \|a_n\|.$$
(CP2) If $P \in \mathcal{U}_n({}^n E; F)$ and $a \in E$, then $P_{a^{n-1}} \in \mathcal{U}(E; F)$ and
$$\|P_{a^{n-1}}\|_{\mathcal{U}} \leq \alpha_2 \max\left\{\left\|\check{P}\right\|_{\mathcal{M}_n}, \|P\|_{\mathcal{U}_n}\right\}\|a\|^{n-1}.$$
(CP3) If $u \in \mathcal{U}(E_n; F)$, $\gamma_j \in E_j'$ for all $j = 1, \ldots, n-1$, then
$$\gamma_1 \cdots \gamma_{n-1} u \in \mathcal{M}_n(E_1, \ldots, E_n; F)$$
and
$$\|\gamma_1 \cdots \gamma_{n-1} u\|_{\mathcal{M}_n} \leq \alpha_3 \|\gamma_1\| \cdots \|\gamma_{n-1}\| \, \|u\|_{\mathcal{U}} .$$
(CP4) If $u \in \mathcal{U}(E; F)$ and $\gamma \in E'$, then $\gamma^{(n-1)} u \in \mathcal{U}_n({}^n E; F)$.

(CP5) P belongs to $\mathcal{U}_n({}^n E; F)$ if, and only if, $\check{P}$ belongs to $\mathcal{M}_n({}^n E; F)$.

Definition 3.2 (Coherent pair of ideals)**.** Let $\mathcal{U}$ be a normed operator ideal and let $N \in \mathbb{N} \cup \{\infty\}$. A sequence $(\mathcal{U}_k, \mathcal{M}_k)_{k=1}^N$, with $\mathcal{U}_1 = \mathcal{M}_1 = \mathcal{U}$, is coherent if there exist positive constants $\beta_1, \beta_2, \beta_3$ such that for all Banach spaces E and F the following conditions hold for $k = 1, \ldots, N-1$:

(CH1) If $T \in \mathcal{M}_{k+1}(E_1, \ldots, E_{k+1}; F)$ and $a_j \in E_j$ for $j = 1, \ldots, k+1$, then
$$T_{a_j} \in \mathcal{M}_k(E_1, \ldots, E_{j-1}, E_{j+1}, \ldots, E_{k+1}; F)$$
and
$$\left\|T_{a_j}\right\|_{\mathcal{M}_k} \leq \beta_1 \|T\|_{\mathcal{M}_{k+1}} \|a_j\|.$$
(CH2) If $P \in \mathcal{U}_{k+1}\left({}^{k+1}E; F\right)$, $a \in E$, then P_a belongs to $\mathcal{U}_k\left({}^k E; F\right)$ and
$$\|P_a\|_{\mathcal{U}_k} \leq \beta_2 \max\left\{\left\|\check{P}\right\|_{\mathcal{M}_{k+1}}, \|P\|_{\mathcal{U}_{k+1}}\right\}\|a\|.$$
(CH3) If $T \in \mathcal{M}_k(E_1, \ldots, E_k; F)$, $\gamma \in E_{k+1}'$, then
$$\gamma T \in \mathcal{M}_{k+1}(E_1, \ldots, E_{k+1}; F)$$
and
$$\|\gamma T\|_{\mathcal{M}_{k+1}} \leq \beta_3 \|\gamma\| \, \|T\|_{\mathcal{M}_k} .$$
(CH4) If $P \in \mathcal{U}_k\left({}^k E; F\right)$ and $\gamma \in E'$, then $\gamma P \in \mathcal{U}_{k+1}\left({}^{k+1}E; F\right)$.

(CH5) For all $k = 1, \ldots, N$, P belongs to $\mathcal{U}_k({}^k E; F)$ if, and only if, $\check{P}$ belongs to $\mathcal{M}_k({}^k E; F)$.

In this section, we will denote the Banach γ_{s,s_1}-summing m-linear operators ideal and the Banach γ_{s,s_1}-summing m-homogeneous polynomials ideal by $\left(\prod_{\gamma_{s,s_1}}^{m,ev}; \pi^{m,ev}(\cdot)\right)$ and $\left(\mathcal{P}_{\gamma_{s,s_1}}^{m,(ev)}; \pi^{m,(ev)}(\cdot)\right)$, respectively. The reason for this is to evidence the linearity/homogeneity of the components of the ideal.

We will study the coherence and the compatibility of the pair $\left(\mathcal{P}_{\gamma_{s,s_1}}^{m,(ev)}, \prod_{\gamma_{s,s_1}}^{m,(ev)}\right)_{m=1}^N$ with the ideal of all absolutely γ_{s,s_1}-summing linear operators $\prod_{\gamma_{s,s_1}}$.

Remark 3.3. For any Banach spaces E and F,

$$\prod_{\gamma_{s,s_1}}^{1,ev}(E;F)=\mathcal{P}^{1,(ev)}_{\gamma_{s,s_1}}(E;F)=\prod_{\gamma_{s,s_1}}(E;F).$$

In the next two propositions, we will check the conditions (CH1) and (CH2) of Definition 3.2

Proposition 3.4. *For each $T\in\prod_{\gamma_{s,s_1}}^{m+1,ev}(E_1,\dots,E_{m+1};F)$ and $(a_1,\dots,a_{m+1})\in E_1\times\cdots\times E_{m+1}$,*

$$T_{a_k}(x_1,\dots,x_{k-1},x_{k+1},\dots,x_{m+1}):=T(x_1,\dots,x_{k-1},a_k,x_{k+1},\dots,x_{m+1})$$

belongs to $\prod_{\gamma_{s,s_1}}^{m,ev}(E_1,\dots,E_{k-1},E_{k+1},\dots,E_{m+1};F)$ and

$$\pi^{m,ev}(T_{a_k})\le\pi^{m+1,ev}(T)\|a_k\|.$$

Proof. Let $T\in\prod_{\gamma_{s,s_1}}^{m+1,ev}(E_1,\dots,E_{m+1};F)$ and $(a_1,\dots,a_{m+1})\in E_1\times\cdots\times E_{m+1}$, $\left(x_j^{(n)}\right)_{j=1}^{\infty}\in\gamma_{s_1}(E_n)$, for $n=1,\dots,k-1,k+1,\dots,m+1$. We will do the computations only for $k=1$. The remaining cases are similar. Thus, for each $b_i\in E_i$ and $\left(x_j^i\right)_{j=1}^{\infty}\in\gamma_{s_1}(E_i)$, $i=2,\dots,m$, consider the null-sequence $\left(x_j^{(1)}\right)_{j=1}^{\infty}\in\gamma_{s_1}(E_1)$; that is, $x_j^{(1)}=0$ for every $j\in\mathbb{N}$. Then,

$$\begin{aligned}&\left(T_{a_1}(b_2+x_j^{(2)},\dots,b_{m+1}+x_j^{(m+1)})-T_{a_1}(b_2,\dots,b_{m+1})\right)_{j=1}^{\infty}\\&\quad=\left(T(a_1+x_j^{(1)},b_2+x_j^{(2)},\dots,b_{m+1}+x_j^{(m+1)})-T(a_1,b_2,\dots,b_{m+1})\right)_{j=1}^{\infty}\in\gamma_s(F).\end{aligned}$$

Thus, $T_{a_1}\in\prod_{\gamma_{s,s_1}}^{m,ev}(E_2,\dots,E_{m+1};F)$. So,

$$\begin{aligned}&\left\|\left(T_{a_1}(b_2+x_j^{(2)},\dots,b_{m+1}+x_j^{(m+1)})-T_{a_1}(b_2,...,b_{m+1})\right)_{j=1}^{\infty}\right\|_{\gamma_s(F)}\\&\le\pi^{m+1,ev}(T)\|a_1\|\left(\|b_2\|+\left\|\left(x_j^{(2)}\right)_{j=1}^{\infty}\right\|_{\gamma_{s_1}(E_1)}\right)\cdots\left(\|b_{m+1}\|+\left\|\left(x_j^{(m+1)}\right)_{j=1}^{\infty}\right\|_{\gamma_{s_1}(E_m)}\right).\end{aligned}$$

Therefore,

$$\pi^{m,ev}(T_{a_1})\le\pi^{m+1,ev}(T)\|a_1\|.$$

□

It follows from Proposition 3.4 the following result.

Proposition 3.5. *For each $P\in\mathcal{P}^{m+1,(ev)}_{\gamma_{s,s_1}}({}^{m+1}E,F)$ and $a\in E$, so P_a belongs to $\mathcal{P}^{m,(ev)}_{\gamma_{s,s_1}}({}^{m}E,F)$ and*

$$\pi^{m,ev}(P_a)\le\pi^{m+1,ev}(\check{P})\|a\|,$$

where

$$P_a(x):=\check{P}(a,x,\overset{m}{\dots},x).$$

The next definition contains an important property that will be used to prove (CH3) and (CH4) of Definition 3.2.

Definition 3.6. Let E be a Banach space and γ_s be a sequence class. We say that the sequence class γ_s is $\mathbb{K}$-closed when, for any $(x_j)_{j=1}^{\infty}\in\gamma_s(\mathbb{K})$ and $(y_j)_{j=1}^{\infty}\in\gamma_s(E)$, the sequence $(z_j)_{j=1}^{\infty}\in\gamma_s(E)$, where $z_j=x_jy_j$ and

$$\left\|(z_j)_{j=1}^{\infty}\right\|_{\gamma_s(E)}\le\left\|(x_j)_{j=1}^{\infty}\right\|_{\gamma_s(\mathbb{K})}\left\|(y_j)_{j=1}^{\infty}\right\|_{\gamma_s(E)}$$

Example 3.7. The sequence classes $\ell_p\langle\cdot\rangle$, $\ell_p(\cdot)$, $\ell_p^{mid}(\cdot)$ and $\ell_p^{w}(\cdot)$ are $\mathbb{K}$-closed.

Definition 3.8. Let γ_s and γ_{s_1} be sequence classes. We say that γ_s and γ_{s_1} are finitely coincident, when for all finite-dimensional linear space E, we have $\gamma_s(E) = \gamma_{s_1}(E)$ and

$$\left\|(x_j)_{j=1}^{\infty}\right\|_{\gamma_s(E)} = \left\|(x_j)_{j=1}^{\infty}\right\|_{\gamma_{s_1}(E)}.$$

Remark 3.9. In the next two propositions we will assume that the sequence class γ_s is $\mathbb{K}$-closed and γ_s and γ_{s_1} are finitely coincident.

Proposition 3.10. *Let $T \in \prod_{\gamma_{s,s_1}}^{m,ev}(E_1, ..., E_m; F)$ and $\varphi \in E'_{m+1}$, so*

$$\varphi T \in \prod_{\gamma_{s,s_1}}^{m+1,ev}(E_1, \dots, E_{m+1}; F)$$

and

$$\pi^{m+1,ev}(\varphi T) \leq \|\varphi\| \pi^{m,ev}(T).$$

Proof. We will do only the case $m = 2$. The other cases are analogous. Let $T \in \prod_{\gamma_{s,s_1}}^{2,ev}(E_1, E_2; F)$, $\varphi \in E'_3$ and $\left(x_j^{(i)}\right)_{j=1}^{\infty} \in \gamma_{s_1}(E_i)$, $a_i \in E_i$, $i = 1, 2, 3$. Thus, because γ_s is linearly stable, finitely determined, $\mathbb{K}$-closed, and because γ_s and γ_{s_1} are finitely coincident, it follows immediately that

$$\left(\varphi T\left(a_1 + x_j^{(1)}, a_2 + x_j^{(2)}, a_3 + x_j^{(3)}\right) - \varphi T(a_1, a_2, a_3)\right)_{j=1}^{\infty} \in \gamma_s(F)$$

and,

$$\begin{aligned}
&\left\|\left(\varphi T(a_1 + x_j^{(1)}, a_2 + x_j^{(2)}, a_3 + x_j^{(3)}) - \varphi T(a_1, a_2, a_3)\right)_{j=1}^{\infty}\right\|_{\gamma_s(F)} \\
&\leq \left\|\left(T\left(a_1, x_j^{(2)}\right)\right)_{j=1}^{\infty}\right\|_{\gamma_s(F)} \left(|\varphi(a_3)| + \left\|\left(\varphi\left(x_j^{(3)}\right)\right)_{j=1}^{\infty}\right\|_{\gamma_{s_1}(\mathbb{K})}\right) + \\
&+ \left\|\left(T\left(x_j^{(1)}, a_2\right)\right)_{j=1}^{\infty}\right\|_{\gamma_s(F)} \left(|\varphi(a_3)| + \left\|\left(\varphi\left(x_j^{(3)}\right)\right)_{j=1}^{\infty}\right\|_{\gamma_{s_1}(\mathbb{K})}\right) + \\
&+ \left\|\left(T\left(x_j^{(1)}, x_j^{(2)}\right)\right)_{j=1}^{\infty}\right\|_{\gamma_s(F)} \left(|\varphi(a_3)| + \left\|\left(\varphi\left(x_j^{(3)}\right)\right)_{j=1}^{\infty}\right\|_{\gamma_{s_1}(\mathbb{K})}\right) + \\
&+ \|T(a_1, a_2)\| \left\|\left(\varphi\left(x_j^{(3)}\right)\right)_{j=1}^{\infty}\right\|_{\gamma_{s_1}(\mathbb{K})} \\
&\leq \pi^{ev}(T)\|a_1\| \left\|\left(x_j^{(2)}\right)_{j=1}^{\infty}\right\|_{\gamma_{s_1}(E_2)} \|\varphi\| \left(\|a_3\| + \left\|\left(x_j^{(3)}\right)_{j=1}^{\infty}\right\|_{\gamma_{s_1}(E_3)}\right) + \\
&+ \pi^{ev}(T)\|a_2\| \left\|\left(x_j^{(1)}\right)_{j=1}^{\infty}\right\|_{\gamma_{s_1}(E_1)} \|\varphi\| \left(\|a_3\| + \left\|\left(x_j^{(3)}\right)_{j=1}^{\infty}\right\|_{\gamma_{s_1}(E_3)}\right) + \\
&+ \pi^{ev}(T) \left\|\left(x_j^{(1)}\right)_{j=1}^{\infty}\right\|_{\gamma_{s_1}(E_1)} \left\|\left(x_j^{(2)}\right)_{j=1}^{\infty}\right\|_{\gamma_{s_1}(E_2)} \|\varphi\| \left(\|a_3\| + \left\|\left(x_j^{(3)}\right)_{j=1}^{\infty}\right\|_{\gamma_{s_1}(E_3)}\right) + \\
&+ \pi^{ev}(T)\|a_1\|\|a_2\|\|\varphi\| \left\|\left(x_j^{(3)}\right)_{j=1}^{\infty}\right\|_{\gamma_{s_1}(E_3)} \\
&\quad = \|\varphi\|\pi^{ev}(T) \left(\prod_{i=1}^{3}\left(\|a_i\| + \left\|\left(x_j^{(i)}\right)_{j=1}^{\infty}\right\|_{\gamma_{s_1}(E_i)}\right) - \|a_1\|\|a_2\|\|a_3\|\right).
\end{aligned}$$

Consequently,

$$\left\|\left(\varphi T\left(a_1+x_j^{(1)},a_2+x_j^{(2)},a_3+x_j^{(3)}\right)-\varphi T(a_1,a_2,a_3)\right)_{j=1}^{\infty}\right\|_{\gamma_s(F)}$$
$$\leq \|\varphi\|\pi^{ev}(T)\prod_{i=1}^{3}\left(\|a_i\|+\left\|\left(x_j^{(i)}\right)_{j=1}^{\infty}\right\|_{\gamma_{s_1}(E_i)}\right).$$

Therefore $\varphi T \in \prod_{\gamma_{s,s_1}}^{m+1,ev}(E_1,\dots,E_{m+1};F)$ and

$$\pi^{m+1,ev}(\varphi T)\leq \|\varphi\|\pi^{m,ev}(T).$$

□

Using the same idea of Proposition 3.10, we get the following result.

Proposition 3.11. *Let $P\in\mathcal{P}_{\gamma_{s,s_1}}^{m,(ev)}({}^mE,F)$ and $\varphi\in E'$. Then*

$$\varphi P\in\mathcal{P}_{\gamma_{s,s_1}}^{m+1,(ev)}({}^{m+1}E;F)$$

and

$$\pi^{m+1,(ev)}(\varphi P)\leq\|\varphi\|\pi^{m,ev}(\check{P}).$$

By Propositions 3.4, 3.5, 3.10, 3.11 and 2.3, the pair

$$\left(\left(\mathcal{P}_{\gamma_{s,s_1}}^{m,(ev)},\pi^{m,(ev)}(\cdot)\right),\left(\prod_{\gamma_{s,s_1}}^{m,ev},\pi^{m,ev}(\cdot)\right)\right)_{m=1}^{\infty}$$

is coherent. Since $\beta_1=1,\beta_2=1$ and $\beta_3=1$, it follows by [17, Remark 3.3] that the pair $\left(\left(\mathcal{P}_{\gamma_{s,s_1}}^{m,(ev)},\pi^{m,(ev)}(\cdot)\right),\left(\prod_{\gamma_{s,s_1}}^{m,ev},\pi^{m,ev}(\cdot)\right)\right)_{m=1}^{\infty}$ is compatible with $\prod_{\gamma_{s,s_1}}$. So, we have the following result.

Theorem 3.12. *The sequence $\left(\left(\mathcal{P}_{\gamma_{s,s_1}}^{m,(ev)},\pi^{m+1,(ev)}(\cdot)\right),\left(\prod_{\gamma_{s,s_1}}^{m,ev},\pi^{m,ev}(\cdot)\right)\right)_{m=1}^{\infty}$ is coherent and compatible with $\prod_{\gamma_{s,s_1}}$.*

It is important to point out that to obtain the proof of Proposition 2.3, 3.4 and 3.5 it is only necessary that the sequence classes be linearly stable and finitely determined. However, to demonstrate the propositions 3.10 and 3.11, extra properties were required for the classes involved; more specifically, the sequence classes should be finitely coincident and the arrival sequence class should be $\mathbb{K}$-closed. These conditions do not appear to be very restrictive because the main classes of the summing ideals existing in the literature are recovered by our work. The next section illustrates our arguments.

4. Applications

For any Banach space E, we will denote $\ell_p\langle E\rangle,\ell_p(E)$ and $\ell_p^w(E)$ the spaces of Cohen strongly p-summing, absolutely p-summing and weakly p-summable E-valued sequences, respectively. In 2014 S. Karn and D. Sinha [13] introduced the space $\ell_p^{mid}(E)$, which was studied in more details by G. Botelho, J. Campos and J. Santos in [4]. This paper established the inclusions

$$\ell_p\langle E\rangle\subset\ell_p(E)\subset\ell_p^{mid}(E)\subset\ell_p^w(E). \tag{4.5}$$

The nature of many operators in the literature is to "improve" the convergence of series. For example, we can cite the absolutely summing operators that transform weakly p-summable sequences into absolutely p-summable sequences. Thinking in this direction, we can define several classes of operators that improve the convergence of the series. In the next two examples, we will present classes that are already known and which are particular cases of our work. In the other examples, we present a few classes of operators that are not yet available in the literature, although they can easily be obtained through the construction presented in this work.

Example 4.1. (a) Let $\mathcal{P}^{m,ev}_{(p,q)}$ be the space of absolutely summing n-homogeneous polynomials and $\prod^{m,ev}_{(p,q)}$ be the space of absolutely summing multilinear operators. For more details about this classes, see [1]. It immediately follows that the pair

$$\left(\left(\mathcal{P}^{m,ev}_{(p,q)}, \|\cdot\|_{ev^2}\right), \left(\prod\nolimits^{m,ev}_{(p,q)}, \|\cdot\|_{ev^2(p,q)}\right)\right)_{m=1}^{\infty}$$

is coherent and compatible with $\prod_{(p,q)}$.

(b) Let $\mathcal{P}^{m,ev}_{Coh,p}$ be the space of the n-homogeneous polynomials Cohen strongly p-summing everywhere and $\mathcal{L}^{m,ev}_{Coh,p}$ be the space of the multilinears operators Cohen strongly p-summing everywhere. For more details about this classes, see [7, 19]. It immediately follows that the pair $\left(\left(\mathcal{P}^{m,ev}_{Coh,p}, \pi^{m,ev}\right), \left(\mathcal{L}^{m,ev}_{Coh,p}, \pi^{m,ev}_{Coh,p}\right)\right)_{m=1}^{\infty}$ is coherent and compatible with $\mathcal{D}_p$.

Note that, the classes of multilinears operator/ homogeneous polynomials that are defined in these two examples consider applications that transform weakly strongly p-summable sequences in strongly p-summable sequences and strongly p-summable sequences in Cohen strongly p-summable. However, due to the inclusions given in (4.5), we can introduce several classes of multilinear operators and homogeneous polynomials that are yet not found in the literature. In this way, this approach establishes an interesting result for this classes. We can consider these examples:

Example 4.2. (a) The classes of multilinears operators and homogeneous polynomials that transform mid p-summable sequences in strongly p-summable operators. In other words, to consider $\gamma_s = \ell_p$ and $\gamma_{s_1} = \ell_p^{mid}$. We denote these classes as multilinear operators and homogeneous polynomials mid strongly p-summing.

(b) The classes of multilinears operators and homogeneous polynomials that transform weakly absolutely p-summable sequences in mid p-summable operators. In other words, to consider $\gamma_s = \ell_p^{mid}$ and $\gamma_{s_1} = \ell_p^{w}$. We denote these classes as multilinear operators and homogeneous polynomials weakly mid p-summing.

(c) The classes of multilinears operators and homogeneous polynomials that transform weakly strongly p-summable and mid p-sommable sequences in Cohen strongly p-summable operators. In other words, to consider $\gamma_s = \ell_p\langle\cdot\rangle$ and $\gamma_{s_1} = \ell_p^{w}, \ell_p^{mid}$. We denote these classes as multilinear operators and homogeneous polynomials weakly Cohen p-summing and mid Cohen p-summing, respectively.

The abstract approach introduced in this paper unifies the way we treat homogeneous polynomial classes that transform sequence spaces. Moreover, one of the advances of this work is to have theories already well established in the literature (see Example 4.1) and also to obtain new classes, as shown in Example 4.2.

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